WAVENUMBER EXPLICIT ANALYSIS FOR TIME-HARMONIC MAXWELL EQUATIONS IN EXTERIOR DOMAINS AND THEIR SPECTRAL APPROXIMATION

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Abstract. This paper is devoted to wavenumber explicit analysis of three-dimensional time-harmonic Maxwell equations in an exterior domain. The infinite domain is first reduced to a finite domain by using an exact transparent boundary condition involving the capacity operator. Remarkably, when the scatterer is a sphere, by using divergence-free vector spherical harmonic expansions of the fields, one can preserve divergence-free property of the electric and magnetic fields, and reduce the Maxwell system to two sequences of decoupled one-dimensional Helmholtz problems (in the radial direction) in a similar setting. This reduction not only leads to more efficient spectral-Galerkin algorithms, but also allows us to carry out, for the first time, wavenumber explicit analysis for 3-D time-harmonic Maxwell equations with exact transparent boundary conditions. We then use the transformed field expansion to deal with more general scatterers, and derive rigorous error estimates for the whole algorithm.

1. Introduction

This paper is devoted to the analysis and spectral-Galerkin approximation of three-dimensional time-harmonic Maxwell equations:

\[-i\omega\mu H + \nabla \times E = 0, \quad -i\omega\varepsilon E - \nabla \times H = F, \quad \text{in} \quad \mathbb{R}^3 \setminus \bar{D};\]

\[E \times n|_{\partial D} = 0; \quad \lim_{r \to \infty} r\left(\sqrt{\frac{\mu}{\varepsilon}} H \times \hat{x} - E\right) = 0,\]

(1.1)

where $D$ is a three-dimensional, simply connected, bounded, perfect conductor (or scatterer), $i = \sqrt{-1}$, $E$, $H$ are respectively the electric and magnetic fields, $F$ is the electric current density, $\mu$ is the magnetic permeability, $\varepsilon$ is the electric permittivity, $\omega$ is the frequency of the harmonic wave, $n$ is the outward normal and $\hat{x} = x/r$ with $r = |x|$. The boundary condition at infinity in (1.1) is known as the Silver-Müller radiation condition.

The above Maxwell equations play an important role in many scientific and engineering applications, including in particular electromagnetic wave scattering, and are also of fundamental mathematical interest (see, e.g., [31, 8, 29]). Despite its seeming simplicity, the system (1.1) is notoriously difficult to solve numerically. Some of the main challenges include: (i) the indefiniteness when $\omega$ is not sufficiently small; (ii) highly oscillatory solutions when $\omega$ is large; (iii) the incompressibility (i.e., $\text{div}(\mu H) = \text{div}(\varepsilon E) = 0$), which is implicitly implied by (1.1); and (iv) the unboundedness of the domain. On the one hand, one needs to construct approximation spaces such that the discrete problems are well posed and lead to good approximations for a wide range of wavenumbers. On the other hand, one needs to develop efficient algorithms for solving the indefinite linear system, particularly for large wavenumbers, resulted from a given discretization. We refer to [29] and the references therein, for various contributions with respect to numerical approximations of the time-harmonic Maxwell equations. The methods of choice for dealing with unbounded domains include the perfectly matched layer (PML) technique [5], boundary integral...
method [23][25], and the artificial boundary condition (ABC) [17][13][16][31]. The last approach is to
enclose the obstacles and inhomogeneities (and nonlinearities at times) with an artificial boundary. A
suitable boundary condition is then imposed, leading to a numerically solvable boundary value problem in
a finite domain. In particular, the ABC is dubbed as a transparent (or nonreflecting) boundary condition
(TBC) (or NRBC), if the solution of the reduced problem coincides with the solution of the original
problem restricted to the finite domain.

In this paper, we consider the reduced problem in a bounded domain with a TBC characterized by the
capacity operator \( \mathcal{T}_b \) (see, e.g., [31]). As usual, we assume that the electric current density \( F \) is compactly
supported in a ball \( B \) of radius \( b \) (that encloses the scatterer \( D \)), and solenoidal, i.e., \( \text{div} \, F = 0 \). Denote
\( k = \omega \sqrt{\mu \varepsilon} \) and \( \eta = \sqrt{\mu / \varepsilon} \). After eliminating the magnetic field \( H \), we can reduce (1.1) to
\[
\nabla \times \nabla \times E - k^2 E = F, \quad \text{in } \Omega := B \setminus D;
\]
\[
E \times n = 0, \quad \text{on } \partial D; \quad (\nabla \times E) \times e_r - ik\mathcal{T}_b [E_S] = h, \quad \text{at } r = b,
\]
where \( e_r = \hat{r} = x/r \), \( \mathcal{T}_b \) is the capacity operator (cf. [31] (5.3.88)), and the tangential field \( E_S = -E \times e_r \times e_r \). Here, we add a boundary data function \( h \) in (1.3) to deal with potentially inhomogeneous boundary conditions.

We shall start with the special case where \( D \) is a ball of radius \( a \):
\[
\Omega = \{(r, \theta, \phi) : a < r < b, \theta \in [0, \pi], \phi \in [0, 2\pi]\},
\]
and the solution \( E \) can be represented in terms of vector spherical harmonics (VSH).

In [31] and other related works, the usual VSH (cf. Appendix A) are used to represent \( E \). In particular, the system (1.2)-(1.3) can be reduced to a coupled system of two components of \( E \), while the other component satisfies the same equation reduced from the Helmholtz equation (cf. [26]):
\[
-\Delta U - k^2 U = F, \quad \text{in } \Omega := B \setminus D,
\]
\[
U|_{\partial D} = 0; \quad \partial_r U - \mathcal{T}_b [U] = H, \quad \text{at } r = b,
\]
where \( \mathcal{T}_b \) is the DtN operator [31] (see (2.1)). The wavenumber explicit analysis for the above Helmholtz
equation has been carried out in [30], but the analysis for the couple system of two other components
appears very difficult. In fact, only the result on well-posedness of (1.2)-(1.3) was obtained in [26].

However, if we use divergence-free vector spherical harmonics [6], the Maxwell system (1.2)-(1.3), in the
case \( D \) is a sphere, can be reduced to two sequences of one-dimensional problems, which are completely
decoupled and the same as those obtained from the Helmholtz equations (1.5) (note: one sequence is
with the boundary conditions (1.6), but the other is with a slightly different boundary condition at
\( r = a \)). Therefore, we can carry out wavenumber explicit analysis for these decoupled problems, leading
to wavenumber explicit estimates for the Maxwell equations.

There has been a longstanding research interest in wavenumber explicit estimates for the Helmholtz
and Maxwell problems. In particular, much effort has been devoted to the Helmholtz equations (see, e.g.,
[12][21][22][4][11][3][9][15][36][18][7][28] as a partial list of literature). Most of the analysis is essentially
based on the Rellich identities [27][9] (also see, e.g., [15][4][28]), which applies only to star-shaped domains
(or the bounded scatterer is of star-shape, see [7]). Moreover, most of the results were established only
for the Helmholtz equation with an approximate boundary condition: \( \partial_r U - ikU = 0 \) instead of the exact
DtN boundary condition. For the Maxwell equations, Hiptmair et al. [20] (and independently by [13])
derived for the first time the wavenumber explicit estimates for the time-harmonic Maxwell equations but
with an approximate boundary condition: \( (\nabla \times E) \times e_r - ikE_S = h \). The analysis can not be applied to
scattering problems with TBC at an artificial boundary, a situation considered in this paper. Indeed, as
shown in [35][7], the presence of the exact DtN boundary condition brought about significant challenges
even for the Helmholtz equations.

The main purposes of this paper are to extend the analysis in [36] to the Maxwell equations, and in the
meantime, provide an essential improvement, which is critical to obtaining the desired estimate for the
Maxwell equations, to an estimate for the Helmholtz equation in [36]. We demonstrate that the spectral
algorithm and analysis for the Maxwell equations in the special domain \( \Omega \) (1.1) are the major component
for dealing with general scatterers using the transformed field expansion (TFE) approach [10].
The rest of the paper is organized as follows. In Section 2, we have a delicate study of the DtN kernel in (2.2), and use the new estimates to derive the improved estimates for the Helmholtz equation (cf. Lemma 2.2 and Theorem 2.2), by removing the factor $k^{3/2}$ in [36] Thm. 3.1. In Section 3, we first reduce the Maxwell system (1.2)-(1.4) to two sequences of decoupled one-dimensional (in the radial direction) Helmholtz problems in a similar setting by using the divergence-free VSH expansions in angular directions. This step is essential for the subsequent analysis. By a delicate analysis of the decoupled one-dimensional problems (cf. Theorems 3.1 and 3.3), we derive the wavenumber explicit error estimates for the complete algorithm. Some concluding remarks are presented in the last section.

Throughout the paper, we shall use the expression $A_0$, which only depends on the domain (in particular independent of $k$), such that $A_0 \leq CB$. The notation $A \sim B$ means $A(\nu) = B(\nu)(1 + o(1))$, when the underlying parameter $\nu$ tends to infinity or zero.

2. Improved wavenumber explicit estimates for the Helmholtz equation

In this section, we improve the a priori estimates for the Helmholtz equation (1.5)-(1.6) in [36] Thm. 3.1, where the DtN operator is defined by

$$T_0[U] = \sum_{l=1}^{\infty} \sum_{|m|=0}^{l} \frac{h_1^{(1)'}(kb)}{h_1^{(1)}(kb)} U_l^m Y_l^m, \text{ where } U_l^m = \int_S U \Big|_{r=b} Y_l^m dS,$$

and $\{Y_l^m\}$ are spherical harmonics defined on the unit spherical surface $S$ (cf. Appendix A).

2.1. Properties of the DtN kernel. The key is to conduct a delicate analysis of the DtN kernel:

$$T_{l,\kappa} := \frac{h_1^{(1)'}(\kappa)}{h_1^{(1)}(\kappa)}, \quad l \geq 1, \quad \kappa > 0. \quad (2.2)$$

Recall that (cf. [36] (2.16))

$$\text{Re}(T_{l,\kappa}) = -\frac{1}{2\kappa} + \frac{J_{\nu}(\kappa)J'_\nu(\kappa) + Y_{\nu}(\kappa)Y'_\nu(\kappa)}{J^2_\nu(\kappa) + Y^2_\nu(\kappa)} , \text{ Im}(T_{l,\kappa}) = \frac{2}{\pi \kappa} \frac{1}{J^2_\nu(\kappa) + Y^2_\nu(\kappa)}, \quad (2.3)$$

for $\nu := l + 1/2$, where $J_{\nu}$ and $Y_{\nu}$ are Bessel functions of the first and second kinds, respectively, of order $\nu$ (cf. [1]). Alternatively, we can formulate

$$\text{Re}(T_{l,\kappa}) = \frac{l+1}{\kappa} - \frac{Y_{\nu+1}(\kappa)}{Y_\nu(\kappa)} - \text{Im}(T_{l,\kappa}) \frac{J_\nu(\kappa)}{Y_\nu(\kappa)} = -\frac{1}{2\kappa} + \frac{Y'_\nu(\kappa)}{Y_\nu(\kappa)} - \text{Im}(T_{l,\kappa}) \frac{J_\nu(\kappa)}{Y_\nu(\kappa)}, \quad (2.4)$$

which can be derived from (2.3) and the properties of Bessel functions. Recall that (see [31] Page 87):

$$-\frac{l+1}{\kappa} \leq \text{Re}(T_{l,\kappa}) < -\frac{1}{\kappa}, \quad 0 < \text{Im}(T_{l,\kappa}) < 1. \quad (2.5)$$

In what follows, let $0 < \theta_0 < 1$ be a prescribed constant, and let

$$\kappa_0 = \sqrt{\theta_0/2} \left( 1 - \theta_0 \right)^{-3/2} \quad (\text{e.g., } \kappa_0 \approx 21.21, \text{ if } \theta_0 = 0.9). \quad (2.6)$$

Based upon asymptotic properties of Bessel functions, we shall carry out the analysis separately for four cases (note: in the course of the analysis, we shall show how these arise (see [43,10])):

$$\rho := \frac{\nu}{\kappa} \in (0, \theta_0) \cup [\theta_0, \vartheta_1] \cup (\vartheta_1, \vartheta_2) \cup [\vartheta_2, \infty) \text{ for } \nu = l + 1/2, \ l \geq 1, \quad (2.7)$$
where $\kappa > \kappa_0$ is fixed, and

$$\vartheta_1 := \vartheta_1(\kappa) = \frac{1}{2} \left( \sqrt{1 + \sqrt{1 + \frac{2}{27\kappa^2}}} + \sqrt{1 - \sqrt{1 + \frac{2}{27\kappa^2}}} \right)^3,$$

$$\vartheta_2 := \vartheta_2(\kappa) = \frac{1}{2} \left( \sqrt{1 + \sqrt{1 - \frac{2}{27\kappa^2}}} + \sqrt{1 - \sqrt{1 - \frac{2}{27\kappa^2}}} \right)^3.$$ (2.8)

**Lemma 2.1.** Let $\theta_0, \kappa_0, \vartheta_1$ and $\vartheta_2$ be the same as above. Then we have

$$0 < \vartheta_1 < 1 < \vartheta_2, \quad \forall \kappa > \sqrt{2/27},$$ (2.9)

and

$$\vartheta_1 = 1 - \frac{1}{\sqrt{2} \kappa^{2/3}} + O(\kappa^{-4/3}), \quad \vartheta_2 = 1 + \frac{1}{\sqrt{2} \kappa^{2/3}} + O(\kappa^{-4/3}).$$ (2.10)

Moreover, if $\kappa > \kappa_0$, then we have $\theta_0 < \vartheta_1$.

**Proof.** We examine the function: $f(t) := \sqrt{1 + t} + \sqrt{1 - t}$, $t \geq 0$, associated with (2.8). One verifies readily that $f'(t) < 0$ for all $t > 0$, $t \neq 0$. Thus, $f(t)$ is monotonically decreasing, and

$$\sqrt{2} \vartheta_1 = f\left(\sqrt{1 + 2/(27\kappa^2)}\right) < f(1) < f\left(\sqrt{1 - 2/(27\kappa^2)}\right) = \sqrt{2} \vartheta_2,$$ (2.11)

which implies (2.9). It is evident that

$$t_1 := \sqrt{1 + \frac{2}{27\kappa^2}} = 1 + \frac{1}{27\kappa^2} + O(\kappa^{-4}).$$ (2.12)

A direct calculation from (2.8) yields

$$2\vartheta_1 = 2 + 3\{ (1 + t_1)^{1/3} (1 - t_1)^{1/3} + (1 + t_1)^{1/3} (1 - t_1)^{2/3} \} = 2 - \frac{3\sqrt{2}}{\kappa^{2/3}}\left( \sqrt{1 + t_1} + \sqrt{1 - t_1} \right)$$

$$= 2 - \frac{\sqrt{2}}{\kappa^{2/3}} \left( \sqrt{2} + \frac{1}{27\kappa^2} - \frac{1}{3\kappa^{2/3}} \right) + O(\kappa^{-2}) = 2 - \frac{\sqrt{2}}{\kappa^{2/3}} \left( \sqrt{2} - \frac{1}{3\kappa^{2/3}} + O(\kappa^{-2}) \right) + O(\kappa^{-2}),$$

which implies the asymptotic estimate of $\vartheta_1$ in (2.10). Similarly, we can derive the estimate of $\vartheta_2$.

We now show that $\theta_0 < \vartheta_1$, for all $\kappa > \kappa_0$ with $\kappa_0$ given by (2.6). Observe from (2.11)-(2.12) that $\sqrt{2} \vartheta_1 = f(t_1)$, so it suffices to show $\sqrt{2} \vartheta_0 < \sqrt{2} \vartheta_1 = f(t_1)$. Using the monotonic decreasing property of $f$, we just require $f^{-1}(\sqrt{2} \vartheta_0) > t_1 = \sqrt{1 + 2/(27\kappa^2)}$, so working out $f^{-1}$, we can obtain $\kappa_0$ in (2.6). □

We have the following estimates of Re$(\mathcal{T}_{l,\kappa})$, and the refined estimates of Im$(\mathcal{T}_{l,\kappa})$ in [36, (2.35)].

**Theorem 2.1.** Let $\theta_0, \vartheta_1, \vartheta_2$ and $\kappa_0$ be defined as above. Denote $\nu = l + 1/2$ and $\rho = \nu/\kappa$. Then for any $\kappa > \kappa_0$, we have the approximation

$$\text{Re}(\mathcal{T}_{l,\kappa}) \sim E_{l,\kappa}^R, \quad \text{Im}(\mathcal{T}_{l,\kappa}) \sim E_{l,\kappa}^I, \quad \forall l \geq 1,$$ (2.13)

where

(i) for $\rho = \nu/\kappa \in (0, \theta_0)$,

$$E_{l,\kappa}^R = -\frac{1}{2\kappa} \left( 1 + \frac{1}{1 - \rho^2} \right), \quad E_{l,\kappa}^I = \sqrt{1 - \rho^2};$$ (2.14)

(ii) for $\rho = \nu/\kappa \in [\theta_0, \vartheta_1]$,

$$E_{l,\kappa}^R = -\frac{1}{2\kappa} \left( 1 + \frac{1}{2(1 - \rho)} \right), \quad E_{l,\kappa}^I = \sqrt{2\rho(1 - \rho)};$$ (2.15)

(iii) for $\rho = \nu/\kappa \in (\vartheta_1, \vartheta_2)$,

$$E_{l,\kappa}^R = -\frac{1}{c_1} \left( \frac{2}{\nu} \right)^{1/3} \left( 1 + 2c_1 t + c_2 t^2 \right) - \frac{1}{2\kappa}, \quad E_{l,\kappa}^I = \sqrt{3c_1} \rho (1 - 2c_1 t) \left( \frac{2}{\nu} \right)^{1/3},$$ (2.16)

where $t = -\sqrt{2}(\kappa - \nu) / \sqrt{2} \nu$ (note: $|t| < 1$), and

$$c_1 = \frac{3}{2} \frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{2}{3}\right)} \approx 0.3645, \quad c_2 = \frac{1 - 16c_1^3}{2c_1} \approx 0.3088;$$ (2.17)
(iv) for $\rho = \nu/\kappa \in [\vartheta_2, \infty)$,

$$
E_{l,\kappa}^R = -\sqrt{\rho^2 - 1} - \frac{1}{2\kappa} \left(1 - \frac{1}{\rho^2 - 1}\right), \quad E_{l,\kappa}^I = \sqrt{\rho^2 - 1} e^{-2\nu \Psi}, \quad \text{where}
$$

$$
\Psi = \ln(\rho + \sqrt{\rho^2 - 1}) - \sqrt{\rho^2 - 1} \rho, \quad \rho > 1.
$$

(2.18)

(2.19)

We provide the proof of this theorem in Appendix 3. In Figure 2.1 we depict in (a)-(b) the graphs of Re($T_{l,\kappa}$) and Im($T_{l,\kappa}$) for various $l$ and $\kappa$, and in (c)-(d), the exact value and approximations in Theorem 2.1 for various samples of $\kappa$.

![Figure 2.1](image)

**Figure 2.1.** (a)-(b): Real and imaginary parts of $T_{l,\kappa}$ with various samples ($l, \kappa \in [0, 120] \times [1, 100]$). (c) Re($T_{l,\kappa}$) (solid line) against $E_{l,\kappa}^R$ for various samples of $T_{l,\kappa}$. (d) Im($T_{l,\kappa}$) versus $E_{l,\kappa}^I$.

### 2.2. Improved estimates for the Helmholtz equation

We first introduce some notation. Let $I := (a, b)$ and $x = r^2$. The weighted Sobolev space $H^s_\omega(I)$ with $s \geq 0$ is defined as usual in Adams [2]. In particular, $L^2_\omega(I)$ is the weighted $L^2$-space with the inner product and norm $(\cdot, \cdot)_\omega$ and norm $\|\cdot\|_\omega$. We also use the anisotropic Sobolev spaces, e.g., $H^s_\omega(S; H^2_\omega(I))$ with the norm characterised by the SPH expansion coefficients $\hat{U}^m_l$ of $U$ (cf. [36] (1.8)). If $\omega \equiv 1$, we drop $\omega$ in the notation.

A weak form of (1.5)-(1.6) is to find $U \in H^1_\omega(S; 0 H^1(I))$ such that (cf. [36] (3.9)):

$$
\mathbb{B}(U, V) = (\partial_r U, \partial_r V)_{x, \Omega} + (\nabla V U, \nabla V)_{x, \Omega} - k^2 (U, V)_{x, \Omega} - b^2 (T_0 U, V)_{x,\Omega} = (F, V)_{x, \Omega} + (H, V)_{x, \Omega}, \quad \forall V \in H^1_\omega(S; 0 H^1(I)), \tag{2.20}
$$

where $0 H^1(I) = \{v \in H^1(I) : v(a) = 0\}$. We expand $U, F, H$ in SPH series as

$$
\{U, F, H\} = \sum_{l=1}^{\infty} \sum_{m=|m|=0} \{\hat{U}^m_l(r), \hat{F}^m_l(r), \hat{H}^m_l(r)\} Y^m_l(\theta, \varphi). \tag{2.21}
$$

Taking $V = \hat{V}^m_l(r) Y^m_l$ in (2.20) and using the property of SPH (cf. Appendix A), we obtain the corresponding weak form for each mode $(l, m)$: find $u := \hat{U}^m_l \in 0 H^1(I)$ such that

$$
\mathbb{B}_l^m(u, v) := (u', v')_{x, \Omega} + \beta_l(u, v) - k^2 (u, v)_{x, \Omega} - b^2 T_0 u v(b) = (f, v)_{x, \Omega} + b^2 h v(b), \quad \forall v \in 0 H^1(I), \tag{2.22}
$$

where $\beta_l = l(l + 1)$ and we have denoted $v := \hat{V}^m_l$, $f := \hat{F}^m_l$, and $h := \hat{H}^m_l$. Here, we drop the weight function $\omega$ in the space $0 H^1(I)$ as it is uniformly bounded below away from $I$.

We have the following improved estimate in the following lemma [36] Lemma 3.1.

**Lemma 2.2.** Let $u$ be the solutions of (2.22). If $f \in L^2(I)$, then we have that for all $k \geq k_0 > 0$ (for some fixed constant $k_0$), and for $l \geq 1, 0 \leq |m| \leq l$,

$$
\|u'\|_{2, \Omega}^2 + \beta_l \|u\|_{2, \Omega}^2 + k^2 \|u\|_{2, \Omega}^2 \lesssim \|f\|_{2, \Omega}^2 + |h|^2. \tag{2.23}
$$
Proof. Taking \( v = u \) in (2.22), we obtain
\[
\|u\|_\infty^2 + \beta_1\|u\|_\infty^4 - k^2\|u\|_\infty^2 - kb^2\text{Re}(T_{1,kb})u(b)^2 = \text{Re}(f, u)_\infty + b^2\text{Re}(h\bar{u}(b)),
\]
(2.24a)
Thus, by choosing suitable \( \beta_1 \), we obtain
\[
-2b^2\text{Im}(T_{1,kb})u(b)^2 = \text{Im}(f, u)_\infty + b^2\text{Im}(h\bar{u}(b)).
\]
(2.24b)
Next taking \( v = (r - a)u \) in (2.22), followed by the derivations in (3.26)-(3.28), we obtain
\[
b^2|I||u'(b)|^2 + \beta_1|I||u(b)|^2 + 2a\|\sqrt{r}'u'(b)\|^2 + k^2b^2\|u(b)\|^2 + 2\text{Re}(f, (r - a)u)_\infty
+ 2b^2|I|\text{Re}(h\bar{u}'(b)) + 2kb^2|I|\text{Re}(T_{1,kb}u(b)\bar{u}'(b)),
\]
where \( |I| = b - a \). Substituting \( |u'|_\infty^2 + \beta_1|u|^2 \) in the identity (2.24a) into the above, and collecting the terms, we obtain
\[
b^2|I||u'(b)|^2 + \{\beta_1|I| - kb^2\text{Re}(T_{1,kb})\}|u(b)|^2 + 2a\|\sqrt{r}'u'(b)\|^2 + 2k^2b^2|I|\|u(b)\|^2 + 2\text{Re}(f, (r - a)u)_\infty
+ 2b^2|I|\text{Re}(h\bar{u}'(b)) + 2kb^2|I|\text{Re}(T_{1,kb}u(b)\bar{u}'(b)),
\]
(2.26)
Hereafter, let \( C \) and \( \{C_t, \epsilon_t\} \) be generic constants independent of \( k, l, m, \) and any function. Using the Cauchy-Schwarz inequality, we obtain
\[
2k^2b^2|I|\text{Re}(T_{1,kb}u(b)\bar{u}'(b))| \leq \epsilon_1b^2|I||u'(b)|^2 + \epsilon_1^{-1}k^2b^2|I||T_{1,kb}|^2|u(b)|^2;
\]
\[
2b^2|I|\text{Re}(h\bar{u}'(b))| \leq \epsilon_2b^2|I||u'(b)|^2 + \epsilon_2^{-1}b^2|I||h|^2;
\]
\[
b^2|\text{Re}(h\bar{u}'(b))| \leq \epsilon_3kb^2|\text{Re}(T_{1,kb})||u(b)|^2 + \frac{b^2}{\epsilon_3k|\text{Re}(T_{1,kb})|}|h|^2;
\]
\[
2|\text{Re}(f, (r - a)u)_\infty| \leq \epsilon_4\|\sqrt{r}'u'^2 + \epsilon_4^{-1}|b|\|f\|_\infty^2;
\]
\[
|\text{Re}(f, u)_\infty| \leq \epsilon_5|\|u\|_\infty^2 + (4\epsilon_5)^{-1}|f|_\infty^2.
\]
Thus, by choosing suitable \( \{\epsilon_t\} \), we obtain from (2.26)-(2.27) that
\[
C_1b^2|I||u'(b)|^2 + D_{l,k}|I||u(b)|^2 + C_2a^2|\sqrt{r}'u'^2 + C_3k^2|u|^2_\infty \lesssim \|f\|_\infty^2 + \left(1 + \frac{1}{k|\text{Re}(T_{1,kb})|}\right)|h|^2,
\]
(2.28)
where \( C_1 = 1 - (\epsilon_1 + \epsilon_2), C_2 = 2 - \epsilon_4, C_3 = 2(1 - a/\xi) - \epsilon_5/k^2 \) with \( \xi \in (a, b) \), and
\[
D_{l,k} := \beta_1 - (1 - \epsilon_3)kb^2|I|^{-1}\text{Re}(T_{1,kb}) - k^2b^2(1 + \epsilon_4^{-1}|T_{1,kb}|^2).
\]
(2.29)
It remains to estimate \( D_{l,k} \), which is negative for e.g., small \( l \). According to the estimates in Lemma (2.1), we conduct the analysis for four different cases as in (2.7).

(i) If \( \rho = \frac{\nu}{kb} \in (0, \theta_0) \) for fixed \( 0 < \theta_0 < 1 \), we obtain from (2.24b) that
\[
k^2b^2|u(b)|^2 \leq \frac{\kappa}{|\text{Im}(T_{1,kb})|}\left\{|\text{Im}(f, u)_\infty| + b^2|\text{Im}(h\bar{u}(b))|\right\}
\leq \frac{\epsilon_7k^2|u|^2_\infty}{2} + \frac{\|f\|_\infty^2}{2|\text{Im}(T_{1,kb})|^2} + \frac{k^2b^2}{2}|u(b)|^2 + \frac{|h|^2}{2|\text{Im}(T_{1,kb})|^2}.
\]
(2.30)
By (2.14), \( |\text{Im}(T_{1,kb})| \) in this range behaves like a constant, so (2.30) implies
\[
k^2b^2|u(b)|^2 \leq \epsilon_7k^2|u|^2_\infty + C(|f|^2_\infty + |h|^2).
\]
(2.31)
By (2.14), \( |T_{1,kb}| \leq C \), so \( D_{l,k} \leq -Ck^2b^2 \). Therefore, using (2.28) and (2.31) leads to
\[
|\sqrt{r}'u'^2 + k^2|u|^2_\infty + k^2|u(b)|^2 \leq C(|f|^2_\infty + |h|^2).
\]
(2.32)
Thus, we derive the desired estimate in this case from (2.24a) and (2.32).
(ii) For $\rho = \bar{C} \in (\theta_0, \theta_1]$, we first show that for any $\bar{c}_0 \in (1 - \theta_0, 1/\sqrt{2})$ and $kb > 1$, there exists a unique $\gamma_0 \in [1/3, 1)$ such that

$$\rho = 1 - \bar{c}_0(kb)^{\gamma_0 - 1}, \quad \text{i.e.,} \quad \gamma_0 = 1 + \frac{\ln((1 - \rho)/\bar{c}_0)}{\ln(kb)}. \quad (2.33)$$

Apparently, $\gamma_0$ decreases with respect to $\rho$, so by $\text{(2.10)}$,

$$\frac{1}{3} \ln(\sqrt{2}\bar{c}_0) - \ln(1) + \frac{\ln(1 + O((kb)^{-2/3}))}{\ln(kb)} = 1 + \frac{\ln((1 - \theta_1)/\bar{c}_0)}{\ln(kb)} \leq \gamma_0 < 1 + \frac{\ln((1 - \theta_0)/\bar{c}_0)}{\ln(kb)}, \quad (2.34)$$

Then one verifies readily that for $\bar{c}_0 \in (1 - \theta_0, 1/\sqrt{2})$, we have $\gamma_0 \in [1/3, 1)$. In view of $\text{(2.33)}$, we can write

$$\nu = kb - \bar{c}_0(kb)^{\gamma_0}. \quad (2.35)$$

Thus, by $\text{(2.19)}$,

$$\text{Re}(T_{l,kb}) \sim -\frac{1}{2\bar{c}_0}(kb)^{-\gamma_0}, \quad \text{Im}(T_{l,kb}) \sim \sqrt{2\bar{c}_0}(kb)^{(\gamma_0 - 1)/2}, \quad |T_{l,kb}|^2 \sim 2\bar{c}_0(kb)^{\gamma_0 - 1}, \quad (2.36)$$

which implies

$$D_{l,k} \sim \nu^2 - \frac{1}{4} + (1 - \varepsilon_3)\frac{b}{2|\bar{T}_|\bar{c}_0}kb^{1 - \gamma_0} - k^2(b^2(1 + \varepsilon_3)2\bar{c}_0(kb)^{\gamma_0 - 1}) \sim -2\bar{c}_0(1 + \varepsilon_3)(kb)^{\gamma_0 + 1}. \quad (2.37)$$

By $\text{(2.24b)}$ and the Cauchy-Schwarz inequality,

$$(kb)^{\gamma_0 + 1}|u(b)|^2 \leq \frac{(kb)^{\gamma_0}}{\text{Im}(T_{l,kb})} \left\{ |\text{Im}(f, u)|^2 + b^2 |\text{Im}(h(u))|^2 \right\} \leq \varepsilon_7 k^2\|u\|^2_{\infty} + \frac{(kb)^{2\gamma_0 - 2}}{2\varepsilon_7 |\text{Im}(T_{l,kb})|^2} \|f\|^2_{\infty} + \frac{(kb)^{\gamma_0 + 1}}{2} |u(b)|^2 + \frac{(kb)^{\gamma_0 - 1}}{2} |\text{Im}(T_{l,kb})|^2 |h|^2. \quad (2.38)$$

Then by $\text{(2.36)}$ and $\text{(2.38)}$,

$$(kb)^{\gamma_0 + 1}|u(b)|^2 \leq \varepsilon_7 k^2\|u\|^2_{\infty} + C((kb)^{\gamma_0 - 1}\|f\|^2_{\infty} + |h|^2). \quad (2.39)$$

Thus, we derive from $\text{(2.28)}$ that

$$\|\sqrt{v}u'\|^2 + k^2\|v\|^2_{\infty} + (kb)^{\gamma_0 + 1}|u(b)|^2 \leq C\|f\|^2_{\infty} + |h|^2. \quad (2.40)$$

Therefore, we obtain $\text{(2.23)}$ from $\text{(2.24a)}$ and $\text{(2.40)}$.

(iii) If $\rho = \bar{C} \in (\theta_1, \theta_2]$, we find from $\text{(2.10)}$ that

$$kb - \sqrt{\frac{k^2}{2} + O(k^{-1/3})} < \nu \leq kb + \sqrt{\frac{k^2}{2} + O(k^{-1/3})}. \quad (2.41)$$

By $\text{(2.16)}$,

$$\text{Re}(T_{l,kb}) \sim -\bar{c}_1(kb)^{-1/3}, \quad \text{Im}(T_{l,kb}) \sim \bar{c}_2(kb)^{-1/3}, \quad |T_{l,kb}|^2 \sim \bar{c}_3(kb)^{-2/3}, \quad (2.42)$$

where $\{\bar{c}_i\}$ are some positive constants independent of $k, l$. We can follow the same procedure as for Case (ii) (but with $\gamma_0 = 1/3$) to derive

$$\|\sqrt{v}u'\|^2 + k^2\|v\|^2_{\infty} + (kb)^{2/3}|u(b)|^2 \leq C\|f\|^2_{\infty} + |h|^2. \quad (2.43)$$

Similarly, $\text{(2.23)}$ follows from $\text{(2.24a)}$ and $\text{(2.43)}$.

(iv) If $\rho = \bar{C} \in (\theta_2, \infty)$, we find from $\text{(2.18)}$ that $\text{Im}(T_{l,kb})$ decays exponentially with respect to $l$, so we cannot get a useful bound of $|u(b)|$ from $\text{(2.24b)}$. We therefore consider two cases:

(a) $\nu = kb + \bar{c}_5(kb)^{\gamma_1}$ with $1/3 < \gamma_1 < 1$;  
(b) $\nu \geq \eta kb$, \quad (2.44)

for constant $\bar{c}_5 \in (\eta - 1, 1/\sqrt{2})$ and $1 < \eta < 1 + 1/\sqrt{2}$. Here, we show that Case (a) can cover $\rho \in (\bar{c}_2, \eta)$. Indeed, similar to $\text{(2.33)}$, we have $\rho = 1 + \bar{c}_5(kb)^{\gamma_1 - 1}$, and

$$\frac{1}{3} - \frac{\ln(\sqrt{2}\bar{c}_5)}{\ln(kb)} + \frac{\ln(1 + O((kb)^{-2/3}))}{\ln(kb)} = 1 + \frac{\ln((\bar{c}_2 - 1)/\bar{c}_5)}{\ln(kb)} \leq \gamma_1 < 1 + \frac{\ln((\eta - 1)/\bar{c}_5)}{\ln(kb)}. \quad (2.45)$$
This implies if $c_5 \in (\eta - 1, 1/\sqrt{2})$ and $1 < \eta < 1 + 1/\sqrt{2}$, then $1/3 < \gamma_1 < 1$ and we can write $\nu$ in the form of (a).

In the first case, we derive from (2.18) that

$$\text{Re}(T_{l,kb}) \sim \sqrt{2c_5(kb)^{(\gamma_1 - 1)/2}}, \quad |T_{l,kb}|^2 \sim 2c_5(kb)^{-1}, \quad D_{l,k} \sim -2c_5(\varepsilon_1^{-1} - 1)(kb)^{-1},$$

where we recall that $\varepsilon_1 < 1$. Noticing that

$$\beta_l \|u\|^2 - k^2 \|u\|^2_H \geq (\beta_l - k^2b^2)\|u\|^2 \geq 0,$$

and $\text{Re}(T_{l,kb}) < 0$, we deduce from (2.24a) that

$$-kb^2 \text{Re}(T_{l,kb})|u(b)|^2 \leq |\text{Re}(f, u)| + b^2 \text{Re}(h\bar{u}(b))|.$$

Using (2.46), (2.48) and following the derivation of (2.38), we can get

$$(kb)^{-1} |u(b)|^2 \leq \varepsilon_8 k^2 \|u\|^2_H + C((kb)^{-1} \|f\|^2_H + |h|^2).$$

We then derive from (2.28) that

$$\|\sqrt{\nu}u'\|^2 + k^2 \|u\|^2_H + (kb)^{-1} |u(b)|^2 \leq C(\|f\|^2_H + |h|^2).$$

Thus, we derive (2.23) for this case from (2.24a) and (2.50).

In the second case of (2.44), we observe from (2.18) that

$$\text{Re}(T_{l,kb}) \sim -\frac{\nu}{kb}, \quad |T_{l,kb}|^2 \sim \frac{\nu^2}{k^2b^2},$$

which implies

$$D_{l,k} \sim \nu^2 - \frac{1}{4} + (1 - \varepsilon_3) \frac{b\nu}{|f|} - k^2b^2 - \varepsilon_1^{-1} \nu^2 \sim -\varepsilon_6 \beta_l.$$

Then, by (2.51) and (2.48),

$$\beta_l |u(b)|^2 \leq \varepsilon_8 \beta_l \|u\|^2 + C(\|f\|^2_H + |h|^2).$$

We then derive from (2.28) that

$$\|\sqrt{\nu}u'\|^2 + k^2 \|u\|^2_H + \beta_l |u(b)|^2 \leq C(\|f\|^2_H + |h|^2).$$

Finally, we obtain (2.23) from (2.24a) and (2.54).

Thanks to the above lemma and the orthogonality of SPH, one can easily derive the following improved result, where a factor of $k^{1/3}$ is removed from the upper bound of [30] Thm. 3.1.

**Theorem 2.2.** Let $U$ be the solution of (2.20). If $F \in L^2(\Omega)$ and $H \in L^2(S)$, then we have

$$\|\nabla U\|_{\Omega} + k\|U\|_{\Omega} \lesssim \|F\|_{\Omega} + \|H\|_{L^2(S)}.$$  

3. A priori estimates for the reduced Maxwell equations

In this section, we perform the wavenumber explicit *a priori* estimates for the Maxwell equations (1.2)-(1.3). The key is to employ a divergence-free vector harmonic expansion of the fields and reduce the problem of interest into two sequences of decoupled one-dimensional Helmholtz problems. This decoupling not only leads to a more efficient numerical algorithm, but also greatly simplifies its analysis.

3.1. Dimension reduction via divergence-free VSH expansions. Let $\{Y^m_l\}$ be the spherical harmonics, and $\{Y^m_l e_r, \nabla_s Y^m_l, T^m_l := \nabla_s Y^m_l \times e_r\}$ be the VSH (see Appendix A). The divergence-free VSH that we shall use are introduced in [3], but it appears that they are rarely discussed, if not at all, in mathematics literature.

Introduce the spaces

$$\mathbb{H}(\text{div}; \Omega) = \{ E \in L^2(\Omega) : \text{div} E \in L^2(\Omega) \}, \quad \mathbb{H}(\text{curl}; \Omega) = \{ E \in (L^2(\Omega))^3 : \nabla \times E \in (L^2(\Omega))^3 \},$$

equipped with the graph norm defined as in [29] P. 52, and

$$\mathbb{H}_0(\text{div}; \Omega) = \{ E \in \mathbb{H}(\text{div}; \Omega) : \text{div} E = 0 \}, \quad \mathbb{H}_0(\text{curl}; \Omega) = \{ v \in \mathbb{H}(\text{curl}; \Omega) : v \times e_r |_{r=\varepsilon} = 0 \}.$$

We have following important properties of solenoidal (or divergence-free) fields.
Proposition 3.1. For \( E \in \mathbb{H}_0(\text{div}; \Omega) \), we can write
\[
E = u_0^0 Y_0^0 e_r + \sum_{l=1}^{\infty} \sum_{m=0}^{l} \left\{ u_{1,l}^m T_l^m + \nabla \times (u_{2,l}^m T_l^m) \right\},
\]
where \( u_0^0 \) satisfies
\[
\left( \frac{d}{dr} + \frac{2}{r} \right) u_0^0 = 0 \quad \text{or} \quad u_0^0 = c/r^2,
\]
for an arbitrary constant \( c \). Equivalently, we can reframe (3.1) as
\[
E = u_0^0 Y_0^0 e_r + \sum_{l=1}^{\infty} \sum_{m=0}^{l} \left\{ u_{1,l}^m T_l^m + \hat{\partial}_r u_{2,l}^m \nabla S Y_l^m + \beta_l^1 \frac{r}{r} u_{2,l}^m Y_l^m e_r \right\},
\]
where \( \beta_l = l(l+1), \hat{\partial}_r = d/dr + 1/r \) and
\[
|u_{1,l}^m(r)| = \beta_l^{-1} \langle E, T_l^m \rangle_S, \quad r^{-1}|u_{2,l}^m(r)| = \beta_l^{-1} \langle E, Y_l^m e_r \rangle_S.
\]
Proof. We first show that if (3.2) holds, then the expansion (3.1) automatically meets \( \text{div} E = 0 \). Note that \( \text{div}(u_{1,l}^m T_l^m) = 0 \) (cf. (A.4)). Acting the divergence operator on (3.1), and using (A.7), we have \( \text{div} E = 0 \), if \( u_0^0 \) satisfies the equation in (3.2) with explicit solution: \( u_0^0 = c/r^2 \).

Thanks to (A.4), the expansion (3.3) follows immediately from (3.1). Then (3.4) is a direct consequence of the orthogonality in (A.1). \( \square \)

Recall that the capacity operator in (1.3) is defined by (cf. (II, 5.3.88)):
\[
\mathcal{T}_l[\Phi] := \eta \mathbf{H} \times e_r |_{r=b} = \sum_{l=1}^{\infty} \sum_{m=0}^{l} \left\{ -i \frac{\hat{\partial}_r h_l^{(1)}(kb)}{h_l^{(1)}(kb)} \phi_{T,l}^m T_l^m + i \frac{h_l^{(1)}(kb)}{\partial_r h_l^{(1)}(kb)} \phi_{Y,l}^m \nabla_S Y_l^m \right\},
\]
where \( h_l^{(1)} \) is the spherical Bessel function of the first kind (cf. [II, and given by
\[
\phi_{T,l}^m = \beta_l^{-1} \langle \Phi, T_l^m \rangle_S, \quad \phi_{Y,l}^m = \beta_l^{-1} \langle \Phi, \nabla_S Y_l^m \rangle_S.
\]
As \( F \) in (1.5) is a solenoidal field, we can expand it as (3.1) with the coefficients \( f_0^0 \) and \( \{f_{1,l}^m, f_{2,l}^m\} \).

We also expand the data \( h \in L^2(S) \) (the space of tangential components) in (1.6) as
\[
h = \sum_{l=1}^{\infty} \sum_{m=0}^{l} \left\{ h_{T,l}^m T_l^m + h_{Y,l}^m \nabla_S Y_l^m \right\},
\]
where the expansion coefficients are given by (3.8) with \( h \) in place of \( \Phi \).

Proposition 3.2. Denote
\[
u_1 = u_{1,l}^m, \quad u_2 = u_{2,l}^m, \quad f_1 = f_{1,l}^m, \quad f_2 = f_{2,l}^m, \quad h_1 = h_{T,l}^m, \quad h_2 = k^{-1}(T_{l,kb} + (kb)^{-1})h_{Y,l}^m,
\]
for \( l \geq 1 \). Then the Maxwell equations (1.2)-(1.3) reduce to \(-k^2 u_0^0 = f_0^0\), and the following two sequences of one-dimensional problems:
\[
- \frac{1}{r^2} (r^2 u_i')' + \frac{\beta_i}{r^2} u_i - k^2 u_i = f_i, \quad r \in I := (a, b); \quad u_i'(b) - kT_{i,kb} u_i(b) = h_i, \quad i = 1, 2,
\]
but with different boundary conditions at \( r = a \):
\[
u_1(a) = 0, \quad u_2(a) + a^{-1} u_2(a) = 0.
\]
Proof. We first consider (1.2). Recall that if \( \text{div} \, \mathbf{u} = 0 \), then we have \( \nabla \times \nabla \times \mathbf{u} = -\Delta \mathbf{u} \). Since \( \text{div} \left( \nabla \times (f T^n_r) \right) = 0 \) (cf. (A.4)), we derive from (3.1) and (A.4)-(A.5) that
\[
\nabla \times \nabla \times (u^m_{1,l} T^n_r) = -\Delta (u^m_{1,l} T^n_r) = -\mathcal{L}_t (u^m_{1,l} T^n_r),
\]
\[
\nabla \times \nabla \times \nabla \times (u^m_{2,l} T^n_r) = -\nabla \times (\Delta (u^m_{2,l} T^n_r)) = -\nabla \times (\mathcal{L}_t (u^m_{2,l} T^n_r)), \tag{3.13}
\]
where the Bessel operator \( \mathcal{L}_t \) is given in (A.3). Thus, using the expansions (3.1), we can reduce (1.2) to
\[
-(\mathcal{L}_t + k^2) w(r) = f(r) \quad \text{for} \quad \{w, f\} = \{u^m_{1,l}, f^m_{1,l}\} \quad \text{or} \quad \{u^m_{2,l}, f^m_{2,l}\}, \tag{3.14}
\]
for \( l \geq 1 \) and \( r \in I \). In addition, we have
\[
-k^2 u^0 = f^0, \quad \text{as} \quad \nabla \times (u^0 Y_0^0 e_r) = \nabla \times (f^0 Y_0^0 e_r) = 0, \tag{3.15}
\]
since \( E \) and \( F \) are solenoidal. This leads to the mode \( u^0 \), so we only consider the modes with \( l \geq 1 \) and \( 0 \leq |m| \leq l \). A direct calculation using (A.2)-(A.3) and (A.4)-(A.5) leads to the reduction of the boundary condition (1.2):
\[
\hat{u}_l u^m_{1,l}(a) = 0, \quad \hat{\partial}_r u^m_{1,l}(a) = 0, \quad \text{where} \quad \hat{\partial}_r := \frac{d}{dr} + \frac{1}{r}. \tag{3.16}
\]
We now turn to the DtN boundary condition (1.3). By (3.1) and (3.13),
\[
\nabla \times E = \sum_{l=1}^{\infty} \sum_{|m|=0}^{l} \left\{ \nabla \times (u^m_{1,l} T^n_r) - \mathcal{L}_t (u^m_{2,l} T^n_r) \right\}. \tag{3.17}
\]
Again from (A.2)-(A.3) and (A.4)-(A.5), we derive
\[
(\nabla \times E) \times e_r |_{r=b} = \sum_{l=1}^{\infty} \sum_{|m|=0}^{l} \left\{ \hat{\partial}_r u^m_{1,l} (\nabla S Y^n_m) \right\} |_{r=b}, \tag{3.18}
\]
\[
E_S |_{r=b} = \sum_{l=1}^{\infty} \sum_{|m|=0}^{l} \left\{ u^m_{1,l} T^n_r + \hat{\partial}_r u^m_{2,l} \nabla S Y^n_m \right\} |_{r=b}. \tag{3.19}
\]
Then, by (3.5) and (3.18),
\[
-ik \mathcal{T}_b [E_S] = \sum_{l=1}^{\infty} \sum_{|m|=0}^{l} \left\{ -k \hat{\partial}_r h^{(1)}_{l_1} (kb) u^m_{1,l} (b) T^n_r + k \frac{h^{(1)}_{l_1} (kb)}{h^{(1)}_{l_2} (kb)} \hat{\partial}_r u^m_{2,l} (b) \nabla S Y^n_m \right\}. \tag{3.20}
\]
Consequently, by (3.9) and (3.18), the DtN boundary condition (1.3) reduces to
\[
\hat{\partial}_r u^m_{1,l} (b) - k \frac{h^{(1)}_{l_1} (kb)}{h^{(1)}_{l_2} (kb)} u^m_{1,l} (b) = h^n_{T,1}, \quad \mathcal{L}_t (u^m_{2,l}) (b) + k \frac{h^{(1)}_{l_1} (kb)}{h^{(1)}_{l_2} (kb)} \hat{\partial}_r u^m_{2,l} (b) = h^n_{T,2} \tag{3.21}
\]
By the equation (3.14) (note: \( f^m_{2,l} (b) = 0 \) as the source field is assumed to be compact supported), we have \( \mathcal{L}_t (u^m_{2,l}) (b) = -k^2 u^m_{2,l} (b) \), so we can simplify (3.20) as
\[
\hat{\partial}_r u^m_{2,l} (b) - k \frac{h^{(1)}_{l_1} (kb)}{h^{(1)}_{l_2} (kb)} u^m_{2,l} (b) = \frac{1}{k} \frac{h^{(1)}_{l_1} (kb)}{h^{(1)}_{l_2} (kb)} h^n_{T,2}. \tag{3.22}
\]
This ends the derivation. \qed

3.2. A priori estimates for \( \{u^m_{1,l}, u^m_{2,l}\} \). A weak form of (3.11)-(3.12) is to find \( u_1 \in H^1(I) \) such that
\[
\mathbb{B}^m_1 (u_1, w) = (f_1, w)_{m} + b^2 h_1 \bar{w} (b), \quad \forall w \in H^1(I), \tag{3.23}
\]
and to find \( v \in H^1(I) \) such that
\[
\mathbb{B}^m_2 (u_2, w) - au_2 (a) w(a) = (f_2, w)_{m} + b^2 h_2 \bar{w} (b), \quad \forall w \in H^1(I), \tag{3.24}
\]
where the bilinear form \( \mathbb{B}^m_i (\cdot, \cdot) \) is defined in (2.22).
Observe that the weak form for \( u_1 \) is the same as that of the Helmholtz equation in (2.22), while (3.23) differ from (3.22) with an extra term: \(-au_2(a)\hat{w}(a)\). As a result, we can obtain the a priori estimates like Lemma 2.2 by using the same argument.

**Theorem 3.1.** Let \( u_1 \) and \( u_2 \) be solutions of (3.22) and (3.23), respectively. If \( f_1, f_2 \in L^2(\Lambda) \), then for all \( k \geq k_0 > 0 \) (for some fixed constant \( k_0 \)), and \( l \geq 1, 0 \leq |m| \leq l \), we have

\[
\| u_i^\prime \|_2^2 + \beta_i \| u_i \|_2^2 + k^2 \| u_i \|_\infty^2 \leq \| f_i \|_2^2 + |h_i|^2, \quad i = 1, 2.
\]

**(Proof.** The estimates in Lemma 2.2 carry over to \( u_1 \), so it suffices to consider \( u_2 \) and deal with the extra term herein. Following the proof of Lemma 2.2, we take two test functions: \( w = u_2 \) and \( w = 2(r-a)u_2 \), and note that the term \( "−au_2(a)\hat{w}(a)" \) vanishes for the second test function. Thus, we only need to deal with the contribution from this extra term as follows:

\[
\| u_2^\prime \|_2^2 + \beta_i \| u_2 \|_2^2 + k^2 \| u_2 \|_\infty^2 \leq \| f_2 \|_2^2 + |h_2|^2.
\]

Using the Sobolev inequality (see, e.g., [31 (B.33)]), we obtain

\[
a|u_2(a)|^2 \leq a \left( 2 + \frac{1}{b-a} \right) \| u_2 \|_2 \| u_2 \|_1 \leq a \left( 2 + \frac{1}{b-a} \right) \left( \| u_2 \|_2^2 + \| u_2 \|_\infty^2 \right)
\]

\[
\leq a^{-3} \left( 2 + \frac{1}{b-a} \right) \left( \| u_2 \|_2^2 + \| u_2 \|_\infty^2 \right).
\]

where we used the simple inequality: \( \sqrt{A^2 + B^2} \leq |A| + |B| \), and the fact \( \varpi/a^2 \geq 1 \). Thus,

\[
a|u_2(a)|^2 \leq \frac{1}{2} \| u_2 \|_2^2 + C \| u_2 \|_\infty^2.
\]

Thus, by (3.25) and (3.27),

\[
\frac{1}{2} \| u_2 \|_2^2 + \beta_i \| u_2 \|_2^2 + k^2 (1 - Ck^{-1}) \| u_2 \|_\infty^2 \leq \| f_2 \|_2^2 + |h_2|^2.
\]

This leads to the desired estimate.

□

It is important to point out that as the expansion in (3.3) involves \( \partial_r u_{2,l}^m \), the direct use of Theorem 3.1 and the orthogonality of VSH only leads to an overly pessimistic estimate: \( \| E \|_\Omega = O(1) \). However, the expected optimal estimate should be \( \| E \|_\Omega = O(k^{-1}) \). In view of this, we next derive an “auxiliary” equation of \( \partial_r u_{2,l}^m \) and apply the analysis similar to that for \( \{ u_{1,l}^m, u_{2,l}^m \} \) in the previous subsection.

### 3.3. A priori estimates for \( \partial_r u_{2,l}^m \).

#### 3.3.1. Equation of \( \partial_r u_{2,l}^m \).

Denote

\[
v_2 := \beta_l u_{2,l}^m/r = \beta_l u_2/r, \quad v_3 := \partial_r u_{2,l}^m = \partial_r u_2, \quad h_Y := -kS_{l,k} h_2 = h_{Y,l}^m,
\]

\[
g_2 := \beta_l f_{2,l}^m/r = \beta_l f_2/r, \quad g_3 := \partial_r f_{2,l}^m = \partial_r f_2,
\]

where the DtN kernel pertinent to (3.5) is defined by

\[
\mathcal{S}_{l,\kappa} := \frac{\hbar_{l}^{(1)}(\kappa)}{\partial_r h_{l}^{(1)}(\kappa)} = -\frac{\hbar_{l}^{(1)}(\kappa)}{\partial_r \hbar_{l}^{(1)}(\kappa)} = -\frac{1}{\mathcal{T}_{l,\kappa} + \kappa^{-1}}, \quad l \geq 1, \quad \kappa > 0.
\]

Recall that \( \mathcal{T}_{l,\kappa} \) is defined in (2.2).

From the equation of \( u_2 \) in Proposition 3.2, we can derive the following “auxiliary” equation.

#### Proposition 3.3.

Let \( v_3 = \partial_r u_2 \). Then we have

\[
-\frac{1}{r^2} (r^2 v_3')' + \frac{\beta_l}{r^2} v_3 - k^2 v_3 - \frac{2}{r^2} v_2 = g_3, \quad r \in I,
\]

\[
v_3(a) = 0, \quad v_3'(b) = k(S_{l,k} - (kb)^{-1}) v_3(b) - b^{-1} v_2(b) = h_Y.
\]

Alternatively, we can replace the boundary condition at \( r = b \) in (3.31) by

\[
v_3'(b) - \frac{\sigma_{l,k} b}{k b S_{l,k}} v_2(b) = \frac{h_Y}{kb S_{l,k}} = -\frac{h_2}{b},
\]

(3.32)
where
\[
\sigma_{l,k} := 1 - \frac{k^2 b^2}{\beta_l} \left( 1 - \frac{1}{k b S_{l,k}} \right) = 1 - \frac{k^2 b^2}{\beta_l} \left( 1 + \frac{T_{l,k}}{k b} + \frac{1}{2 k^2 b^2} \right).
\] (3.33)

**Proof.** One verifies readily that \( \hat{\partial}_r v_3 = \hat{\partial}_r (\hat{\partial}_r u_2) = r^{-2} (r^2 u_2'')' \), so by (3.11),
\[
-\hat{\partial}_r v_3 + \frac{\beta_l}{r^2} u_2 - k^2 u_2 = f_2, \quad r \in I.
\] (3.34)

Applying \( \hat{\partial}_r \) on both sides of the above equation, we obtain the first equation in (3.31) by a direct calculation. Since \( v_3(a) = \hat{\partial}_r u_2(a) \), the boundary condition \( v_3(a) = 0 \) is a direct consequence of (3.12). Noting that \( u_2'(b) = v_3(b) - u_2(b)/b \), we obtain from (3.30) and the boundary condition in (3.11) that
\[
u_2(b) + \frac{S_{l,k}}{k} v_3(b) = \frac{S_{l,k}}{k} b_2 = -\frac{h_y}{k^2}.
\] (3.35)

Taking \( r = b \) in (3.34) (note: \( f_2(b) = 0 \)), we obtain
\[
u_2(b) = -k^{-2} (v_3'(b) + b^{-1} v_3(b) - b^{-1} v_2(b)).
\] (3.36)

Inserting (3.36) into (3.35) yields the boundary condition at \( r = b \) in (3.31). More precisely, solving out \( v_3(b) \) from (3.35), and using the fact \( u_2(b) = b v_2(b)/\beta_l \), we can obtain (3.32)-(3.33) from (3.31).

3.3.2. Properties of the DtN kernel \( S_{l,\kappa} \). By (3.30), we have that for integer \( l \geq 1 \) and real \( \kappa > 0 \),
\[
\text{Re}(S_{l,\kappa}) = -\frac{\text{Re}(T_{l,\kappa}) + \kappa^{-1}}{(\text{Re}(T_{l,\kappa}) + \kappa^{-1})^2 + (\text{Im}(T_{l,\kappa}))^2}; \quad \text{Im}(S_{l,\kappa}) = \frac{\text{Im}(T_{l,\kappa})}{(\text{Re}(T_{l,\kappa}) + \kappa^{-1})^2 + (\text{Im}(T_{l,\kappa}))^2}.
\] (3.37)

which, together with (2.5), implies
\[
\text{Re}(S_{l,\kappa}) > 0, \quad \text{Im}(S_{l,\kappa}) > 0, \quad \text{for} \quad l \geq 1, \quad \kappa > 0.
\] (3.38)

In Figure 3.1 (a)-(b), we depict the graphs of \( \text{Re}(S_{l,\kappa}) \) and \( \text{Im}(S_{l,\kappa}) \) for various samples \( (l, \kappa) \in [0, 120] \times [1, 100] \), which shows a quite different behaviour, compared with that of \( T_{l,\kappa} \) in Figure 2.1.

![Figure 3.1](image)

(a) \( \text{Re}(S_{l,\kappa}) \)  
(b) \( \text{Im}(S_{l,\kappa}) \)  
(c) \( \text{Re}(S_{l,\kappa}) \) vs \( S_{l,\kappa}^R \)  
(d) \( \text{Im}(S_{l,\kappa}) \) vs \( S_{l,\kappa}^I \)

**Figure 3.1.** (a)-(b) graphs of real and imaginary parts of \( S_{l,\kappa} \) for various \( (l, \kappa) \in [0, 120] \times [1, 100] \). (c) \( \text{Re}(S_{l,\kappa}) \) (solid line) against \( S_{l,\kappa}^R \); (d) \( \text{Im}(S_{l,\kappa}) \) (solid line) against \( S_{l,\kappa}^I \) with \( \kappa = 30, 50, 70, 90 \) (note: “+” for \( \rho = \nu/\kappa \in (0, \theta_0) \), “o” for \( \rho \in [\theta_0, \theta_1] \), “x” for \( \rho \in (\theta_1, \vartheta_2) \) and “*” for \( \rho \in [\vartheta_2, \infty) \)).

Thanks to (3.37) and the estimates in Theorem 2.1 we can analyze the behaviour of \( S_{l,\kappa} \). In Figure 3.1 (c)-(d), we plot the exact value and approximations in Theorem 3.2 below for various samples of \( \kappa \).

**Theorem 3.2.** Let \( \theta_0, \vartheta_1, \vartheta_2 \) and \( \kappa_0 \) be as before. Denote \( \nu = l + 1/2 \) and \( \rho = \nu/\kappa \). Then for any \( \kappa > \kappa_0 \),
\[
\text{Re}(S_{l,\kappa}) \sim S_{l,\kappa}^R, \quad \text{Im}(S_{l,\kappa}) \sim S_{l,\kappa}^I, \quad \forall l \geq 1, \quad \text{where}
\] (3.39)

(i) for \( \rho = \nu/\kappa \in (0, \theta_0) \),
\[
S_{l,k}^R = \frac{1}{2k} \left( \frac{\rho}{1 - \rho^2} \right)^2, \quad S_{l,k}^I = \frac{1}{\sqrt{1 - \rho^2}};
\] (3.40)
where $\rho$ is a generic positive constant independent of $t$.

Theorem 3.3. Let $\theta_0$ and $\theta_1 \in \mathbb{R}$ be the same as in (2.1)–(2.8) and $g_2, g_3 \in L^2(I)$, then we have that for all $k \geq k_0 > 0$ (for some fixed constant $k_0$), and $l \geq 1$, $0 \leq |m| \leq l$,

$$
\|v_3\|_{H^1}^2 + \beta l \|v_3\|^2 + k^2 \|v_3\|_{H^1}^2 \leq C_{l,k} \left( \frac{1}{2l} \|g_2\|_{H^1}^2 + \|g_3\|_{H^1}^2 \right) + C \left( 1 + \frac{\beta^2}{k^4} \right) \|h_Y\|^2,
$$

where $C$ is a generic positive constant independent of $k, l, m$ and $v_3$, and

$$
C_{l,k} = C \begin{cases} 1, & \text{if } \rho = \nu/(kb) \in (0, \theta_0) \cup (\theta_2, \infty), \\ (kb)^{1-\gamma}, & \text{if } \rho = \nu/(kb) \in (\theta_0, \theta_2]. \end{cases}
$$

Note that for $\rho \in (\theta_0, \theta_2]$, we have $\rho = 1 + \xi(kb)^{-\gamma}$ or $\nu = l + 1/2 = kb + \xi(kb)^{-1}$, for $1/3 \leq \gamma < 1$, and some constant $\xi$. We postpone the derivation of the above estimates to Appendix C.

Remark 3.1. With some careful calculations, one can verify that

$$
\min \{ H^R(t) \} = H^R(t = -1) \approx 0.2493, \quad \max \{ H^R(t) \} = H^R(t \approx 0.8004) \approx 1.9291,
$$

$$
\min \{ H^l(t) \} = H^l(t \approx 0.2479, \quad \max \{ H^l(t) \} = H^l(t = 0) = 1.
$$

Thus, we roughly have $0.2493 \leq H^R(t) \leq 1.9291$ and $0.2479 \leq H^l(t) \leq 1$. \hfill \Box
Proof. Taking \( w = v_3 \) in (3.46), we obtain
\[
\|v_3\|_2^2 + \beta_1^2 \|v_3\|_2^2 - k^2 \|v_3\|_w^2 - k \|v_3\|_w^2 + b \|v_3\|_w^2 + b^2 \text{Re}(S_{l,kb})|v_3(b)|^2 + b|v_3(b)|^2
= b \text{Re}(v_2(b) \bar{v}_3(b)) + 2b \text{Re}(v_2, v_3) + b^2 \text{Re}(h_Y \bar{v}_3(b)) + \text{Re}(g_3, v_3)_w,
\]
\( (3.50a) \)
\[-k \|v_3\|_w^2 - k \|v_3\|_w^2 = b \text{Im}(v_2(b) \bar{v}_3(b)) + 2b \text{Im}(v_2, v_3) + b^2 \text{Im}(h_Y \bar{v}_3(b)) + \text{Im}(g_3, v_3)_w.\]
\( (3.50b) \)

Next taking \( w = 2(r-a)v_4 \) in (3.46), and following the derivation of (2.25)–(2.26), we can obtain
\[
b^2 |I|^2 \|v_3(b)|^2 + (\beta_1 |I| + b) \|v_3(b)|^2 + 2a \|\sqrt{r}v_3\|_w^2 + 2k^2 \int_0^b \left[ 1 - \frac{a}{r} \right] |v_3|^2 r^2 dr
= (k^2 b^2 |I| + kb^2 \text{Re}(S_{l,kb})) |v_3(b)|^2 + 2k^2 |I| \text{Re}(S_{l,kb}) (\|v_3\|_w^2 + |v_3(b)|^2 + 2 \text{Re}(v_2, v_3) + \text{Re}(g_3, v_3)_w + 2b |I| \text{Re}(v_2 \bar{v}_3(b)) + 2b^2 |I| \text{Re}(h_Y \bar{v}_3(b)) + 2 \text{Re}(g_4, (r-a)v_4)_w).
\]
\( (3.51) \)

Then we can derive the estimate similar to (2.28) (by noting that \( S_{l,kb} - (kb)^{-1} \) should be in place of \( T_{l,kb} \) and the term of the left endpoint \( r = a \) is not involved):
\[
b^2 |I|^2 \|v_3(b)|^2 + D_{l,k} |I| \|v_3(b)|^2 + a \|\sqrt{r}v_3\|_w^2 + k^2 \|v_3\|_w^2 \leq C \left( \|v_2\|_w^2 + \|v_2(b)\|_w^2 + |g_3(3.56)|_w^2 + |h_Y|^2 \right),
\]
where
\[
D_{l,k} := \beta_1 - (1 - \varepsilon_3) |I|^{-1} k b^2 \text{Re}(S_{l,kb}) - k b^2 b^2 (1 + \varepsilon_1^{-1} |S_{l,kb} - (kb)^{-1}|^2).
\]

Thus, it remains to bound the term \( D_{l,k} |I| \|v_3(b)|^2 \) (note: it is negative for some range of \( l \)), and to estimate the terms of \( v_2 \) by using that of \( u_2 \) in Theorem 3.11 and its proof. Following the proof of Theorem 3.11, we proceed with four cases.

(i) If \( \rho = \frac{k}{kb} \in (0, \theta_0) \) for fixed \( 0 < \theta_0 < 1 \), we find from (3.40) that both \( k \beta_1 \text{Re}(S_{l,kb}) \) and \( \text{Im}(S_{l,kb}) \) behave like constants. Thus, from (3.50b), we can obtain the bound like (2.31):
\[
k^2 b^2 \|v_3(b)|^2 \leq k \varepsilon k^2 \|v_3\|_w^2 + C \left( \|v_2\|_w^2 + \|v_2(b)\|_w^2 + |g_3(3.56)|_w^2 + |h_Y|^2 \right).
\]
\( (3.54) \)

Noting from (3.40) and (3.53) that
\[
D_{l,k} \sim \beta_1 - C k^2 b^2,
\]
we infer from (3.52) that
\[
b^2 |I|^2 \|v_3(b)|^2 + \beta_1 |I| \|v_3(b)|^2 + a \|\sqrt{r}v_3\|_w^2 + k^2 \|v_3\|_w^2 \leq C \left( \|v_2\|_w^2 + \|v_2(b)\|_w^2 + |g_3(3.56)|_w^2 + |h_Y|^2 \right).
\]
\( (3.56) \)

Recall from (3.29) that \( h_2 = -h_Y/(k, S_{l,kb}), u_2 = r \beta_1^{-1} v_2 \) and \( f_2 = r \beta_1^{-1} g_2 \). Then by (2.32),
\[
|v_2|_w^2 + |v_2(b)|^2 \leq C \left( \frac{1}{k^2} \|g_2\|_w^2 + \frac{\beta_2^2}{k^4} |h_Y|^2 \right) \leq C \left( \frac{1}{\beta_1} \|g_2\|_w^2 + \frac{\beta_2^2}{k^4} |h_Y|^2 \right).
\]
\( (3.57) \)

Thus, using (3.50a), (3.54), (3.56), (3.57) and the Cauchy-Schwarz inequality, we can obtain (3.48).

(ii) If \( \rho = \frac{k}{kb} \in [\theta_0, \theta_1] \), we start with (2.35), and find from (3.41) that
\[
\text{Re}(S_{l,kb} - (kb)^{-1}) \sim \frac{\varepsilon_1}{\varepsilon_0} (kb)^{-2} \gamma, \quad \text{Im}(S_{l,kb}) \sim \frac{1}{\sqrt{\varepsilon_0}} (kb)^{(1-2)\gamma}/2,
\]
\( (3.58) \)

where \( 1/3 \leq \gamma_0 < 1 \). Thus, by (3.55)–(3.58), \( D_{l,k} \sim -C (kb)^{-2} \gamma \). As with (2.37)–(2.39), we can derive
\[
(kb)^{-2} \gamma \|v_3(b)|^2 \leq k^2 \|v_3\|_w^2 + C \left( (kb)^{-2} \gamma (\|v_2\|_w^2 + |g_3|_w^2 + |v_2(b)|^2 + |h_Y|^2 \right).
\]
\( (3.59) \)

Therefore, we have
\[
\|\sqrt{r}v_3\|_w^2 + k^2 \|v_3\|_w^2 + (kb)^{-2} \gamma |v_3(b)|^2 \leq C \left( (kb)^{-2} \gamma (\|v_2\|_w^2 + |g_3|_w^2 + |v_2(b)|^2 + |h_Y|^2 \right).
\]
\( (3.60) \)

Like (3.57), we derive from (2.40) (note: \( h_2 = -h_Y/(k, S_{l,kb}), u_2 = r \beta_1^{-1} v_2, f_2 = r \beta_1^{-1} g_2 \)) and (3.58) that
\[
k^{-2} \gamma |v_2|_w^2 + |v_2(b)|^2 \leq C \left( \frac{1}{k^1 + \gamma} \|g_2\|_w^2 + \frac{\beta_2^2}{k^4} |h_Y|^2 \right) \leq C \left( \frac{k^{-1} + \gamma}{\beta_1} \|g_2\|_w^2 + \frac{\beta_2^2}{k^4} |h_Y|^2 \right).
\]
\( (3.61) \)

Thus, as with the previous case, we can obtain the desired estimate.
(iii) If \( \rho = \frac{\ell}{k} \in (\theta_1, \theta_2) \), we have the range in (2.41). Using (3.42)-(3.43), we can show that in this range, the bound is the same as (2.50) with \( \gamma_0 = 1/3 \):

\[
\|\sqrt{r}v_3^r\|_2^2 + k^2\|v_3\|_\infty^2 + (kb)^{\beta/k}\|v_3(b)\|^2 \leq C\left((kb)^{2/3}(\|v_2\|_2^2 + \|g_3\|_\infty^2) + \|v_2(b)\|^2 + |h_Y|^2\right). \tag{3.62}
\]

Similarly, we can bound the terms involving \( v_2 \) by (3.61) with \( \gamma_0 = 1/3 \).

(iv) If \( \rho = \frac{\ell}{k} \in (\theta_2, \infty) \), we find from (3.45) that \( \text{Im}(\zeta_{l,kb}) \) decays exponentially with respect to \( e \). However, since \( \text{Re}(\zeta_{l,kb} - (kb)^{-1}) > 0 \), we do not have (2.48) to bound the term \( D_{l,k}|v_3(b)|^2 \) (note: \( D_{l,k} < 0 \)), as opposed to the estimate of \( u_2 \) in Theorem 3.1. For this purpose, we use the equivalent boundary condition (3.32)-(3.33). Correspondingly, we modify the weak form (3.46) as

\[
(v_3', w') + \beta_1(v_3, w) - k^2(v_3, w) = b\sigma_{l,kb}v_2(b)\bar{w}(b) + 2(v_2, w)
\]

\[
+ (g_3, w) + b^2\frac{h_Y}{k}\bar{\sigma}_{l,kb}\bar{w}(b), \quad w \in \mathcal{H}_1^1(\Lambda).
\] \tag{3.63}

Taking \( w = v_3 \) in (3.63), leads to

\[
\|v_3\|_2^2 + \beta_1\|v_3\|^2 - k^2\|v_3\|_\infty^2 = b\text{Re}(\sigma_{l,kb}v_2(b)v_3(b))
\]

\[
+ \text{Re}(g_3, v_3) + 2\text{Re}(v_2, v_3) + b^2\text{Re}\left(\frac{h_Y}{k}\bar{\sigma}_{l,kb}\bar{v}_3(b)\right), \tag{3.64}
\]

Next taking \( w = 2(r - a)v_3' \) and following the same procedure in deriving (2.25)-(2.26), we have

\[
b^2|I|v_3'(b)|^2 + (\beta_1 - k^2b^2)I|v_3(b)|^2 + 2a\sqrt{r}v_3'^2 + 2k^2\int_a^b \left(1 - \frac{a}{r}\right)|v_3'|^2 r^2 dr
\]

\[
= 2b|I|\text{Re}(\sigma_{l,kb}v_2(b)v_3'(b)) + b\text{Re}(\sigma_{l,kb}v_2(b)v_3'(b)) + 4\text{Re}(v_2, (r - a)v_3') + 2\text{Re}(v_2, v_3)
\]

\[
+ 2\text{Re}(g_3, (r - a)v_3') + \text{Re}(g_3, v_3) + 2b^2|I|\text{Re}\left(\frac{h_Y}{k}\bar{\sigma}_{l,kb}\bar{v}_3(b)\right) + 2b^2\text{Re}\left(\frac{h_Y}{k}\bar{\sigma}_{l,kb}\bar{v}_3(b)\right).
\] \tag{3.65}

Using the Cauchy-Schwarz inequality, we can derive

\[
|v_3'(b)|^2 + (\beta_1 - k^2b^2)|v_3(b)|^2 + \|\sqrt{r}v_3'(b)\|^2 + \|v_2\|_\infty^2 \leq C\left\{\text{Re}(\sigma_{l,kb})^2 (1 + (\beta_1 - k^2b^2)^{-1})|v_3(b)|^2
\]

\[
+ \|v_2\|_\infty^2 + \|g_3\|_\infty^2 + \frac{1}{(kb)^2}\|\zeta_{l,kb}\|^2 (1 + (\beta_1 - k^2b^2)^{-1})|h_Y|^2\right\}. \tag{3.66}
\]

We first consider the range (a) in (2.44), i.e., \( \nu \sim kb + \tilde{c}_5(kb)^{\gamma_1} \) for \( 1/3 \leq \gamma_1 < 1 \) and some constant \( \tilde{c}_5 > 0 \). From (3.33) and (3.44), one verifies

\[
\beta_1 - k^2b^2 \sim 2\tilde{c}_5(kb)^{1 + \gamma_1}, \quad \sigma_{l,kb} \sim \text{Re}(\sigma_{l,kb}) \sim \frac{1}{\sqrt{2\tilde{c}_5(kb)^{1 - \gamma_1}}}, \quad |\sigma_{l,kb}| \sim 2\tilde{c}_5(kb)^{-\gamma_1 - 1}. \tag{3.67}
\]

Then we obtain from (3.66)-(3.67) that

\[
k^2\|v_3\|_\infty^2 \leq C\left((kb)^{2(\gamma_1 - 1)}|v_2(b)|^2 + \|v_2\|_\infty^2 + \|g_3\|_\infty^2 + (kb)^{-(1 + \gamma_1)}|h_Y|^2\right). \tag{3.68}
\]

Recalling that \( h_2 = -h_Y/(k\zeta_{l,kb}) \), \( u_2 = r\beta_1^{-1}v_2 \) and \( f_2 = r\beta_1^{-1}g_2 \), we have from (2.50) and (3.67) that

\[
\|v_2\|_\infty^2 + (kb)^{(2\gamma_1 - 1)}|v_2(b)|^2 \leq C\left(\frac{1}{\beta_1}\|g_2\|_\infty^2 + \frac{\beta_1^2}{k^2}|h_Y|^2\right). \tag{3.69}
\]

As \( v_3 \in \mathcal{H}_1^1(I) \), one verifies readily that

\[
|v_3(b)| \leq \int_a^b |v_3'(r)| dr \leq C\|v_3\|_\infty. \tag{3.70}
\]

Thus, using (3.64) and the Cauchy-Schwarz inequality, we can obtain the same upper bound as (3.68) for \( \|v_3\|^2 + \beta_1\|v_3\|^2 \). This leads to the desired estimate for this case.

We then consider the range (b) in (2.44), i.e., \( \nu > \eta kb \) with \( \eta > 1 \). Once again, by (3.33) and (3.44),

\[
\sigma_{l,kb} \sim \text{Re}(\sigma_{l,kb}) \sim \frac{kb}{\nu\sqrt{1 - \eta^{-2}}}, \quad \sigma_{l,kb} \sim 1 - \eta^{-2}. \tag{3.71}
\]
It is evident that
\[ \beta_1 \|v_3\|^2 - k^2 \|v_3\|^2 \geq (\beta_1 - k^2 \beta_1^2) \|v_3\|^2 \geq \beta_1 (1 - \eta^{-2}) \|v_3\|^2. \] (3.72)

Using the Cauchy-Schwarz inequality, and \(3.70\), \(3.72\), we have from \(3.64\) that
\[ \|v_3\|^2 + \beta_1 \|v_3\|^2 \leq C \left( (v_2(b))^2 + \beta_3^{-1} \|v_2\|^2 + \beta_3^{-1} \|g_1\|^2 + \frac{\beta_3}{k \beta_3} |h_Y|^2 \right). \] (3.73)

Then by \(2.23\), \(2.54\) and the fact that \(h_2 = -h_Y/(k S_{l,kb})\), \(u_2 = r \beta_3^{-1} v_2\) and \(f_2 = r \beta_3^{-1} g_2\), we obtain
\[ |v_2(b)|^2 + \beta_3^{-1} \|v_2\|^2 \leq C \left( \frac{1}{\beta_3} \|g_2\|^2 + \frac{\beta_3^2}{k \beta_3} |h_Y|^2 \right). \] (3.74)

Then we can derive the desired estimates.

\[ \Box \]

**Remark 3.2.** It is seen from \(3.24\) that \(\|u_2\|_\infty = O(1)\), while by \(3.48\), \(\|u_2\|_\infty = O(k^{-1} \sqrt{c_{l,k}})\) by noting that \(v_3 = \hat{\eta} u_2\).

\[ \Box \]

### 3.4. Main result on a priori estimates of \(E\)

We are in a position to derive a priori estimates for the Maxwell equations. A weak form of \((1.2), (1.3)\) is to find \(E \in V := H_0(\text{curl}; \Omega) \cap H_0(\text{div}; \Omega)\) such that
\[
\mathcal{B}(E, \Psi) := (\nabla \times E, \nabla \times \Psi)_\Omega - k^2 (E, \Psi)_\Omega - ikb \langle \mathcal{S}_k E_S, \Psi_S \rangle_S = (F, \Psi)_\Omega + b^2 \langle h, \Psi_S \rangle_S, \quad \forall \Psi \in V. \] (3.75)

Its well-posedness can be established using the property: \(\text{Re} \langle \mathcal{S}_k E_S, E_S \rangle_S > 0\) (see, e.g., Nédélec [31] Chapter 5 and Monk [29] Chapter 10). By \(31\) (5.3.47), the surface divergence of \(h\) (with the expansion \(3.9\)) can be expressed as
\[ \text{div}_S h = -\sum_{l=1}^{\infty} \sum_{|m|=0}^{l} \beta_l h_{Y,l}^m Y_l^m, \quad \text{so} \quad \|\text{div}_S h\|^2_{L^2(S)} = \sum_{l=1}^{\infty} \sum_{|m|=0}^{l} \beta_l^2 \|h_{Y,l}^m\|^2. \] (3.76)

**Theorem 3.4.** Let \(E\) be the solution to \(3.75\). If \(F \in L^2(\Omega), h \in L^2(T)\) and \(\text{div}_S h \in L^2(S)\), then we have \(E \in H_0(\text{curl}; \Omega)\) and
\[ \|\nabla \times E\|_\Omega + \kappa \|E\|_\Omega \leq C \left( k \|F\|_\Omega + \|h\|_{L^2(S)} + \kappa^{-2} \|\text{div}_S h\|_{L^2(S)} \right). \] (3.77)

for all \(k \geq k_0 > 0\) (\(k_0\) is some positive constant), where \(C\) is independent of \(k, E, F\) and \(h\).

**Proof.** With the notation in \(3.29\), we can rewrite the field \(E\) in \(3.3\) as
\[ E = u_0^Y Y_0^0 e_r + \sum_{l=1}^{\infty} \sum_{|m|=0}^{l} \left\{ u_{l,l}^m T_l^m + v_{l,l}^m Y_l^m e_r + v_{l,l}^m \nabla S Y_l^m \right\}, \] (3.78)

where we recall (cf. Proposition \(3.2\)) \(-k^2 u_0^Y = f_0^Y\). Thus, by the orthogonality and \(A.1\),
\[ \|E\|^2_{\Omega} = \|u_0^Y\|^2_{\infty} + \sum_{l=1}^{\infty} \sum_{|m|=0}^{l} \beta_l \left\{ \|u_{l,l}^m\|^2_{\infty} + \|v_{l,l}^m\|^2_{\infty} + \|v_{l,l}^m\|^2_{\infty} \right\}. \] (3.79)

Working out \(\nabla \times E\) via \(3.78\) and \(A.4\) \(- A.5\), we obtain from \(A.1\) that
\[
\|\nabla \times E\|^2_{\Omega} = \sum_{l=1}^{\infty} \sum_{|m|=0}^{l} \beta_l \left\{ \|\nabla u_{l,l}^m\|^2_{\infty} + \beta_l \|u_{l,l}^m\|^2_{\infty} + \|v_{l,l}^m/r - \hat{\eta} v_{l,l}^m\|^2 \right\}. \] (3.80)

Noting that \(\beta_l \geq 2 \leq 2 \beta_1\) and \(\|\nabla u_{l,l}^m\|^2_{\infty} \leq 2 \left( \|u_{l,l}^m\|^2_{\infty} + \|u_{l,l}^m\|^2_{\infty} \right),\) we obtain from \(3.79\), \(3.80\) that
\[ \|\nabla \times E\|^2_{\Omega} + k^2 \|E\|^2_{\Omega} \leq \|u_0^Y\|^2_{\infty} + \sum_{l=1}^{\infty} \sum_{|m|=0}^{l} \beta_l \left\{ 2 \left( \|u_{l,l}^m\|^2_{\infty} + \beta_l \|u_{l,l}^m\|^2_{\infty} \right) + k^2 \|u_{l,l}^m\|^2_{\infty} \right\} \]
\[ + \sum_{l=1}^{\infty} \sum_{|m|=0}^{l} \beta_l \left\{ 2 \|v_{l,l}^m\|^2 + k^2 \beta_l^{-1} \|v_{l,l}^m\|^2_{\infty} \right\} + \sum_{l=1}^{\infty} \sum_{|m|=0}^{l} \beta_l \left\{ 4 \left( \|v_{l,l}^m\|^2_{\infty} + \|v_{l,l}^m\|^2_{\infty} \right) + k^2 \|v_{l,l}^m\|^2_{\infty} \right\}. \]
Similarly, using the orthogonality of VSH, we have
\[
\|F\|_\Omega^2 = \|f_0^0\|_\Omega^2 + \sum_{l=1}^{\infty} \sum_{|m|=0} \beta_l \left\{ \|f_{l,1}^m\|_\Omega^2 + \beta_l^{-1} \|g_{l,1}^m\|_\Omega^2 + \|g_{l,3}^m\|_\Omega^2 \right\},
\]
(3.81)
\[
\|h\|_{L^2(S)}^2 = \sum_{l=1}^{\infty} \sum_{|m|=0} \beta_l \left\{ |h_{l,1}^m|^2 + |h_{l,3}^m|^2 \right\}.
\]
Recall from (3.29) that \( h_{l,1}^m = -h_{l,1}^m/(k \kappa_{l,k}) \), \( u_{l,1}^m = r \beta_l^{-1} v_{l,1}^m \) and \( f_{l,1}^m = r \beta_l^{-1} g_{l,1}^m \). Then by Theorem 3.1,
\[
\|v_{l,1}^m\|^2 + 2 \beta_l^{-1}\|v_{l,1}^m\|^2 \leq C \left\{ \|\beta_l^{-1} g_{l,1}^m\|^2 + k^{-4} \beta_l^2 |h_{l,1}^m|^2 \right\},
\]
(3.82)
where we have used the fact \( |\kappa_{l,k}|^{-2} \leq C |\kappa_l/k|^2 \) for all the ranges of \( l, k \) in the proof of Theorem 3.3. We further derive from Theorems 3.1, 3.3 and (3.82) that
\[
\|\nabla \times E\|_\Omega^2 + k^2 \|E\|_\Omega^2 \leq k^{-2} \|f_0^0\|_\Omega^2^2 + C \sum_{l=1}^{\infty} \sum_{|m|=0} \beta_l \left\{ \|f_{l,1}^m\|_\Omega^2^2 + |h_{l,1}^m|^2 \right\} + C \sum_{l=1}^{\infty} \sum_{|m|=0} \beta_l \left\{ \|\beta_l^{-1} g_{l,1}^m\|^2 + |h_{l,1}^m|^2 \right\} + k^{-4} \beta_l^2 |h_{l,1}^m|^2
\]
\[
+ k^{-4} \beta_l^2 |h_{l,1}^m|^2 + \sum_{l=1}^{\infty} \sum_{|m|=0} \beta_l \left\{ C_{l,k} (\|\beta_l^{-1} g_{l,1}^m\|^2 + |g_{l,3}^m|^2) + C (1 + k^{-4} \beta_l^2) |h_{l,1}^m|^2 \right\}.
\]
Finally, the desired estimate follows from (3.76), (3.81) and the above. \(\square\)

4. Spectral-Galerkin approximation and its wavenumber explicit analysis

In this section, we consider the analysis of spectral-Galerkin approximation to (3.75). We look for the approximation of \( E \) in the form
\[
E_{N}^l = -k^{-2} f_0^0 \mathbf{y}_0^0 \mathbf{e}_r + \sum_{l=1}^{L} \sum_{|m|=0} \{ u_{1,l}^N \mathbf{T}_{l}^m + \nabla \times (u_{2,l}^N \mathbf{T}_{l}^m) \},
\]
(4.1)
where \( u_{1,l}^N := u_{1,l}^N \) and \( u_{2,l}^N := u_{2,l}^N \) are respectively the solutions of the spectral-Galerkin schemes:

(i) Find \( u_{1}^N \in 0P_N := 0H^1(I) \cap P_N \) (where \( P_N \) is the space of polynomials of degree at most \( N \)) such that
\[
\mathbf{B}_l^m (u_{1}^N, \phi) = \left( f_{l,1} \phi \right)_\Omega + b^2 h_{l,1} \tilde{\phi}(b), \quad \forall \phi \in 0P_N,
\]
(4.2)
(ii) Find \( u_{2}^N \in P_N \) such that
\[
\mathbf{B}_l^m (u_{2}^N, \psi) - au_{2}^N (a) \tilde{\psi}(a) = \left( f_{l,1} \psi \right)_\Omega + b^2 h_{l,1} \tilde{\psi}(b), \quad \forall \psi \in P_N.
\]
(4.3)
Here, the bilinear forms \( \mathbf{B}_l^m \) is defined in (2.22). It is evident that by Proposition 3.1, the expansion in (4.1) preserves the divergence-free property of the continuous field.

**Theorem 4.1.** Theorem 3.1 hold when \( u_{1}^N, u_{2}^N \) are in place of \( u_1, u_2 \) in (3.24), respectively.

**Remark 4.1.** The algorithm in the recent work [26] was based on VSH expansion in [31], so the divergence-free condition could only be fulfilled approximately. Moreover, one had to deal three components where two were coupled. In a nutshell, the above algorithm is much more efficient. \(\square\)

4.1. Error estimates. As before, we start with the schemes (4.2)-(4.3) in one dimension. To describe the errors more precisely, we introduce the weighted Sobolev space
\[
X^s(I) := \left\{ u \in L^2(I) : \|(r-a)(b-r)\|_{L^2(I)} \leq a \|u\|_{L^2(I)}, \quad 1 \leq l \leq s \right\}, \quad s \in \mathbb{N},
\]
with the norm and semi-norm
\[
\|u\|_{X^s(I)} = \left( \|u\|_2^2 + \sum_{l=1}^{s} \|(r-a)(b-r)\| / 2 \right)^{1/2}, \quad \|u\|_{X^s(I)} = \left( \|u\|_{L^2(I)} \right)^{1/2}.
\]
Following the proof of [26] Thm 4.2 (but using the improved estimates in Theorem 3.1), we have the following error estimate for the scheme (4.2).
Lemma 4.1. Let $u_1$ and $u_N^s$ be the solution of (3.22) and (4.2), respectively, and define $e_N^s = u_1 - u_N^s$. If $u_1 \in H^1(I) \cap X^s(I)$ with integer $s \geq 1$, then for all $k \geq k_0$ (where $k_0$ is a certain constant), we have

$$\| (e_N^s) \|_\infty + \sqrt{\beta_1} \| e_N^s \| + k \| e_N^s \|_\infty \lesssim (\sqrt{\beta_1} + k^2 N^{-1}) N^{1-s} |u_1|_{X^s(I)}, \quad (4.4)$$

where $\beta_1 = l/(l+1)$ and $\infty = r^2$ as before.

Now, we turn to (4.3). Consider the orthogonal projection $\pi_N^1 : H^1(I) \rightarrow \mathcal{P}_N$ defined by

$$((\pi_N^1 v - v), \phi) = 0, \quad \forall \phi \in \mathcal{P}_N. \quad (4.5)$$

Noting that the weight function $\infty$ is uniformly bounded below and above, we follow the argument in [34] Ch. 3, and derive the following estimate.

Lemma 4.2. For any $v \in X^s(I)$ with $s \in \mathbb{N}$, we have

$$\| (\pi_N^1 v - v) \|_\infty + N \| \pi_N^1 v - v \|_\infty \lesssim N^{1-s} |v|_{X^s(I)}. \quad (4.6)$$

Lemma 4.3. Let $u_2$ and $u_N^s$ be the solution of (3.23) and (4.3), respectively, and define $e_N^s = u_2 - u_N^s$. If $u_2 \in X^s(\Lambda)$ with $s \in \mathbb{N}$, then for all $k \geq k_0$ (where $k_0$ is a certain constant), then the estimate (4.4) holds when $u_2$ and $e_N^s$ are in place of $u_1$ and $e_N^s$, respectively.

Proof. Let $\hat{e}_N = u_N^s - \pi_N^1 u_2$ and $\hat{e}_N = u_2 - \pi_N^1 u_2$. Then $e_N^s = \hat{e}_N - \hat{e}_N$. By (3.23) and (4.3),

$$\mathbb{B}_l^m (e_N^s, \psi) - ae_N^s (\hat{e}_N, \psi) \hat{e}_N = 0 = \mathbb{B}_l^m (\hat{e}_N, \psi) - ae_N^s (\hat{e}_N, \psi) \hat{e}_N - \mathbb{B}_l^m (\hat{e}_N, \psi) + ae_N^s (\hat{e}_N, \psi) \hat{e}_N, \quad \forall \psi \in \mathcal{P}_N. \quad (4.7)$$

Thus, by (4.5),

$$\mathbb{B}_l^m (\hat{e}_N, \psi) - ae_N^s (\hat{e}_N, \psi) \hat{e}_N = \mathbb{B}_l^m (\hat{e}_N, \psi) - ae_N^s (\hat{e}_N, \psi) \hat{e}_N = \beta_l (\hat{e}_N, \psi) - (k^2 + 1) (\hat{e}_N, \psi) \infty = a e_N^s (\hat{e}_N, \psi) \hat{e}_N - k b^2 T_k b \hat{e}_N (b) \hat{e}_N, \quad \forall \psi \in \mathcal{P}_N. \quad (4.7)$$

Compared with the analysis for (4.2), the only difference is the presence of the extra term $"-a e_N^s (\hat{e}_N, \psi) \hat{e}_N"$, which is akin to the situation in the proof of Theorem 3.1. We omit the details, as one can refer to the proofs of [36] Thm 4.2 and Theorem 3.1.

We now estimate the error between the electric field and its spectral approximation in (4.1)-(4.3). We first introduce suitable functional spaces to characterize the regularity of the electric field. For any $E \in L^2(\Omega)$, we write

$$E = v_{2,0}^0 (r) Y_0^0 e_r + \sum_{l=1}^{\infty} \sum_{m=0}^{l} \left\{ v_{1,l}^m (r) T_l^m + v_{2,l}^m (r) Y_l^m e_r + v_{3,l}^m (r) \nabla s Y_l^m \right\}. \quad (4.8)$$

We introduce the anisotropic Sobolev space $H^t(S; H_x^s(I))$ for $t \geq 0$ and integer $s \geq 0$, equipped with the norm:

$$\| E \|_{H^t(S; H_x^s(I))} = \left( \| v_{2,0}^0 \|_{H_x^s(I)}^2 + \sum_{l=1}^{\infty} \sum_{m=0}^{l} \beta_l^{1+t} \left( \| v_{1,l}^m \|_{H_x^s(I)}^2 + \beta_l^{-1} \| v_{2,l}^m \|_{H_x^s(I)}^2 + \| v_{3,l}^m \|_{H_x^s(I)}^2 \right) \right)^{1/2}. \quad (4.9)$$

Note that $H^0(S; H_x^0(I)) = L^2(\Omega)$. Here, we are interested in the divergence-free fields. In this case, like Proposition 3.1, we can rewrite $E \in H_0(\text{curl}; \Omega)$ in the divergence-free form:

$$E = \frac{c}{r^2} Y_0^0 e_r + \sum_{l=1}^{\infty} \sum_{m=0}^{l} \left\{ u_{1,l}^m (r) T_l^m + \nabla \times (u_{2,l}^m (r) T_l^m) \right\}, \quad (4.10)$$

where $c$ is an arbitrary constant, and for $l \geq 1$,

$$v_{1,l}^m (r) = u_{1,l}^m (r), \quad v_{2,l}^m (r) = \frac{\beta_l}{r} u_{2,l}^m (r), \quad v_{3,l}^m (r) = \left( \frac{d}{dr} + \frac{1}{r} \right) u_{2,l}^m (r). \quad (4.11)$$

Note that we can substitute (4.11) into (4.9) to express the norm in (4.9) in terms of $\{u_{1,l}^m, u_{2,l}^m\}$.
Theorem 4.2. If $E \in \mathbb{H}_0(\text{curl};\Omega) \cap L^2(S;H^s_\omega(I)) \cap H^s(S;L^2_w(I))$ with $s \in \mathbb{N}$, then
\[ \|E - E_N\|_{\Omega} \lesssim (1 + k^{-1}N)(L + k^2N^{-1})N^{-s}\|E\|_{L^2(S;H^s_\omega(I))} + L^{-s}\|E\|_{H^s(S;L^2_w(I))}, \] (4.12)
for all $k \geq k_0$ with $k_0$ being a positive constant.

Proof. By (3.1) and (4.1),
\[ E - E_N = \sum_{l=1}^{L} \sum_{|m|=0}^l \left\{ (u_{1,l}^m - u_{1,l}^{N,m}) T_l^m + \nabla \times (u_{2,l}^m - u_{2,l}^{N,m}) T_l^m \right\} \]
\[ + \sum_{l=L+1}^{\infty} \sum_{|m|=0}^l \left\{ u_{1,l}^m T_l^m + \nabla \times (u_{2,l}^m T_l^m) \right\} := S_1 + S_2, \]
where $S_2$ counts the error from truncating the VSH series. It is clear that by the orthogonality of VSH, (4.9) and (4.11),
\[ \|S_2\|_{\Omega}^2 = \sum_{l=L+1}^{\infty} \sum_{|m|=0}^l \beta_l \left\{ \|u_{1,l}^m - u_{1,l}^{N,m}\|_{\Omega}^2 + \|\hat{\partial}_l u_{2,l}^m\|_{\Omega}^2 + \beta_l \|u_{2,l}^m - u_{2,l}^{N,m}\|_{\Omega}^2 \right\} \]
\[ \lesssim \sum_{l=L+1}^{\infty} \sum_{|m|=0}^l \beta_l \left( \sqrt{\beta_l} + k^2N^{-1} \right)^2 k^{-2}N^{-2s} \|u_{1,l}^m\|_{\Omega}^2 \]
\[ \lesssim \sum_{l=L+1}^{\infty} \sum_{|m|=0}^l \beta_l \left( \sqrt{\beta_l} + k^2N^{-1} \right)^2 N^{-2s} \|u_{1,l}^m\|_{\Omega}^2. \] (4.14)

Next, by (3.79), Lemma 4.1, Lemma 4.3 and (4.11),
\[ \|S_1\|_{\Omega}^2 \lesssim \sum_{l=1}^{L} \sum_{|m|=0}^l \beta_l \left\{ \|u_{1,l}^m - u_{1,l}^{N,m}\|_{\Omega}^2 + \|u_{2,l}^m - u_{2,l}^{N,m}\|_{\Omega}^2 + \beta_l \|u_{2,l}^m - u_{2,l}^{N,m}\|_{\Omega}^2 \right\} \]
\[ \lesssim \sum_{l=1}^{L} \sum_{|m|=0}^l \beta_l \left( \sqrt{\beta_l} + k^2N^{-1} \right)^2 k^{-2}N^{-2s} \|u_{1,l}^m\|_{\Omega}^2 \]
\[ + \sum_{l=1}^{L} \sum_{|m|=0}^l \beta_l \left( \sqrt{\beta_l} + k^2N^{-1} \right)^2 N^{-2s} \|u_{2,l}^m\|_{\Omega}^2. \] (4.15)

By (4.11) and a direct calculation,
\[ |u_{2,l}^m|_{\Omega}^2 \lesssim \|\partial_{\varphi}^r u_{2,l}^m\|_{L^2(S;L^2_w(I))}^2 = \|\partial_{\varphi}^r (\hat{\partial}_l u_{2,l}^m) - \partial_{\varphi}^r (u_{2,l}^m/r)\|_{L^2(I)}^2 \]
\[ \lesssim |\partial_{\varphi}^r (\hat{\partial}_l u_{2,l}^m)|_{L^2(I)}^2 + |\partial_{\varphi}^r (u_{2,l}^m/r)|_{L^2(I)}^2 \]
\[ = \|\partial_{\varphi}^r u_{3,l}^m\|_{L^2(I)}^2 + \beta_l^{-2} \|\partial_{\varphi}^r u_{3,l}^m\|_{L^2(I)}^2. \] (4.16)

As the weight $\omega$ is uniformly bounded below and above for $r \in (a,b)$, we derive from (4.9), (4.11) and (4.15)–(4.16) that
\[ \|S_1\|_{\Omega} \lesssim (1 + k^{-1}N)(L + k^2N^{-1})N^{-s}\|E\|_{L^2(S;H^s_\omega(I))}. \] (4.17)

A combination of (4.14) and (4.17) leads to the desired estimate. \hfill \Box

Remark 4.2. Note that the estimate in (4.12) is in the $L^2$-norm, not in the usual energy norm. For the continuous problem, we were able to obtain the bound for the energy norm through a further estimate of $\partial_l u_{1,l}^m$ in Subsection 3.3. However, this approach does not carry over to the discrete problem, as the second test function does not belong to the finite dimensional space for the spectral-Galerkin approximation of (3.46). We shall derive below a sub-optimal error estimate in the energy norm through a different approach. \hfill \Box

In what follows, we will derive a bound for the error $\nabla \times (E - E_N)$.

Theorem 4.3. If $E \in L^2(S;H^s_\omega(I)) \cap H^{s-1}(S;H^s_\omega(I)) \cap H^s(S;L^2_w(I))$ with $s \geq 3$, then
\[ \|\nabla \times (E - E_N)\|_{w,\Omega} \lesssim (N + 1 + kN^{-1})(L + k^2N^{-1})N^{1-s}\|E\|_{L^2(S;H^s_\omega(I))} \]
\[ + L^{-s}\left\{ \|E\|_{H^{s-1}(S;H^s_\omega(I))} + \|E\|_{H^s(S;L^2_w(I))} \right\}, \]
(4.18)
for all $k \geq k_0$ with $k_0$ being a positive constant, where $w = (b - r)(r - a)$. \hfill \Box
Proof. For notational convenience, let $e_{lm}^N = u_{lm}^N - u_{lm}^N (i = 1, 2)$. By (4.13), (A.1) and (A.5)-(A.4),
\begin{align*}
\left\| \nabla \times (E - E_N^L) \right\|_{w, \Omega}^2 & \leq \sum_{l=1}^{L} \sum_{|m|=0}^{l} \beta_l \left\{ \| \hat{\partial}_r e_{lm}^N \|_w^2 + \beta_l \| e_{lm}^N \|_w^2 + \| r L_i(e_{lm}^N) \|_w^2 \right\} \\
& + \sum_{l=L+1}^{\infty} \sum_{|m|=0}^{l} \beta_l \left\{ \| \hat{\partial}_r u_{lm}^N \|_w^2 + \beta_l \| u_{lm}^N \|_w^2 + \| r L_i(u_{lm}^N) \|_w^2 \right\} := T_1 + T_2. \tag{4.19}
\end{align*}

We first estimate $T_2$. It is clear that by (4.19) and (4.11),
\begin{align*}
\| r \hat{\partial}_r u_{lm}^N \|^2_w & + \beta_l \| u_{lm}^N \|^2_w \lesssim \| v_{lm}^N \|^2_{H^1_\omega(I)} + \| v_{lm}^N \|^2_{L^2_\omega(I)}, \\
\| r L_i(u_{2l}^m) \|^2_w & = \| \hat{\partial}_r^2 u_{2l}^m - \beta L_i^{-1} u_{2l}^m \|^2_w = \| \hat{\partial}_r^2 u_{2l}^m - \beta r L_i^{-1} u_{2l}^m \|^2_w = \| r \hat{\partial}_r v_{3l}^m - v_{2l}^m \|^2_w \lesssim \| v_{3l}^m \|^2_{H^1_\omega(I)} + \| v_{2l}^m \|^2_{L^2_\omega(I)},
\end{align*}
so we have
\begin{align*}
T_2 & \leq \sum_{l=L+1}^{\infty} \sum_{|m|=0}^{l} \beta_l \left\{ \| v_{1l}^m \|^2_{H^1_\omega(I)} + \| v_{2l}^m \|^2_{L^2_\omega(I)} + \| v_{3l}^m \|^2_{H^1_\omega(I)} \right\} \\
& + \sum_{l=L+1}^{\infty} \sum_{|m|=0}^{l} \beta_l^2 \| v_{1l}^m \|^2_{L^2_\omega(I)} \lesssim \beta_l^{-1} \left\{ \| E \|^2_{H^{-1}(S; H^1_\omega(I))} + \| E \|^2_{H^1(S; L^2_\omega(I))} \right\}. \tag{4.21}
\end{align*}

We next turn to estimating $T_1$. We see that it is necessary to obtain $H^2$-estimate of $e_{lm}^N$. To simplify the notation, we will drop $l, m$ from the notations if no confusion may arise. Taking $v = w(r) \hat{e}_N^\alpha (\in \mathcal{P}_N)$ with $w(r) = (r - a)(b - r)$ in (4.7), and using integration by parts, we obtain
\begin{align*}
B_l^m(\hat{e}_N, w \hat{e}_N^\alpha) = -(r^2 \hat{e}_N^\alpha)'(r^2 \hat{e}_N^\alpha)' + \beta_1(\hat{e}_N, w \hat{e}_N^\alpha) - k^2(r^2 \hat{e}_N, w \hat{e}_N^\alpha)' \\
= \beta(\hat{e}_N, w \hat{e}_N^\alpha) - (k^2 + 1)(r^2 \hat{e}_N, w \hat{e}_N^\alpha). \tag{4.22}
\end{align*}
Using integration by parts again, we derive from a direct calculation that
\begin{align*}
- \text{Re}(r^2 \hat{e}_N^\alpha)'(r^2 \hat{e}_N^\alpha)' & = -\| \hat{e}_N^\alpha \|^2_w - 2 \text{Re}(r \hat{e}_N^\alpha, w \hat{e}_N^\alpha) = -\| \hat{e}_N^\alpha \|^2_w + \int_a^b |r \hat{e}_N^\alpha|^2 (rw)' \, dr; \\
\text{Re}(\hat{e}_N, w \hat{e}_N^\alpha) & = -\| \hat{e}_N \|^2_w - \text{Re} \int_a^b \hat{e}_N \overline{w} \, dr = -\| \hat{e}_N \|^2_w - \frac{1}{2} \| \hat{e}_N \|^2_w + \frac{1}{2} \int_a^b |r \hat{e}_N|^2 w'' \, dr \\
& = -\| \hat{e}_N \|^2_w + \frac{b - a}{2} (|\hat{e}_N(a)|^2 + |\hat{e}_N(b)|^2) - \| \hat{e}_N \|^2_w; \\
- \text{Re}(r^2 \hat{e}_N, w \hat{e}_N^\alpha) & = \| r \hat{e}_N \|^2_w + \frac{1}{2} \| \hat{e}_N \|^2_w - \frac{1}{2} \int_a^b |r \hat{e}_N|^2 (rw)'' \, dr \\
& = \| r \hat{e}_N \|^2_w - \frac{b - a}{2} (a^2 |\hat{e}_N(a)|^2 + b^2 |\hat{e}_N(b)|^2) - \frac{1}{2} \int_a^b |r \hat{e}_N|^2 (rw)'' \, dr,
\end{align*}
and further by the Cauchy-Schwartz inequality,
\begin{align*}
|\langle \hat{e}_N, w \hat{e}_N^\alpha \rangle| & \leq \int_a^b |(w \hat{e}_N')| |r \hat{e}_N'| \, dr \leq \frac{1}{2} \| \hat{e}_N' \|^2 + \frac{1}{2} \| (w \hat{e}_N') \|^2 \leq \frac{1}{2} \| \hat{e}_N' \|^2 + c(\| \hat{e}_N \|^2 + \| \hat{e}_N^\alpha \|^2); \\
|\langle r^2 \hat{e}_N, w \hat{e}_N^\alpha \rangle| & \leq \int_a^b |(r^2 w \hat{e}_N')| |r \hat{e}_N'| \, dr \leq \frac{1}{2} \| \hat{e}_N' \|^2 + \frac{1}{2} \| (r^2 w \hat{e}_N') \|^2 \leq \frac{1}{2} \| \hat{e}_N' \|^2 + c(\| \hat{e}_N \|^2 + \| \hat{e}_N^\alpha \|^2).
\end{align*}
Thus, we obtain from (4.22) and the above estimates that
\begin{align*}
\| r \hat{e}_N^\alpha \|^2_w & \lesssim (\beta_l + k^2)(\| \hat{e}_N \|^2_{H^1_\omega(I)} + \| \hat{e}_N \|^2_{H^1_\omega(I)}). \tag{4.23}
\end{align*}
Recall that \( \hat{e}_N = u^N_2 - \pi_N^2 u_2, \hat{e}_N = u_2 - \pi_N^1 u_2 \) and \( e^m_m = \hat{e}_N - \hat{e}_S \), so we derive from Lemma \ref{lem:approximation} and Lemma \ref{lem:transformation} that

\[
\| (e^m_m)^\prime\|_w^2 \lesssim \| (e^m_m)^\prime\|_w^2 + (\beta_1 + k^2)(\| e^m_m \|_{H^1(I)}^2 + \| \hat{e}_S \|_{H^1(I)}^2) \\
\lesssim \| (u_2 - \pi_N^1 u_2)^\prime\|_w^2 + (\beta_1 + k^2)(\sqrt{\beta_1} + k^2 N^{-1})^2 N^{-2s} |u_2|_{X_s}^2. \tag{4.24}
\]

In order to estimate \( \| (u_2 - \pi_N^1 u_2)^\prime\|_w^2 \), we need to use the orthogonal projection \( \pi_N^2 : H^2(I) \to P_N \), and recall its approximation result (cf. \cite{34, Ch. 4}): for any \( v \in X^s(I) \),

\[
\| \pi_N^2 v - v\|_{H^s(I)} \lesssim N^{\mu-s} |v|_{X_s(I)}, \quad \mu = 0, 1, 2, \quad s \geq 2. \tag{4.25}
\]

Applying the inverse inequality (cf. \cite{34, Thm 3.33}) and the above approximation result, we obtain

\[
\| (\pi_N^1 v - \pi_N^2 v)^\prime\| \lesssim N^s |v|_{X_s(I)}, \quad s \geq 2.
\]

Therefore, we have

\[
\| (\pi_N^1 v - \pi_N^2 v)^\prime\| \leq \| (\pi_N^1 v - \pi_N^2 v)^\prime\| + \| (v - \pi_N^2 v)^\prime\| \lesssim N^{3-s} |v|_{X_s(I)}. \tag{4.26}
\]

From (4.24) and (4.26), we have

\[
\| (e^m_m)^\prime\|_w^2 \lesssim \{ N^4 + (\beta_1 + k^2)(\sqrt{\beta_1} + k^2 N^{-1})^2 \} N^{-2s} |u_2|_{X_s}^2. \tag{4.27}
\]

Now, we are ready to estimate \( T_1 \) in (4.19). Using Lemma \ref{lem:transformation}, we obtain

\[
\| r L (e^m_m)\|_w^2 \lesssim \| (e^m_m)^\prime\|_w^2 + \beta_1^2 |e^m_m|_{X_s}^2 \lesssim \{ N^4 + (\beta_1 + k^2)(\sqrt{\beta_1} + k^2 N^{-1})^2 \} N^{-2s} |u_2|_{X_s}^2. \tag{4.28}
\]

Therefore, we derive from Lemma \ref{lem:approximation}, \ref{lem:transformation} and \ref{lem:estimation},

\[
T_1 \lesssim \sum_{l=1}^{L} \sum_{m=0}^{l} \beta_1 (\sqrt{\beta_1} + k^2 N^{-1})^2 N^{-2s} |u_1, l|_{X_s}^2 \tag{4.29}
\]

A combination of (4.19), (4.21) and (4.29) leads to the desired estimate. \( \square \)

5. General scatterers through transformed field expansion

We consider now a general scatterer enclosed by \( \hat{\Omega} = \{(r, \theta, \phi) : 0 < r < a + g(\theta, \phi), \theta \in [0, \pi], \phi \in [0, 2\pi]\} \), for some \( a > 0 \) and given \( g \). Let us choose the radius \( b \) of the artificial spherical boundary such that \( b > \max_{r, \phi}\{a + g(\theta, \phi)\} \), and consider the Maxwell equations (1.2)-(1.3) in the domain \( \hat{\Omega} = \{a + g(\theta, \phi) < r < b\} \). An effective approach to deal with scattering problems in general domains with moderately large wave numbers is the so-called transformed field expansion (TFE) \cite{10}. It has been successfully applied to various situations, including in particular acoustic scattering problems in 2-D \cite{32} and 3-D \cite{14}.

In our recent work \cite{26}, we applied the TFE approach to the Maxwell equation (1.2)-(1.3) in \( \hat{\Omega} \). We outline below the essential steps of this approach, and refer to \cite{26} for more details.

- The first step is to transform the general domain \( \hat{\Omega} = \{a + g < r < b\} \) to the spherical shell \( \hat{\Omega} = \{a < r' < b\} \) in \( \hat{\Omega} \) with the change of variables:

\[
r' = \frac{(b-a)r - a g(\theta, \phi)}{b-a - g(\theta, \phi)}, \quad \theta' = \theta, \quad \phi' = \phi. \tag{5.1}
\]

With this change of variable, the Maxwell equation (1.2)-(1.3) in \( \hat{\Omega} \) is transformed to a Maxwell equation in \( \hat{\Omega} \) which can still be written in the form (1.2)-(1.3) with the understanding that all new terms (induced by the transform) are included in \( F \) and \( H \) (cf. \cite{26} (3.6)). With a slight abuse of notation, we shall still use \( r \) to denote \( r' \) and the same notations to denote the transformed functions.
The second step is to assume \( g(\theta, \phi) = \varepsilon f(\theta, \phi) \) and expand the solution \( E \) in \( \varepsilon \):

\[
E(r, \theta, \phi) = \sum_{n=0}^{\infty} E_n(r, \theta, \phi)\varepsilon^n.
\] (5.2)

Similarly, we can expand \( F \) (the original source function) and \( h \) as

\[
F(r, \theta, \phi) = \sum_{n=0}^{\infty} F_n(r, \theta, \phi)\varepsilon^n, \quad h(\theta, \phi) = \sum_{n=0}^{\infty} h_n(\theta, \phi)\varepsilon^n.
\] (5.3)

One can then derive a recursion formula for \( E_n \) (for \( n \geq 0 \)):

\[
\nabla \times \nabla \times E_n - k^2 E_n = F_n + G_n, \quad \text{in } \Omega;
\]

\[
E_n \times e_r = 0, \quad \text{at } r = a; \quad (\nabla \times E_n) \times e_r - i k b \left[(E_n)_b\right] = h_n, \quad \text{at } r = b,
\] (5.4) (5.5)

where \( G_n \) and \( h_n \) are given by explicit recurrence formulae in [33 Appendix B].

The third step is to obtain approximation \( E_{n, M}^L \) (in the form of (4.1)) to \( E_n \) (for \( 0 \leq n \leq M \)) by solving the above Maxwell equations (5.4)-(5.5) in the spherical shell \( \Omega \) using the decoupled method presented in Section 4. Then, we define our approximation to \( E \) by

\[
E_{n, M}^L(r, \theta, \phi) = \sum_{n=0}^{M} E_{n,N}^L(r, \theta, \phi)\varepsilon^n.
\] (5.6)

Next, we shall use the general convergence theory developed in [33] to give an error estimate for \( E - E_{n, M}^L \). Using essentially the same argument as in the proof of [33 Thm 5.5] for the Helmholtz equation, we can prove the following bounds.

**Proposition 5.1.** Let \( F \in (H^{s-2}(\Omega))^3 \), \( f \in H^s(\Omega) \) and \( h \in (H^{s-3/2}(\Omega))^2 \) for an integer \( s \geq 2 \). Then, the expansion (5.2) converges strongly, i.e., there exists \( C_1, C_2 > 0 \) such that

\[
\|E_n\|_{(H^s(\Omega))^3} \leq C_1 \left( \|F\|_{(H^{s-2}(\Omega))^3} + \|h\|_{(H^{s-3/2}(\Omega))^2} \right) B^n, \quad \text{for some } B > C_2 \|f\|_{H^s(\Omega)}.
\] (5.7)

On the other hand, it can be shown that the space with the norm in (4.9) satisfies \( H^l(S; H^s_\infty(I)) \subseteq (H^{s+l}(\Omega))^3 \). Therefore, with the above result and Theorems 4.2-4.3 at our disposal, we can then apply Theorem 2.1 in [33] to obtain the following:

**Theorem 5.1.** Let \( E \) be the solution of the Maxwell equations in \( \hat{\Omega} \) and \( E_{n, M}^L \) be its approximation defined in (5.6). Then, under the condition of Proposition 5.1 and Theorems 4.2-4.3, we have

\[
\|E - E_{n, M}^L\|_{\hat{\Omega}} \lesssim (B\varepsilon)^{M+1} + \left\{ (1 + k^{-1}N)(L + k^2N^{-1})N^{-s} + L^{-s} \right\} \left( \|F\|_{(H^{s-2}(\hat{\Omega}))^3} + \|h\|_{(H^s(\Omega))^2} \right),
\]

and

\[
\|\nabla \times (E - E_{n, M}^L)\|_{w,\hat{\Omega}} \lesssim (B\varepsilon)^{M+1} + \left\{ (N + (1 + kN^{-1})(L + k^2N^{-1}))N^{-s} + L^{-s} \right\} \left( \|F\|_{(H^{s-2}(\hat{\Omega}))^3} + \|h\|_{(H^s(\Omega))^2} \right).
\]

for any \( B > C_2 \|f\|_{H^s(\Omega)} \), where \( C_2 \) is the constant in Proposition 5.1.

6. CONCLUDING REMARKS

We summarize below some of the main contributions of this paper.

First, we considered the special case with the scatterer being a sphere:

- We reduced the Maxwell system into two sequences of decoupled one-dimensional problems by using divergence-free vector spherical harmonics. This reduction not only led to a more efficient spectral-Galerkin algorithm, but also greatly simplified its analysis.

- We derived wavenumber explicit bounds for the (continuous) Maxwell system with (exact) transparent boundary conditions, and wavenumber explicit error estimates for its spectral-Galerkin approximation.

- We derived optimal wavenumber explicit \textit{a priori} bounds and error estimates for the Helmholtz equation, which improved the results in [30].
Then, we applied the transformed field expansion (TFE) approach\cite{10} to deal with general scatterers. By using the general framework developed in \cite{33}, we derived rigorous wavenumber explicit error estimates for the complete algorithm for general scatterers. To the best of our knowledge, these are the first such estimates for Maxwell system with transparent boundary conditions.

Note that the scattering problems with transparent boundary conditions at an artificial boundary can not be dealt with the usual approach in \cite{27,9,18,20,28}, for the underlying domain is not of star shape, the method presented here provides a viable approach to deal with an important class of scattering problems with transparent boundary conditions at an artificial boundary.

Acknowledgement: The authors would like to thank Dr. Xiaodan Zhao at the National Heart Center Singapore, for earlier attempts on the error analysis.

APPENDIX A. Properties of vector spherical harmonics

We adopt the notation and normalization of spherical harmonics in Nédélec \cite{31}. Let $(r, \theta, \varphi)$ (with $\theta \in [0, \pi]$ and $\varphi \in [0, 2\pi]$) be the spherical coordinates. Then the (right-handed) orthonormal coordinate basis consists of $\{e_r, e_\theta, e_\varphi\}$. Denote by $\nabla_S$ and $\Delta_S$ the tangent gradient operator and the Laplace-Beltrami operator on $S$ (the unit spherical surface). We denote by $\{Y_{lm}^m(\theta, \varphi)\}$ the (scalar) spherical harmonics which are eigenfunctions of $\Delta_S$, and form an orthonormal basis of $L^2(S)$ with the inner product: $\langle u, v \rangle_S = \int_S u \overline{v} \, dS$.

We use the family of VSH: $\{Y_{lm}^m e_r, \nabla_S Y_{lm}^m, T_{lm}^m := \nabla_S Y_{lm}^m \times e_r\}$ in the SpherePack \cite{37} (also see \cite{30}). They are mutually orthogonal in $L^2(S)$ (for vector fields), and normalised such that

$$\langle T_{lm}^m, T_{lm}^m \rangle_S = l(l+1), \quad \langle \nabla_S Y_{lm}^m, \nabla_S Y_{lm}^m \rangle_S = l(l+1), \quad \langle Y_{lm}^m e_r, Y_{lm}^m e_r \rangle_S = 1. \quad \text{(A.1)}$$

We have

$$T_{lm}^m \times e_r = -\nabla_S Y_{lm}^m, \quad \nabla_S Y_{lm}^m \times e_r = T_{lm}^m, \quad Y_{lm}^m e_r \times e_r = 0. \quad \text{(A.2)}$$

Define the differential operators:

$$d^\pm_l = \frac{d}{dr} \pm \frac{l+1}{r}, \quad L_l = \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{l(l+1)}{r^2}, \quad \hat{\partial}_r = \frac{d}{dr} + \frac{1}{r}. \quad \text{(A.3)}$$

Let $f$ be a scalar function of $r$. The following properties can be derived from \cite{19}:

$$\text{div}(fT_{lm}^m) = 0, \quad \Delta(fT_{lm}^m) = L_l(f)T_{lm}^m, \quad \nabla \times (fT_{lm}^m) = \hat{\partial}_r f \nabla_S Y_{lm}^m + l(l+1)\frac{f}{r} Y_{lm}^m e_r, \quad \text{(A.4)}$$

$$\nabla \times (f\nabla_S Y_{lm}^m) = -\hat{\partial}_r f T_{lm}^m, \quad \nabla \times (fY_{lm}^m e_r) = \frac{f}{r} T_{lm}^m. \quad \text{(A.5)}$$

Moreover, we have

$$\text{div}(f\nabla_S Y_{lm}^m) = \frac{l(l+1)}{2l+1} (d_{l-1}^- - d_{l+2}^+) f Y_{lm}^m = -l(l+1)\frac{f}{r} Y_{lm}^m, \quad \text{(A.6)}$$

$$\text{div}(fY_{lm}^m e_r) = \frac{1}{2l+1} (ld_{l-1}^- + (l+1)d_{l+2}^+) f Y_{lm}^m = \left( \frac{d}{dr} + \frac{2}{r} \right) f Y_{lm}^m. \quad \text{(A.7)}$$

APPENDIX B. Proof of Theorem 2.1

Case (i) $\rho = \nu/\kappa \in (0, \theta_0)$. Set $\sec \beta = \kappa/\nu = \rho^{-1}$, i.e., $\cos \beta = \rho$ with $0 < \beta < \pi/2$. One verifies

$$\sin \beta = \sqrt{1 - \rho^2}, \quad \tan \beta = \frac{\sqrt{1 - \rho^2}}{\rho}, \quad \cot \beta = \frac{\rho}{\sqrt{1 - \rho^2}}, \quad 0 < \rho < \theta_0 < 1. \quad \text{(B.1)}$$

Recall the formulas (cf. \cite{11} (9.3.15-9.3.20)))

$$J_\nu(\nu \sec \beta) = \frac{2}{\pi \nu \tan \beta} (L_1 \cos \psi + M_1 \sin \psi), \quad Y_\nu(\nu \sec \beta) = \sqrt{\frac{2}{\pi \nu \tan \beta}} (L_1 \sin \psi - M_1 \cos \psi),$$

$$J'_\nu(\nu \sec \beta) = -\sqrt{\frac{2\beta}{\pi \nu}} (L_2 \cos \psi + M_2 \sin \psi), \quad Y'_\nu(\nu \sec \beta) = \sqrt{\frac{2\beta}{\pi \nu}} (L_2 \cos \psi - M_2 \sin \psi),$$

$$K_\nu(\nu \sec \beta) = \frac{2}{\pi \nu \tan \beta} (L_1 \sin \psi + M_1 \cos \psi), \quad \Psi_\nu(\nu \sec \beta) = \sqrt{\frac{2}{\pi \nu \tan \beta}} (L_1 \cos \psi - M_1 \sin \psi),$$

$$K'_\nu(\nu \sec \beta) = -\sqrt{\frac{2\beta}{\pi \nu}} (L_2 \sin \psi - M_2 \cos \psi), \quad \Psi'_\nu(\nu \sec \beta) = \sqrt{\frac{2\beta}{\pi \nu}} (L_2 \sin \psi + M_2 \cos \psi).$$
where $\psi = \nu (\tan \beta - \beta) - 1/4$, and $L_i = L_i(\nu, \beta), M_i = M_i(\nu, \beta), i = 1, 2$ are given in [1] P. 366-367. Inserting them into (2.3) leads to

$$\text{Re}(T_{i, \kappa}) = -\frac{1}{2\kappa} - \sin \beta \frac{L_1 M_2 + L_2 M_1}{L_1^2 + M_1^2}, \quad \text{Im}(T_{i, \kappa}) = \frac{\rho \tan \beta}{L_1^2 + M_1^2}. \quad (B.2)$$

We find it suffices to take the leading term of $L_i, M_i, i = 1, 2$ in [1] P. 366-367, that is,

$$L_1 \sim 1, \quad L_2 \sim 1, \quad M_1 \sim \frac{3 \cot \beta + 5 \cot^3 \beta}{24\nu}, \quad M_2 \sim \frac{9 \cot \beta + 7 \cot^3 \beta}{24\nu}. \quad (B.3)$$

By a direct calculation and using (B.1), we obtain

$$\sin \beta (L_1 M_2 + L_2 M_1) \sim \sin \beta \frac{\cot \beta + \cot^3 \beta}{2\nu} = \frac{1}{2\kappa} \frac{1}{1 - \rho^2}, \quad (B.4)$$

and

$$M_i^2 \sim \frac{3 + 5\rho^2}{192(1 - \rho^2)^2} \frac{1}{\kappa^2}, \quad \frac{1}{L_i^2 + M_i^2} \sim 1 - M_i^2 = 1 + O(\kappa^{-2}). \quad (B.5)$$

Then we obtain (2.14) from (B.2) and the above.

**Cases (ii)-(iii)** $\rho = \nu/\kappa \in [\theta_0, \theta_1] \cup (\theta_1, \theta_2)$. We adopt the asymptotic formulas [1] (9.3.23-9.3.28):

$$J_\nu(\nu + z\sqrt{\nu}) \sim \left(\frac{2}{\nu}\right)^{1/4} \text{Ai}(-\sqrt{2}z) + O(\nu^{-1}), \quad Y_\nu(\nu + z\sqrt{\nu}) \sim -\left(\frac{2}{\nu}\right)^{1/4} \text{Bi}(-\sqrt{2}z) + O(\nu^{-1}),$$

$$J'_\nu(\nu + z\sqrt{\nu}) \sim -\left(\frac{2}{\nu}\right)^{3/4} \text{Ai}'(-\sqrt{2}z) + O(\nu^{-3/4}), \quad Y'_\nu(\nu + z\sqrt{\nu}) \sim \left(\frac{2}{\nu}\right)^{3/4} \text{Bi}'(-\sqrt{2}z) + O(\nu^{-3/4}), \quad (B.6)$$

where $\text{Ai}(t)$ and $\text{Bi}(t)$ are Airy functions of the first and second kinds, respectively. Set

$$t := -\sqrt{2}z, \quad \kappa = \nu + z\sqrt{\nu} \text{ (i.e., } z = (\kappa - \nu)/\sqrt{\nu}). \quad (B.7)$$

We obtain from (2.6) and (2.3) that

$$\text{Re}(T_{i, \kappa}) \sim -\frac{1}{2\kappa} - \left(\frac{2}{\nu}\right)^{1/3} T_R(t), \quad \text{Im}(T_{i, \kappa}) \sim \frac{2}{\pi\kappa} \left(\frac{\nu}{2}\right)^{2/3} T_i(t), \quad (B.8)$$

where

$$T_R(t) = \frac{\text{Ai}(t)\text{Ai}'(t) + \text{Bi}(t)\text{Bi}'(t)}{\text{Ai}^2(t) + \text{Bi}^2(t)}, \quad T_i(t) = \frac{1}{\text{Ai}^2(t) + \text{Bi}^2(t)}. \quad (B.9)$$

Note that the Airy functions have different asymptotic behaviours for $t \leq -1$ and $-1 < t < 1$ (see, e.g., [1] 3.38). We therefore solve the equations: $t = -\sqrt{2}z = -\sqrt{2}(\kappa - \nu)/\sqrt{\nu} = \pm 1$, that is,

$$\nu + 2^{-\frac{3}{2}}\nu^{\frac{1}{2}} - \kappa = 0, \quad \nu - 2^{-\frac{3}{2}}\nu^{\frac{1}{2}} - \kappa = 0. \quad (B.10)$$

Both are cubic equations in $\nu^{\frac{1}{2}}$ with only one real root each. We find the real root of the first equation is $\kappa \theta_1$, while that of the second one is $\kappa \theta_2$, where $\theta_1$ and $\theta_2$ are given in (2.8).

(a) For $\rho \in [\theta_0, \theta_1]$ (note: $t = -\sqrt{2}z \leq -1$), we recall the asymptotic formulas (see [1] (10.4.60)))

$$\text{Ai}(t) \sim \frac{1}{\sqrt{-\pi t^{3/2}}} \left(\sin \xi - \frac{5}{72\eta} \cos \xi\right), \quad \text{Ai}'(t) \sim -\frac{t}{\pi \eta^2} \left(\cos \xi - \frac{7}{72\eta} \sin \xi\right),$$

$$\text{Bi}(t) \sim \frac{1}{\sqrt{-\pi t^{3/2}}} \left(\cos \xi + \frac{5}{72\eta} \sin \xi\right), \quad \text{Bi}'(t) \sim \frac{t}{\pi \eta^2} \left(\sin \xi + \frac{7}{72\eta} \cos \xi\right), \quad (B.11)$$

where

$$\xi = \eta + \frac{\pi}{4}, \quad \eta = \frac{2}{3}(-t)^{3/2} = \frac{2}{3}(\sqrt{2}z)^{3/2}.$$  

Thus, a direct calculation leads to

$$\text{Ai}(t)\text{Ai}'(t) + \text{Bi}(t)\text{Bi}'(t) \sim \frac{1}{6\pi \eta} = \frac{1}{4\pi}(-t)^{-3/2},$$

$$\text{Ai}^2(t) + \text{Bi}^2(t) \sim \frac{1}{\pi \sqrt{-t}} \left(1 + \left(\frac{5}{72\eta}\right)^2\right) = \frac{1}{\pi \sqrt{-t}} + O((-t)^{-7/2}). \quad (B.12)$$
Inserting them into (B.9), we obtain
\[ T_R(t) \sim -\frac{1}{4t} = \frac{1}{4\sqrt[4]{2}} \frac{\sqrt{\nu}}{\kappa - \nu}, \quad T_I(t) \sim \frac{\pi \sqrt{-t}}{1 + O((-t)^{-3})} \sim 2^{1/6} \pi \left( \frac{\kappa - \nu}{\nu^{1/3}} \right)^{1/2}. \] (B.13)

We derive from (B.8) that
\[ \text{Re}(\mathcal{T}_{l,k}) \sim -\frac{1}{2\kappa} - \frac{1}{4(\kappa - \nu)}, \quad \text{Im}(\mathcal{T}_{l,k}) \sim \frac{\nu}{\kappa^2} \left( -\frac{\kappa}{\nu} - 1 \right). \] (B.14)

This yields (2.15).

(b) For \( \rho \in (\vartheta_1, \vartheta_2) \) (note: \( |t| = \sqrt{\nu}|z| < 1 \)), we approximate \( T_R(t) \) and \( T_I(t) \) in (B.9) by their Taylor expansions at \( t = 0 \), which requires to evaluate \( \text{Ai}(m)(0) \) and \( \text{Bi}(m)(0) \) for \( m \geq 1 \). Recall that the Airy functions satisfy the Airy equation: \( w''(t) - tw(t) = 0, \ t \in \mathbb{R} \), and some special values are
\[ \text{Ai}(0) = \frac{1}{3^\frac{1}{3} \Gamma\left(\frac{2}{3}\right)}, \quad \text{Ai}'(0) = -\frac{1}{3^\frac{1}{3} \Gamma\left(\frac{2}{3}\right)}, \quad \text{Bi}(0) = \frac{1}{3^\frac{1}{3} \Gamma\left(\frac{2}{3}\right)}, \quad \text{Bi}'(0) = \frac{3^\frac{1}{3}}{\Gamma\left(\frac{2}{3}\right)}. \] (B.15)

With these and some tedious calculation, we can obtain
\[ T_R(t) = T_R(0) + T_R'(0) t + \frac{T_R''(0)}{2} t^2 + O(t^3), \quad T_I(t) = T_I(0) + T_I'(0) t + \frac{T_I''(0)}{2} t^2 + O(t^3), \]
with
\[ c_1 := T_R(0) = \frac{3^\frac{1}{3}}{\sqrt{2} \Gamma\left(\frac{2}{3}\right)} \approx 0.3645, \quad T_R'(0) = 2c_1^2, \quad T_R''(0) = 1 - 16c_1^3, \]
\[ T_I(0) = \frac{3^\frac{1}{3}}{4} \left( \Gamma\left(\frac{2}{3}\right) \right)^2 = \sqrt{3} \pi c_1, \quad T_I'(0) = -2\sqrt{3} \pi c_1^2, \quad T_I''(0) = 0. \]

Noting that \( t = -\sqrt{\nu}(\kappa - \nu)/\sqrt{\nu} \), Thus, we derive from (B.8) and (B.9) that
\[ \text{Re}(\mathcal{T}_{l,k}) \sim -\frac{1}{2\kappa} - \frac{1}{4(\kappa - \nu)}, \quad \text{Im}(\mathcal{T}_{l,k}) \sim 2^{1/3} \sqrt{3} \pi c_1 \left( \frac{\nu}{\kappa} \right)^{1/3} \left( 1 - 2c_1 t \right), \quad \text{where} \ t = -\sqrt{\nu} \frac{\kappa - \nu}{\sqrt{\nu}}. \] (B.17)

Hence, we obtain the desired estimates for this case.

**Case (iv) \( \rho = \nu/\kappa \in [\vartheta_2, \infty) \).** Set sech \( \alpha = \rho^{-1} \), i.e., \( \cosh \alpha = \rho \) with \( \alpha > 0 \). One verifies
\[ \sinh \alpha = \sqrt{\rho^2 - 1}, \quad \tanh \alpha = \sqrt{\rho^2 - 1} / \rho, \quad \Psi := \alpha - \tanh \alpha > 0. \] (B.18)

Recall the asymptotic formulas [1] (9.3.7-9.3.8):
\[ J_\nu(\nu \text{sech} \alpha) \sim \frac{e^{-\nu \Psi}}{\sqrt{2\pi \nu \tanh \alpha}} \{ 1 + O(\nu^{-1}) \}; \quad Y_\nu(\nu \text{sech} \alpha) \sim -\frac{e^{\nu \Psi}}{\sqrt{\pi / 2} \nu \tanh \alpha} \{ 1 + O(\nu^{-1}) \}. \] (B.19)

Note that by (B.18),
\[ \Psi(\rho) = \arccosh \rho - \sqrt{1 - \rho^{-2}} = \ln(\rho + \sqrt{\rho^2 - 1}) - \sqrt{\rho^2 - 1} / \rho, \quad \rho > 1, \] (B.20)
which is monotonically increasing with respect to \( \rho \). By (2.10), we have
\[ \Psi(\vartheta_2) \sim \ln(1 + \tau + \sqrt{2\tau + \tau^2}) - \sqrt{2\tau + \tau^2} / (1 + \tau) \sim \tau + \sqrt{2\tau + \tau^2} - \sqrt{2\tau + \tau^2} / (1 + \tau) \]
\[ = \tau + \tau \sqrt{2\tau + \tau^2} / (1 + \tau) \sim \tau, \quad \text{where} \ \tau := 1 / \sqrt{2} \kappa^{2/3}. \] (B.21)
Thus, we observe from (B.19) that in the range of interest, $J_\nu, J'_\nu$ decay exponentially, while $Y_\nu, Y'_\nu$ grow exponentially. By (2.3) and (B.19),

$$\text{Im}(\mathcal{T}_{l,\kappa}) = \frac{2}{\pi \kappa} J_\nu^2(\kappa) + Y_\nu^2(\kappa) \sim \frac{4\nu}{\kappa} \tanh \alpha \frac{e^{-2\nu \psi}}{4 + e^{-4\nu \psi}} \sim \sqrt{\rho^2 - 1} e^{-2\nu \psi},$$

(B.22)

which leads to the estimate of the imaginary part in (2.18). As $\text{Im}(\mathcal{T}_{l,\kappa})$ decays exponentially with respect to $l$. We derive from (2.4) that

$$\text{Re}(\mathcal{T}_{l,\kappa}) = \frac{l}{\kappa} - \frac{Y_{\nu+1}(\kappa)}{Y_\nu(\kappa)} - \text{Im}(\mathcal{T}_{l,\kappa}) \frac{J_\nu(\kappa)}{Y_\nu(\kappa)} \sim \frac{l}{\kappa} - \frac{Y_{\nu+1}(\kappa)}{Y_\nu(\kappa)}.$$

(B.23)

In order to obtain better estimate, we resort to the asymptotic approximation of the ratio (cf. [24]):

$$\frac{Y_{\nu+1}(\kappa)}{Y_\nu(\kappa)} = \frac{1 + \sqrt{1 - \rho^2}}{\rho^{-1}} \left\{ 1 - \frac{1}{2} \frac{1}{1 - \rho^2} \frac{1}{1 + O\left(\frac{1}{\nu^2}\right)} \right\},$$

(B.24)

which is valid for $\nu > \kappa$ and $\kappa \sim \nu$. In fact, as shown in [24], it is derived from the formula (B.19) with more terms. Inserting (B.24) into (B.23) leads to the estimate of the real part in (2.18).

**APPENDIX C. PROOF OF THEOREM 3.2**

**Case (i)** $\rho = \nu/\kappa \in (0, \theta_0)$. By (3.37) and (2.14),

$$\text{Re}(\mathcal{S}_{l,\kappa}) \sim \frac{\rho^2}{2\kappa} \frac{1 - \rho^2}{(1 - \rho)^3 + \kappa^2 \rho^2} \sim \frac{\rho^2}{2\kappa} \frac{1}{(1 - \rho)^2}, \quad \text{Im}(\mathcal{S}_{l,\kappa}) \sim \frac{(1 - \rho^2)^2 \sqrt{1 - \rho^2}}{(1 - \rho^2)^3 + 4\kappa^2 \rho^4} \sim \frac{1}{\sqrt{1 - \rho^2}}.$$

This leads to (3.40).

**Case (ii)** $\rho = \nu/\kappa \in [\theta_0, \theta_1]$. By (3.37) and (2.15),

$$\text{Re}(\mathcal{S}_{l,\kappa}) \sim \frac{1}{2\kappa} \left( 1 + \frac{1}{2(1 - \rho)} \right) \left( \frac{1}{4\kappa^2} \left( 1 + \frac{1}{2(1 - \rho)} \right)^2 + 2\rho(1 - \rho) \right)^{-1} \sim \frac{1}{4\rho(1 - \rho)} \left( 1 + \frac{1}{2(1 - \rho)} \right),$$

and

$$\text{Im}(\mathcal{S}_{l,\kappa}) \sim \frac{\sqrt{2\rho(1 - \rho)}}{2\pi^2 (1 + \frac{1}{2(1 - \rho)})^2 + 2\rho(1 - \rho)} \sim \frac{1}{\sqrt{2\rho(1 - \rho)}}.$$

so (3.41) follows.

**Case (iii)** $\rho = \nu/\kappa \in (\theta_1, \theta_2)$. By (3.37) and (2.16),

$$\text{Re}(\mathcal{S}_{l,\kappa}) \sim \frac{\sqrt{2/\nu} (c_1 + 2c_1 t + \frac{1}{2} (1 - 16c_1^2 t)^2) - 1/(2\kappa)}{(\sqrt{2/\nu} (c_1 + 2c_1 t + \frac{1}{2} (1 - 16c_1^2 t)^2) + 1/(2\kappa))^2} + \left( \frac{\sqrt{2/\nu} \sqrt{3c_1} (1 - 2c_1 t)}{2} \right)^2$$

$$\sim \frac{1}{c_1} \left( \frac{\nu}{2} \right)^{1/3} \left( 1 + 2c_1 t + c_2 t^2 \right) \left( 1 + 2c_1 t + c_2 t^2 \right)^2 + 3\rho^2 (1 - 2c_1 t)^2 \sim \frac{1}{c_1} \left( \frac{\nu}{2} \right)^{1/3} \frac{1 + 2c_1 t + c_2 t^2}{(1 + 2c_1 t + c_2 t^2)^2 + 3(1 - 2c_1 t)^2}$$

$$= \frac{1}{4c_1} \left( \frac{\nu}{2} \right)^{1/3} \frac{1 + 2c_1 t + c_2 t^2}{1 - 2c_1 t + (4c_1^2 + c_2^2)t^2 + c_1 c_2 t^3 + c_2^2 t^4/4}.$$

where $c_2 = (1 - 16c_1^2)/(2c_1) \approx 0.3088$. In the above, we dropped the term $-1/(2\kappa)$, and used $\rho \approx 1$.

Similarly, we can derive

$$\text{Im}(\mathcal{S}_{l,\kappa}) \sim \frac{\sqrt{3}}{4c_1} \left( \frac{\nu}{2} \right)^{1/3} \frac{1 - 2c_1 t}{1 - 2c_1 t + (4c_1^2 + c_2^2)t^2 + c_1 c_2 t^3 + c_2^2 t^4/4}.$$

Thus, we obtain (3.42).

**Case (iv)** $\rho = \nu/\kappa \in [\theta_2, \infty)$. Noticing from (2.18) that $\text{Im}(\mathcal{T}_{l,\kappa})$ is exponentially small in this range, we obtain from (3.37) and (2.18) that

$$\text{Re}(\mathcal{S}_{l,\kappa}) \sim \frac{\left( \sqrt{\rho^2 - 1} - \frac{1}{2\kappa} \left( 1 + \frac{1}{\rho^2 - 1} \right) \right)^{-1}}{\sqrt{\rho^2 - 1} \left( \frac{1}{2\kappa \sqrt{\rho^2 - 1}} \left( 1 + \frac{1}{\rho^2 - 1} \right) \right)^{-1}} \sim \frac{1}{\sqrt{\rho^2 - 1}} \left( 1 + \frac{1}{\rho^2 - 1} \right),$$

and
\[ \text{where we used} \ (1-y)^{-1} \sim 1+y, \ (1-y)^{-2} \sim 1+2y \ \text{for} \ y \approx 0. \ \text{This ends the proof.} \]

**References**


