Probability

Chapter 2, Unit 3

Objective

- We enter the world of uncertainties
- We will answer questions such as:
  - What are the chances of getting an A in the class?
  - What are the chances that it will rain tomorrow?
- To answer such questions we provide or compute a **probability** measure
- A probability is a numerical value that measures the likelihood that an event occurs.

Definitions

- A set is a collection of elements
- An empty set contains no element and it is denoted by $\varnothing$
- The universal set is the set containg everything

Definitions

- An **experiment** is any process that generates outcomes
  - Exp1: Coin toss
    - The outcomes: a Head (H) or a Tail (T)
  - Exp2: Roll a die
    - Outcomes: 1,2,3,4,5,6
- **Sample space**: is the set of all outcomes of an experiment
  - S={H,T}
  - S={1,2,3,4,5,6}
- An **outcome** or a **sample point**: is any element of the sample space
  - H is a sample point
  - 1 is a sample point

Probability requirements

- Let E_i = an experiment outcome

1. $$0 \le P(E_i) \le 1, \quad \forall i \in [1, n]$$

2. $$\sum_{i=1}^{n} P(E_i) = P(E_1) + P(E_2) + \ldots + P(E_n) = 1$$

How do we measure the probabilities?

- Three methods

1. Classical method
2. Relative frequency method
3. Subjective method

Classical method

It assumes that all outcomes are equally likely

- Exp3
  - Roll a die

    $S=\{1,2,3,4,5,6\}$

    $P(1)= P(2)= P(3)= P(4)= P(5)= P(6)$

    $P(1)+ P(2)+ P(3)+ P(4)+ P(5)+ P(6)=1$

  ➔ $P(1)= P(2)= P(3)= P(4)= P(5)=P(6)=1/6$

  - Toss a coin

    $S=\{H,T\}$

    $P(H)=P(T)$

    $P(H)+P(T)=1$

  ➔ $P((H)=P(T)=1/2$

In general, if we have *n outcomes* $P(E_i) = 1 / n$

Relative frequency method

you run an experiment, collect data and infer the probabilities from the data collected.

- Exp4:

  You want to test the market for a new product. You randomly select say 500 people and ask them what they think of the product (they like it or not). Say 200 out of the 500 said YES they like it and 300 said NO they do not like it.

  ➔ P(YES)=200/500 = 2/5

  ➔ P(NO)=300/500 = 3/5

  Note that: P(YES)< 1 and P(NO)< 1 and P(YES)+P(NO)=1

Subjective method

used when

1. little information is available
2. Outcomes are not equally likely

So you have to rely on intuition or experience

- Exp5: Purchase of a house

How do we measure the probabilities?

- Generally the best probability estimates are obtained by combining the estimates from the classical or relative frequency methods with the objective probability estimates.

Event

- **Definition:** an event is a collection of sample points
  - Exp6: Roll a die
    - $S=\{1,2,3,4,5,6\}$
    - We are interested in the event A that all outcomes are even
      - A={2,4,6}

    - If we ran the experiment and all outcomes are even we say that event A has occurred

$$P(A)=\frac{\textit{number} \text{ of elements in the set of event A}}{\textit{number} \text{ of elements in the sample space S}}$$

    - P(A)=3/6

      = P(2)+P(4)+P(6)

      1/6+1/6+1/6 =1/2

Complement of an event

- The complement of an event *A* is an event consisting of all sample points that are not in *A*. It is denoted as A^c (or $\overline{A}$)

- Complement of an event must satisfy the following:

$$P(A) + P(\overline{A}) = 1 \quad or \quad P(A) = 1 - P(\overline{A})$$

- Exp7: If we know that the 0.9 probability that items produced are not defective then we can conclude that the probability that an item produced is defective is equal to

  1-0.9=0.1

Addition (Union) Law

- Addition Law is used when we want to compute the probability that at least one of two event (event *A* or event *B* or both) will occur.
- This is translated into computing $P(A \cup B)$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Exp8

- Students are tested twice (exam1, exam2). You conduct a survey and find that among the 500 students questioned:
  - 250 students passed exam 1
  - 400 students passed exam 2
  - 200 students passed both exams

- Compute the prob. of giving a passing grade to students who passed at least one test?

$$P(A \cap B) = 200/500 = 0.4$$

Exp8

- Let A be the event: students passed exam 1
- Let B be the event: students passed exam 2
  - *P(A)=250/500 = 0.5*
  - *P(B)=400/500 = 0.8*

- Let C= event of giving a passing grade to students who passed at least one test ➔ $C = A \cup B$

$$P(C) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cap B) = 200 / 500 = 0.4$$

$$P(C) = P(A) + P(B) - P(A \cap B)$$
$$= 0.5 + 0.8 - 0.4$$
$$= 0.9$$

➔ There is a 90% chance that a student passed the course when he/she only passed one exam

Mutually Exclusive Events

- Events A and B are mutually exclusive then

$$P(A \cap B) = 0$$

$$A \cap B = \varnothing$$

  - $P(A \cup B) = P(A) + P(B)$
  - $P(A \cup \overline{A}) = P(A) + P(\overline{A})$

$$= P(S)$$

$$= 1$$

Conditional Probability

- Is a measure of the probability that an event occurs is dependent on the information you have
- The conditional probability of event A occurring given the occurrence of event B is:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \text{ assuming } P(B) \neq 0$$

Conditional Probability

- Exp 9: Country X is pushing to start 100 new projects in computer development and telecommunications. Two US giants IBM and AT&T have signed contracts for these projects, 40 (30 projects in computer development and 10 projects in telecommunications) projects for IBM and 60 (40 projects in computer development and 20 projects in telecommunications) for AT&T. Given that a randomly chosen project is in telecommunications, what is the probability that it is undertaken by IBM?

Conditional Probability

- Exp 9: Country X is pushing to start 100 new projects in computer development and telecommunications. Two US giants IBM and AT&T have signed contracts for these projects, 40 (30 projects in computer development and 10 projects in telecommunications) projects for IBM and 60 (40 projects in computer development and 20 projects in telecom.) for AT&T. Given that a randomly chosen project is in telecommunications, what is the probability that it is undertaken by IBM?

- ANS.
  - Let IBM: be the event that an IBM project is chosen
  - Let T: be the event that a telecommunication project is chosen
  - $P(IBM \cap T) = 10/100$

    $P(T) = 30/100$
  - $$P(IBM \mid T) = \frac{P(IBM \cap T)}{P(T)} = \frac{10/100}{30/100} = 1/3$$

Conditional Probability

- Variation of the conditional probability of event A occurring given the occurrence of event B:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A \cap B) = P(A \mid B)P(B)$$

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

$$P(A \cap B) = P(B \mid A)P(A)$$

Independence of events

- Events A and B are independent of each others if

$$P(A \mid B) = P(A)$$
$$P(B \mid A) = P(B)$$

The most useful

$$P(A \cap B) = P(A)P(B)$$

The joint probability of A and B

Independence of events

- Exp: The probability that a consumer will see the advertisement of a product on TV is 0.04

  The probability that a consumer will hear the advertisement of a product on the radio is 0.02

  The two events are assumed to be independent.

1. What is the prob. that the consumer will be exposed to the adv. on TV and on the radio.
2. What is the prob. that the consumer will be exposed to at least one of the ads.

Independence of events

- ANS. Let A be the event that a consumer sees the advertisement of a product on TV

  Let B be the event that a consumer sees the advertisement of a product on the radio

1. Since A and B are independent then

$$P(A \cap B) = P(A)P(B) = 0.04 * 0.02 = 0.0008$$

2. the prob. that the consumer will be exposed to at least one of the ads. is by definition the probability of the union of A and B

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= 0.04 + 0.02 - 0.0008$$
$$= 0.0592$$

Law of total probability

- Consider 2 events A and B, we can always compute:

$$P(A) = P(A \cap B) + P(A \cap \overline{B})$$

- In general if *B* is partitioned in *n* events

$$P(A) = \sum_{i=1}^{n} P(A \cap B_i)$$

Law of total probability

- Exp: Let A be the event that a picture card is drawn from the set of 52 cards. The picture cards are aces, kings, queens and jacks.

  Let H= be the event that a heart is drawn

  Let C= be the event that a club is drawn

  Let D= be the event that a diamond is drawn

  Let S= be the event that a spade is drawn

The probability of a picture card is drawn is

$$\begin{aligned} P(A) &= P(A \cap H) + P(A \cap C) + P(A \cap D) + P(A \cap S) \\ &= 4/52 + 4/52 + 4/52 + 4/52 \\ &= 16/52 \end{aligned}$$

Law of total probability

- The law of total probability using conditional probability
  - 2 set case

$$P(A) = P(A \mid B)P(B) + P(A \mid \overline{B})P(\overline{B})$$

Law of total probability

- Exp. An analyst believes the stock market has a 0.75 chance of going up if the economy does well and a 0.3 chances of going up if the economy does not do well in the next year. He also believes that there is a 0.8 chance that the economy will do well in the coming year. What is the prob. That the stock market will go up next year?
- ANS.

Let U= event that the stock market will go up

Let W= event that the economy will do well

$$P(U) = P(U \mid W)P(W) + P(U \mid \overline{W})P(\overline{W})$$

$$= 0.75 * 0.8 + 0.3 * 0.2 = 0.66$$

Law of total probability

- The law of total probability using conditional probability
  - Case where there are *n* sets in the partition B

$$P(A) = \sum_{i=1}^{n} P(A \mid B_i) P(B_i)$$

Bayes Theorem

- Allows us to compute the probability of B given A from the probability of A given B

$$P(B \mid A) = \frac{P(A \mid B)P(B)}{P(A \mid B)P(B) + P(A \mid \overline{B})P(\overline{B})}$$

$P(B)$ and $P(\overline{B})$ are called prior probabilities of the event B

P(B|A) is called the posterior probability of B

Bayes Theorem

- Consider a test for illness:
  1. When an ill person is tested, the test will indicate so with a prob. Of 0.92
  2. When a non ill person is tested, the test will indicate so with a prob. Of 0.04

  The illness is supposed to affect only 0.1% of the population. If a person is randomly selected, tested and the result is positive, what is the posterior prob. (posterior to the test result) that the person is ill (Prob(a person is ill given that the test result is >0))?

Bayes Theorem

ANS.

Let Z= event that the test result is positive

Let I= event that the person is ill

Then

$$P(I) = 0.001,\ P(\bar{I}) = 1 - 0.001 = 0.999,$$

$$P(Z \mid I) = 0.92,\ P(Z \mid \bar{I}) = 0.04$$

We need to compute P(I|Z). But we have P(Z|I). So using Bayes rule

$$P(I \mid Z) = \frac{P(Z \mid I)P(I)}{P(Z \mid I)P(I) + P(Z \mid \bar{I})P(\bar{I})}$$

$$= \frac{(0.92)(0.001)}{(0.92)(0.001) + (0.04)(0.999)}$$

$$= 0.0225$$

Extended Bayes Theorem

- The probability of one of the sets j in partition B

$$P(B_j \mid A) = \frac{P(A \mid B_j)P(B_j)}{\sum_{i=1}^{n} P(A \mid B_j)P(B_j)}, \quad j = 1, 2, \ldots.., n$$

Definitions

- **Random variable**:
  - is an uncertain quantity whose values depends on chance.
  - It is a numerical description of the outcome of an experiment
  - It is defined over the sample space
- Exp. Toss a coin
  - Define X: a random variable that takes
    - 0 if the outcome is head
    - 1 if the outcome is tail
- A random variable can take a discrete value ➔ a discrete random variable
- A random variable can take a continuous value ➔ a continuous random variable

Probability distribution

- A r. v. has a probability law: a rule that assigns probabilities to the different values of the r. v.
- The probability law (probability assignment) is called probability distribution of the r. v.
- A r. v. is usually denoted by a capital letter ‘X’
- The probability distribution is then denoted by P(X).
- We denote by small letter the values that the r.v. take on P(X=x) or P(x)

- Exp.

Sample Space	**r. v.**
H	X=0
T	X=1

P(X=0)=P(0)=1/2

P(X=1)=P(1)=1/2

Probability distribution

- Exp.

Sales values	Nb. Of days
no sales	54
1 car	117
2	72
3	42
4	12
5	3
	Total=300

Probability distribution

- Exp. X=number of cars sold

x	**P(x)** (using frequency method)
0	54/300=0.18
1	117/300=0.39
2	72/300=0.24
3	42/300=0.14
4	12/300=0.04
5	3/300=0.01
	Total=1.0

Discrete random variable

Discrete random variables

- A discrete random variable can assume at most a countable number of values

- The probability distribution function (pdf) of a discrete random variable X must satisfy the following 2 conditions

1. $P(x) \geq 0$ for all values x
2. $\sum_{all\ x} P(x) = 1$

Discrete random variables

- Cumulative distribution function (cdf) F(x) of a r. v. x is the probability that the value of the r.v. is at most some value x.

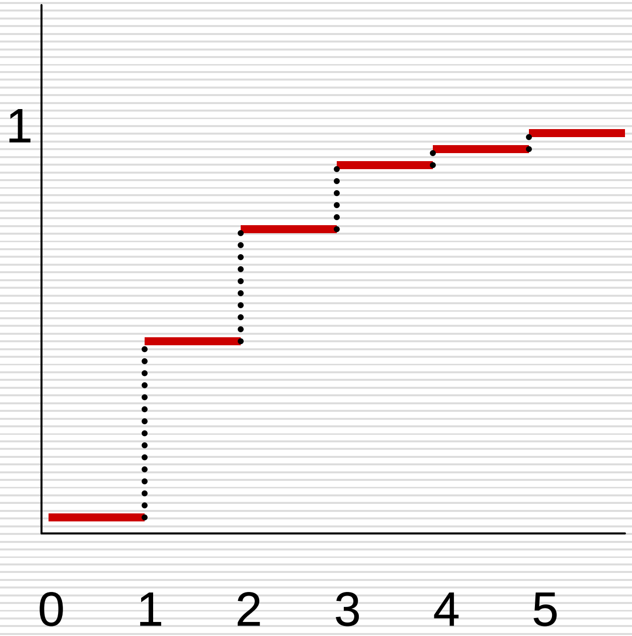
$$F(x) = P(X \leq x) = \sum_{all \ \mathrm{i} \leq \mathrm{x}} \mathrm{P(i)}$$

Cumulative distribution function

- Exp. X=number of cars sold

x	P(x)	F(x)=P(X<=x)
0	0.18	0.18
1	0.39	0.57
2	0.24	0.81
3	0.14	0.95
4	0.04	0.99
5	0.01	1.00

Total=1.0



Expected Value

- The expected value of a discrete random variable X is

$$\mu = E(X) = \sum_{all\ x} xP(x)$$

- It is a measure of the centrality of the pdf
- It is the value we expected the random variable to take on average

Expected Value

- Exp. X=number of cars sold

x	P(x)	F(x)	E(X)
0	0.18	0.18	0
1	0.39	0.57	0.39
2	0.24	0.81	0.48
3	0.14	0.95	0.42
4	0.04	0.99	0.16
5	0.01	1.00	0.05
	Total=1.0		Sum=1.50

➔ E(X)=1.5 ➔the dealer is expected to sell 1.5 cars/day

Variance & Standard deviation

- The variance of a discrete random variable X is

$$\sigma^2 = V(X) = E\left[(X-\mu)^2\right] = \sum_{all\ x} (x-\mu)^2 P(x)$$

- Computation formula

$$\sigma^2 = V(X) = E\left[X^2\right] - \left[E(X)\right]^2$$

- It is a measure of the variability and dispersion of the r. v.

Variance & Standard deviation

- Exp. X=number of cars sold

x	P(x)	E(X)	$E(X^2)$
0	0.18	0	0
1	0.39	0.39	0.39
2	0.24	0.48	0.96
3	0.14	0.42	1.26
4	0.04	0.16	0.64
5	0.01	0.05	0.25
		Sum=1.5	Sum=3.5

➔ $\sigma^2 = V(X) = E\left[X^2\right] - \left[E(X)\right]^2$ = 3.5 - 1.5^2 = 1.25

Variance & Standard deviation

- The standard deviation of a discrete random variable X is

$$SD(X) = \sigma = \sqrt{V(X)}$$

Variance & Standard deviation

- Exp. X=number of cars sold

x	**P(x)**	**E(X)**	$E(X^2)$
0	0.18	0	0
1	0.39	0.39	0.39
2	0.24	0.48	0.96
3	0.14	0.42	1.26
4	0.04	0.16	0.64
5	0.01	0.05	0.25
		Sum=1.5	Sum=3.5

➔ $\sigma^2 = V(X) = 3.5 - 1.5^2 = 1.25$

$$SD(X) = \sqrt{1.25} = 1.118$$

➔ 1.118 cars will be sold per day. Easier interpretation than using the variance

Binomial Probability Distribution

- Experiments that satisfy the following conditions are called binomial experiments

1. The experiment consists of *n* trials
2. Each trial has 2 possible outcomes called *success* or *failure*
3. *The probability of success is p* and it remains constant from trial to trial. The probability of a failure is $q=1-p$
4. The *n* trials are independent. The outcome of one trial does not affect the outcome of other trials.

Binomial Probability Distribution

- A r.v. X that counts the number of successes in n trials, where p is the probability of success in any given trial, is said to follow the binomial distribution with parameters n and p.

$$X \sim Bin(n,p)$$

- We call X a binomial r.v.

Binomial Probability Distribution

- The binomial probability distribution P(x) associated with X is

$$P(x) = \binom{n}{x} p^x q^{n-x} = \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

- x= the number of successes
- n = total number of trials
- p = the probability of success in a single trial
- q=1-p

Binomial Probability Distribution

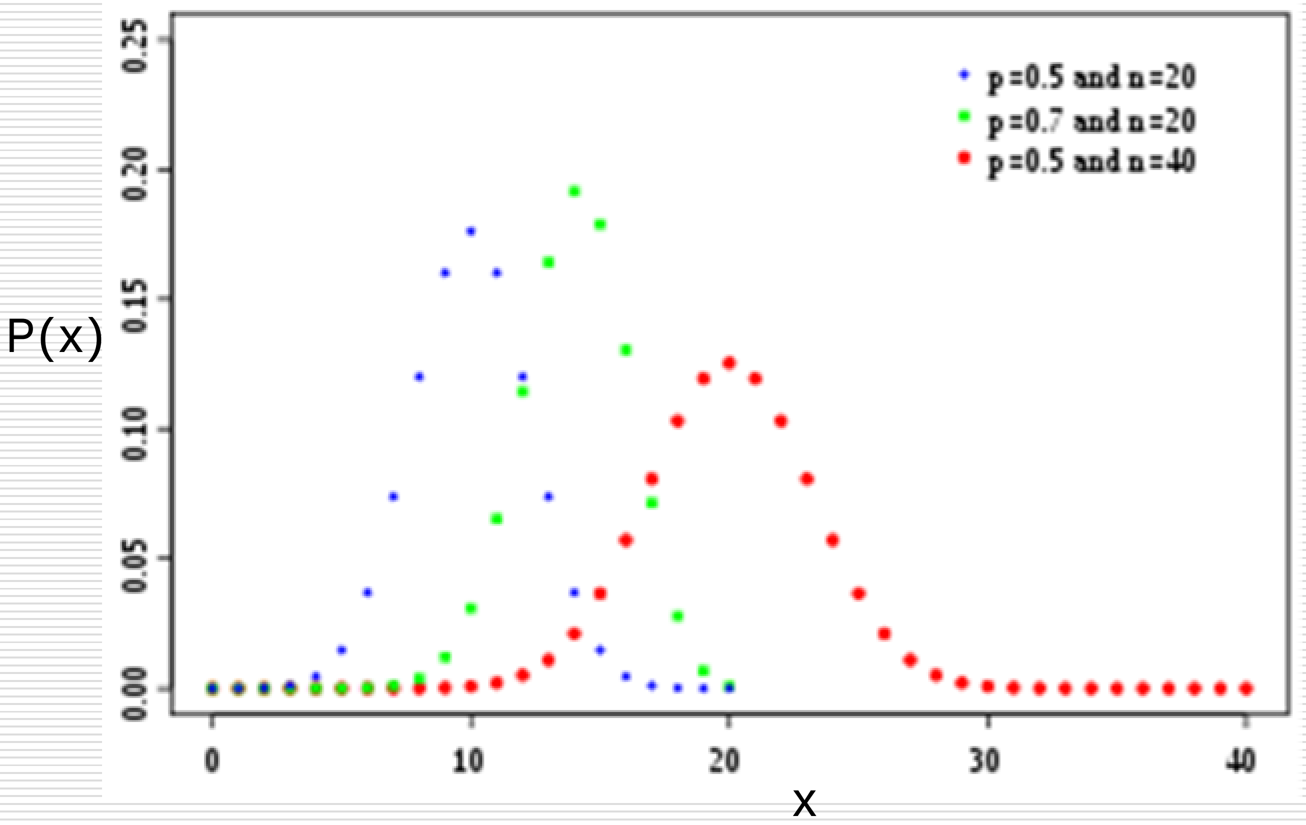
- Exp. Toss a coin. Assume we have a biased coin and P(H)=1/3 and P(T)=2/3.
- Define a success as getting a head.
- What is the prob. Of having 5 successes in 10 trials.

- ANS. p=1/3, q=2/3, n=10, x=5

$$P(5) = \binom{10}{5}(1/3)^5(2/3)^{10-5}$$

$$= \frac{10!}{5!(10-5)!}(1/3)^5(2/3)^5 = 0.0097$$

Graphical Representation



Binomial Probability Distribution

- The mean

$$\mu = E(X) = np$$

- The variance

$$\sigma^2 = V(X) = npq$$

- The standard deviation

$$\sigma = SD(X) = \sqrt{npq}$$

Binomial Probability Distribution

- Exp. Toss a coin. Assume we have a biased coin and P(H)=1/3 and P(T)=2/3.
- Define a success as getting a head.
- Compute the E(X), V(X) and SD(X) where the total number of trials is 10.

ANS. P=1/3, q=2/3, n=10, x=5

- E(X)=10(1/3)=10/3
- V(X)=10(1/3)(2/3)=20/9
- $SD(X) = \sqrt{20/9} = \frac{2\sqrt{5}}{3}$

Binomial Probability Distribution

- Cumulative distribution function (cdf)
- Exp. p=q=1/2, n=5

x	$P(x)=\binom{5}{x}(0.5)^x(0.5)^{5-x}$	$F(x)=P(X \le x)=\sum_{all\ i \le x} P(i)$
0	0.031	0.031
1	0.156	0.187
2	0.313	0.500
3	0.313	0.813
4	0.156	0.969
5	0.031	1.000

$$P(x=3)=F(3)-F(2)=\sum_{i=0}^{3}P(i)-\sum_{i=0}^{2}P(i)$$

Poisson Distribution

- Used in Operations research, probability modeling
- It useful in describing a r.v. that counts the number of occurrences in a given time interval
  - Number of machine failure per week
  - The number of customers in a queue per day (in a bank, hospital,...)
  - Number of accidents per year

Poisson Distribution

- pdf.

$$P(x) = \frac{\mu^x e^{-\mu}}{x!}, \quad \text{for } x = 0,1,2,\ldots$$

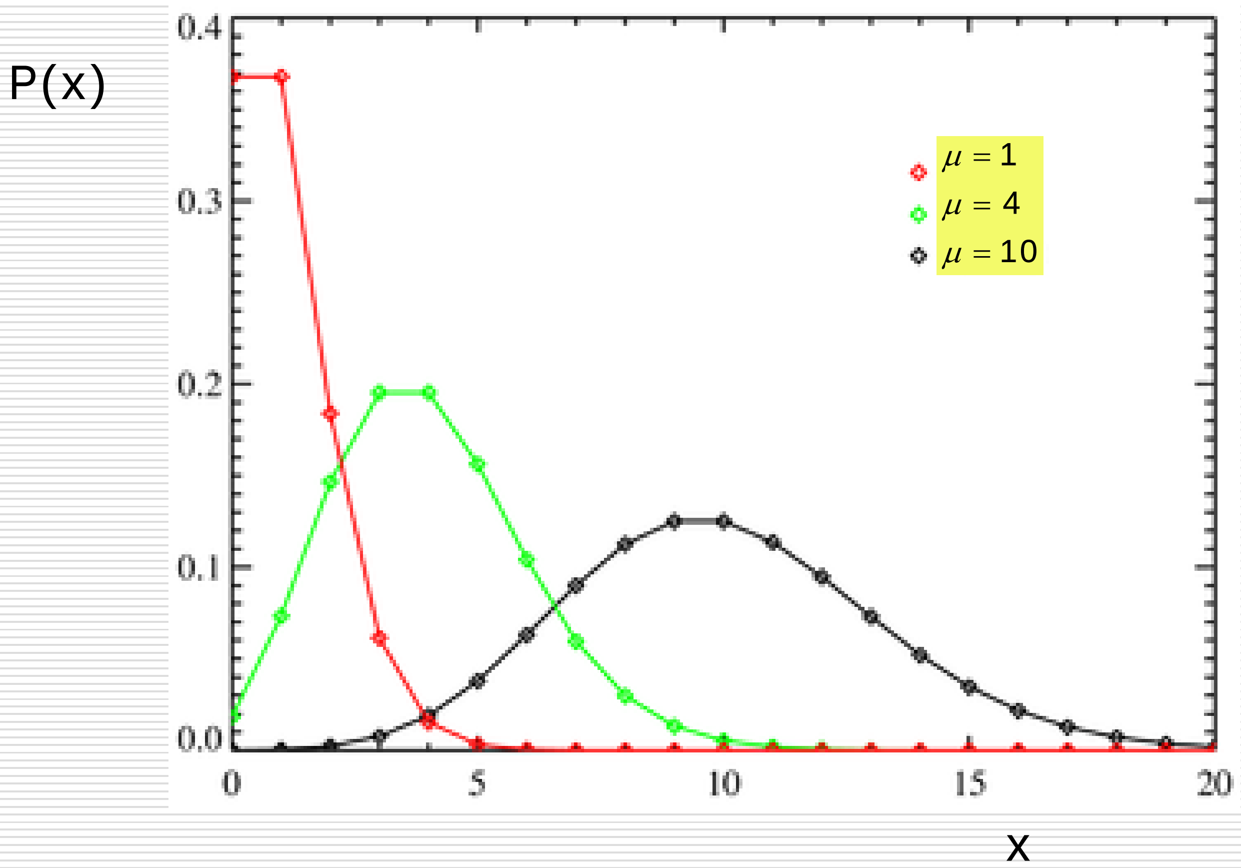
where x is the number of occurrences in a given time interval

μ is the mean or average number of occurrences of the distribution in a given time interval

μ is also the variance of the distribution

e (=2.7118..) is the base of the natural logarithm

Graphical Representation



Poisson Distribution

- If the following two assumptions hold, the Poisson distribution is applicable
  1. In a very small interval, the probability that two events will occur is close to zero
  2. The probability of any number of events occurring (or non-occurring) over a given interval is independent of the number of events that occurred (or not occurred) prior to the interval

Poisson Distribution

- Exp. We are interested in the number of arrivals at a bank during the time frame from 8:00 am to 8:15 am. Assume that
  1. In a very small interval, the probability that two arrivals is close to zero
  2. The probability of any number of arrivals (or non-arrivals) over a given interval is independent of the number of arrivals (or non-arrivals) that occurred prior to the interval

→ The Poisson distribution is applicable

Poisson Distribution

- Assume that the number of arrivals in a period of 15 minutes is 10 customers

➔ μ = 10 customers/15 minutes

$$P(x) = \frac{10^x e^{-10}}{x!}, \quad \text{for } x=0,1,2,\ldots$$

$$P(x = 5) = \frac{10^5 e^{-10}}{5!}$$

Poisson Distribution

- It is a good approximation of the binomial distribution for large *n (n >19)* and small *p (p <= .05)*
- *In the case of approximating the binomial by the Poisson distribution*

$$\mu = np$$

Continous random variable

Continuous random variables

- A continous r.v. may take on any value in an interval of numbers (1, 2, 2.1231, 1.301,...)
- The probability distribution function *f(x)* of a continuous random variable X has the following 3 properties

1. $f(x) \geq 0$ for all x
2. The probability that X will be bewteen two numbers a and b is equal to the area under f(x) between a and b
3. The total area under the entire curve of f(x) is equal to 1

Continuous random variables

- P(a<X<b) = Area under f(x) between a and b.
- But the Area

$$Area = \int_a^b f(x)dx$$

- The cumulative distribution function of a continuous r.v.

$F(x) = P(X \leq x) = Area$ under f(x) between the smallest possible value of X and x

$$Area = \int_a^b f(x)dx = F(b) - F(a)$$

Continuous random variables

- Remark

$$P(X = x) = 0$$

$$P(X \le x) = F(x)$$

$$P(X \ge x) = 1 - P(X \le x)$$
$$= 1 - F(x)$$

Expected Value & Variance

- The expected value of a continuous random variable X is

$$\mu = E(X) = \int_{-\infty}^{+\infty} x f(x)$$

- The variance of a continuous random variable X is

$$\sigma^2 = V(X) = E\left[X^2\right] - \left[E(X)\right]^2$$

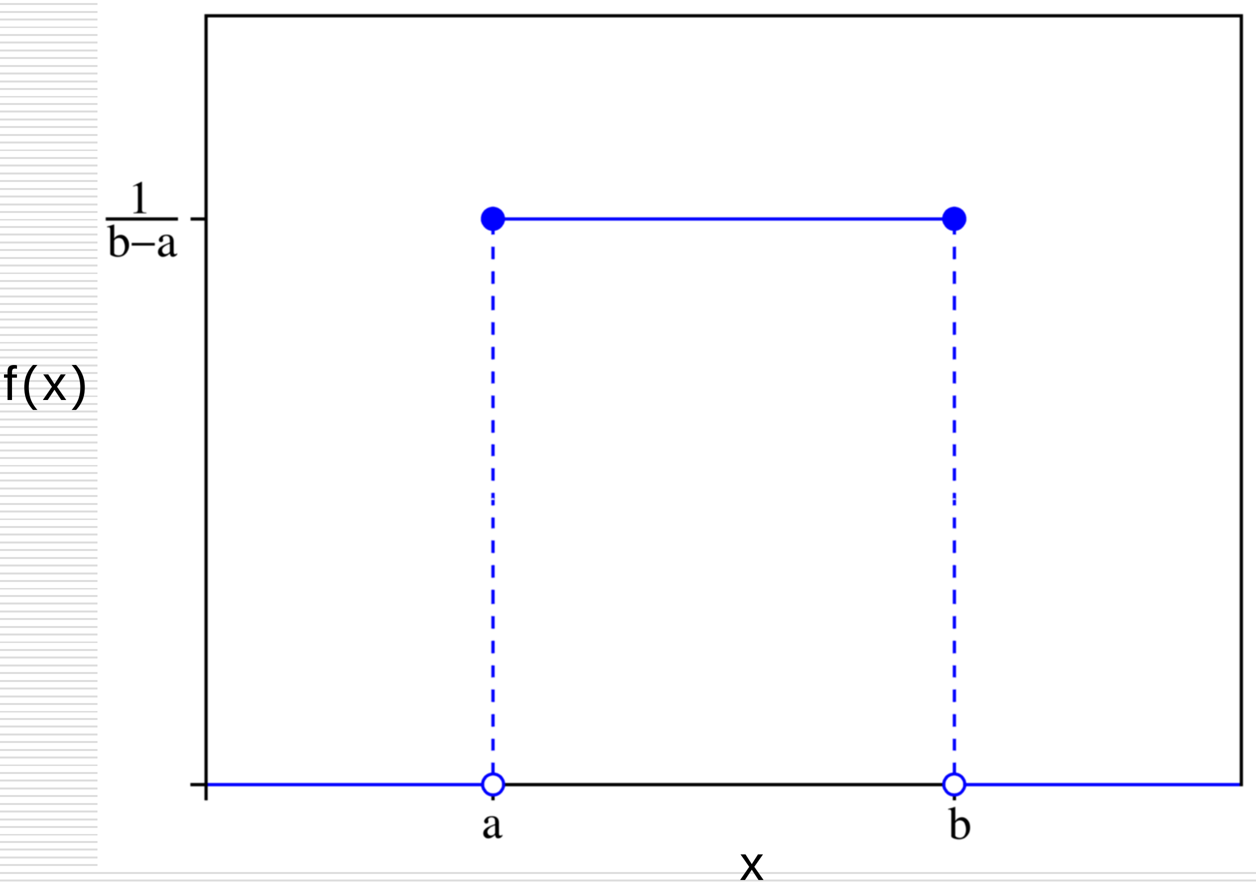
$$E(X^2) = \int_{-\infty}^{+\infty} x^2 f(x)$$

Uniform Distribution

- A r.v. has the continuous uniform distribution over an interval I=[a b] if it equally likely to be in any subinterval of I as in any subinterval of the same length.
- The probability density function of a uniform variable over I=[a b] is

$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{for } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

Graphical Representation



Exponential Distribution

- The r. v. that measures the time between 2 occurrences that have a Poisson distribution is an exponential r. v.
- The density function

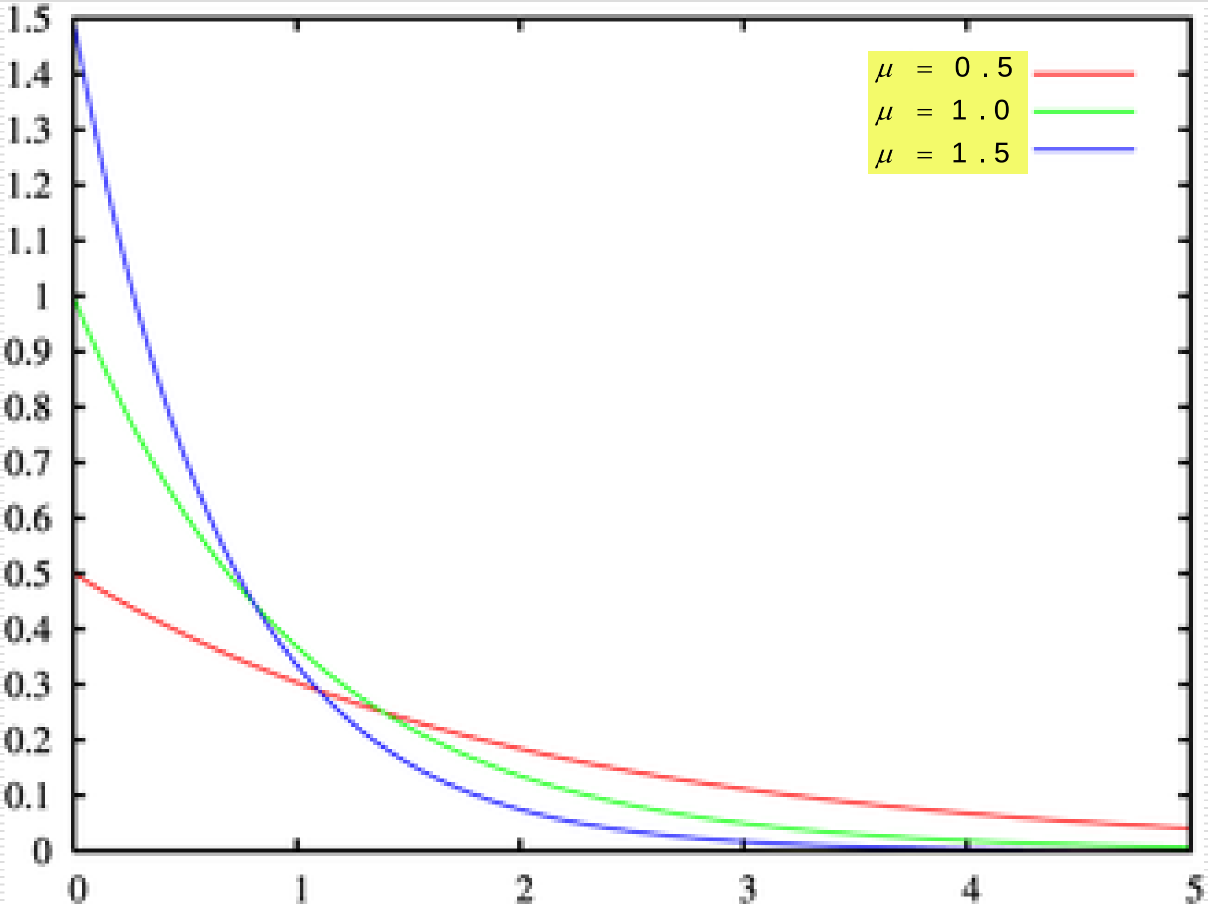
$$f(x) = \mu e^{-\mu x} \quad \text{for } x \geq 0 \text{ and } \mu > 0$$

- The mean = standard deviation = $1 / \mu$

- The cumulative distribution

$$F(x) = 1 - e^{-\mu x} \quad \text{for } x \geq 0$$

Graphical Representation



Exponential Distribution

- Exp. The time between breakdown of a machine is known to have an exponential distribution with parameter 2. Time is measured in hours.

1. What is the probability that the machine will work continuously for at least 1 hour?
2. What is the average time between breakdowns?

- ANS.

1. We have to compute the area under the curve of f(x) to the right of x=1. That is we need to compute P(X>=1).

$$P(X \geq 1) = 1 - P(X \leq 1) = 1 - F(1)$$

$$= 1-(1-e^{-2(1)})$$

$$= e^{-2} = 0.1353$$

2. The average time between breakdowns is=

$$1/\mu = 1/2 \ hours$$

Normal distribution

- The normal distribution can be used to describe, at least approximately, any variable that tends to cluster around the mean.
  - For example, the heights of adult males in the United States are roughly normally distributed, with a mean of about 70 in (1.8 m).
  - By the central limit theorem, the sum of a large number of independent random variables is distributed approximately normally.
  - For this reason, the normal distribution is used throughout statistics, natural science, and social science as a simple model for complex phenomena. For example, the observational error in an experiment is usually assumed to follow a normal distribution, and the propagation of uncertainty is computed using this assumption.
  - It is the limit of the Binomial distribution

Normal distribution

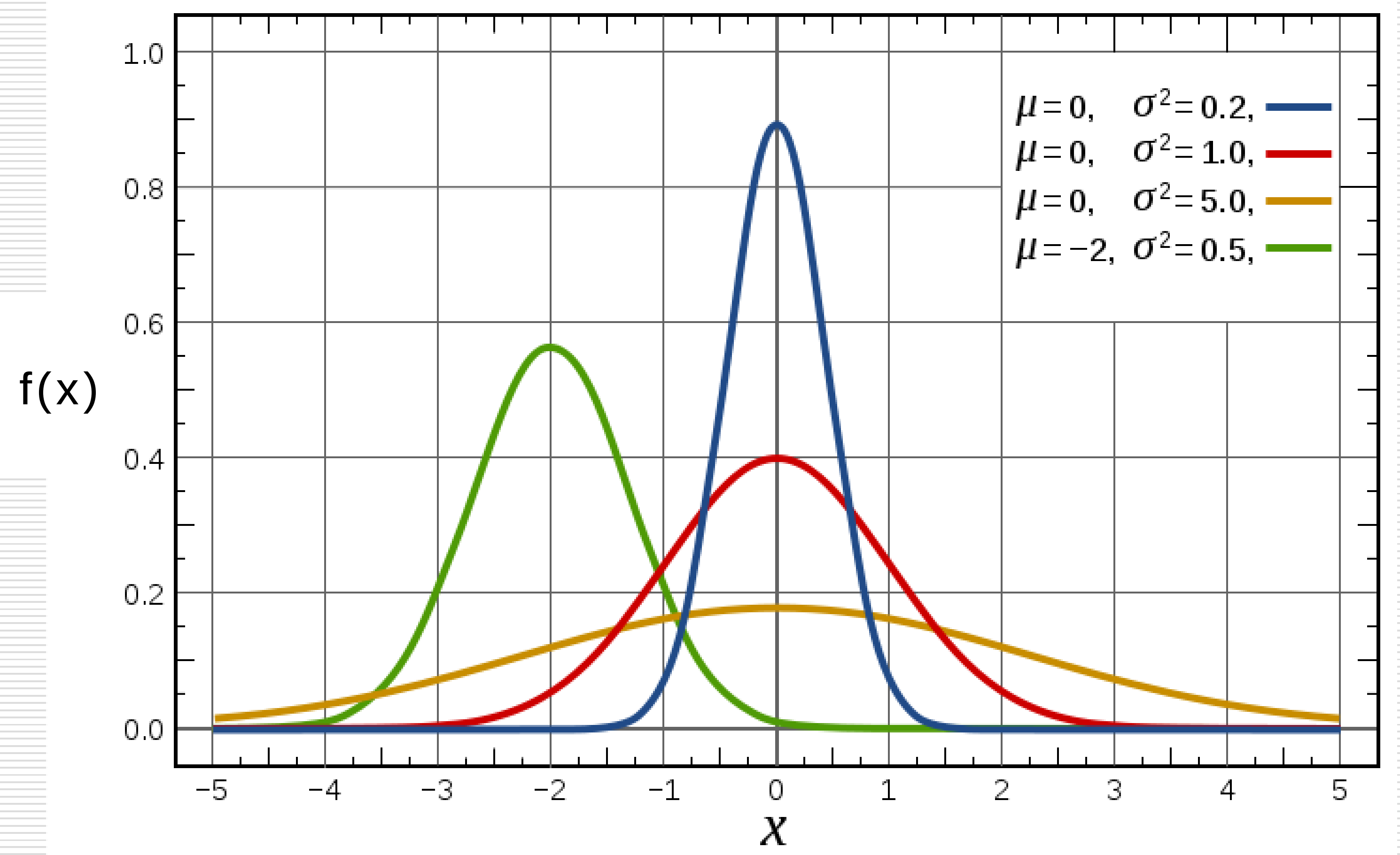
- The probability density function of a normally distributed r.v. X ---

$$X \sim N(\mu, \sigma^2)$$

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)} \quad \text{for } -\infty < x < +\infty$$

where e=2.718… and π=3.141…

Graphical Representation



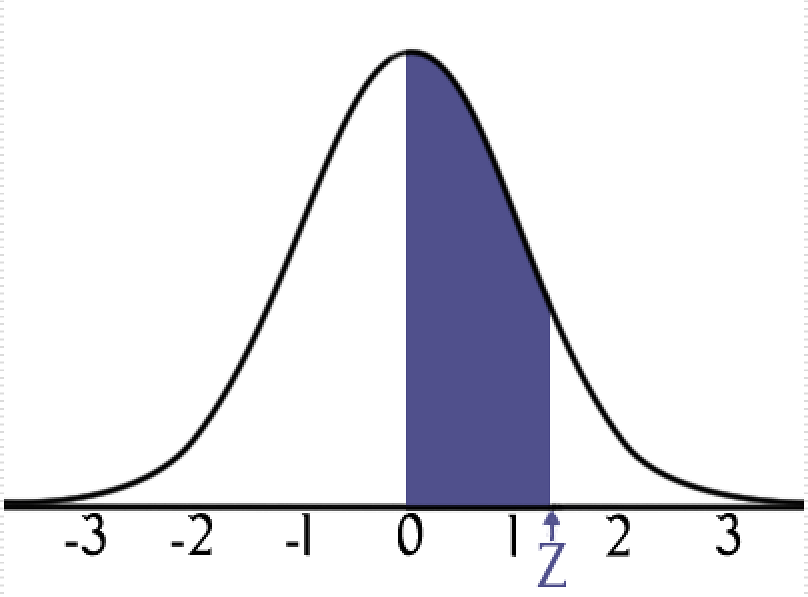
Standard Normal distribution

- The probability density function of a normally distributed r.v. Z ---

$$Z \sim N(0,1)$$

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \quad \text{for } -\infty < z < +\infty$$

There are standard normal probability tables



Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	P(Z<z)
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	**0.1736**	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.7	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2967	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990	0.4990

Standard Normal distribution

- The table in previous slide can be used to find the area under the curve from the central line to any "Z-value" value up to 3, in steps of 0.1

- This will then tell you what portion of the population are within "Z" standard deviations of the mean.

- For example, to determine the area under the curve between 0 and 0.45, start at the row for 0.4, and read along until 0.45 - there is the value 0.1736

- Because the curve is symmetrical, the same table can be used for values going either direction, so a negative 0.45 also has an area of 0.1736.

- The area to the left of -0.45 is the same as the area to the right of 0.45 ➔

  $$P(Z<-0.45)=P(Z>0.45)=0.5-P(Z<0.45)$$

Standard Normal distribution

- $P(Z<-2.47) = P(Z>2.47)$
  $= 0.5 - P(Z<2.47)$
  $= 0.5-0.4932=0.0068$

- $P(1<Z<2)=P(Z<2)-P(Z<1)=0.4772-0.3413=0.1359$

- Exp. Normal distributions can be transformed to standard normal distributions as follows:

$$X \sim N(2, 5^2) \text{ then } Z = \frac{X-\mu}{\sigma} \sim N(0,1)$$

$$Z = \frac{X-\mu}{\sigma} = \frac{X-2}{5}$$

$P(X>4.5) = P(Z>(4.5-2)/5) = P(Z>0.5)$
$= 0.5-P(Z<0.5) = 0.5 - 0.1915=0.3085$