Review of Basic Mathematics

This notes review prerequisite math for MGT 201.

1 Summation

Throughout this course we will be concerned with computations that involve potentially large data sets. As a result, it is useful to develop notation designed to express these lengthy computations in a precise but succinct way. On such notation is called sigma ($\sum$) notation.

Sigma notation is useful to indicate a sum. Frequently you will see statements like

$$\sum_{i=1}^{3} (\text{expression which depends on } i)$$

The variable i is termed the *variable or index of summation*. In this case, i is to begin with the value 1 and we will terminate the summation after $i = 3$. The basic idea is simple:

1. Substitute the current value of i into the expression which depends upon i;
2. Increase i by 1 and return to step (1).

Examples

(1)

$$\sum_{i=1}^{3} i = 1 + 2 + 3 = 6$$

(2)

$$\begin{aligned}\sum_{i=1}^{3}(2i-1) &= [2(1)-1] + [2(2)-1] + [2(3)-1] \\ &= 1 + 3 + 5 \\ &= 9\end{aligned}$$

(3)

$$\sum_{j=1}^{2} \frac{2^j}{j!} =$$

(4)

$$\sum_{k=1}^{3} x_k =$$

Note: If x_k represents the value of the k-th data point in a data set, then (4) represents the sum of the first three data points. $j!$ is the *factorial* of j: $j! = 1 \times 2 \times \cdots \times j$.

1.1 Rules for Using Sigma $\sum$ Notation

Rule 1: For any constant c

$$\sum_{i=1}^{n} c = nc \qquad (1)$$

Proof.

$$\begin{aligned}\sum_{i=1}^{n} c &= \underbrace{c + c + \cdots + c}_{n} \\ &= \end{aligned}$$

□

Examples:

(a) $\sum_{s=1}^{3} 6 = (6 + 6 + 6) = 3(6) = 18$

(b) $\sum_{y=2}^{4} 3a =$

Rule 2:

$$\sum_{i=1}^{n} cx_i = c\sum_{i=1}^{n} x_i \qquad (2)$$

Proof.

$$\begin{aligned}\sum_{i=1}^{n} cx_i &= cx_1 + cx_2 + \cdots + cx_n \\ &= \\ &= c\sum_{i=1}^{n} x_i\end{aligned}$$

□

Examples:

(a) $\sum_{k=2}^{5} (3x_k) =$

(b) $\sum_{y=1}^{4} x^2 y =$

Rule 3:

$$\sum_{i=1}^{n}(x_i + y_i) = \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i \tag{3}$$

Proof.

$$\begin{aligned}\sum_{i=1}^{n}(x_i + y_i) &= (x_1 + y_1) + (x_2 + y_2) + \cdots + (x_n + y_n) \\ &= \\ &= \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i\end{aligned}$$

□

Rule 4:

$$\sum_{i=1}^{n}\sum_{j=1}^{m} x_{ij} = \sum_{i=1}^{n}(x_{i1} + x_{i2} + \cdots + x_{im}) \tag{4}$$

$$= (x_{11} + x_{12} + \cdots + x_{1m}) \tag{5}$$

$$+ (x_{21} + x_{22} + \cdots + x_{2m}) \tag{6}$$

$$+ \ldots \tag{7}$$

$$+ (x_{n1} + x_{n2} + \cdots + x_{nm}). \tag{8}$$

For example, for the matrix

$$X = (x_{ij}) = \begin{pmatrix} 11 & 12 & 13 \\ 21 & 22 & 23 \\ 31 & 32 & 33 \end{pmatrix}, \tag{9}$$

we have

$$\sum_{i=1}^{3}\sum_{j=2}^{2} x_{ij} = (12) + (22) + (32) =$$

$$\sum_{i=2}^{3}\sum_{j=1}^{2} x_{ij} = (21 + 22) + (31 + 32) =$$

$$\sum_{i=1}^{2}\sum_{j=2}^{3} x_{ij} = (12 + 13) + (22 + 23) =$$

2 Integration

It should come as no surprise that we will be extremely interested in computing probabilities in this class. One elegant way of obtaining a probability is to determine the area under a carefully chosen curve or function. Fortunately, it is frequently easy to find the area under a curve by integration.

Suppose we wish to find the area under $f(x) = x^2$, for $x \in [0, 1]$. We can use the fact that the area under a curve of the form $f(x) = x^n$ is of the form:

$$\text{Area} = \int x^n dx = \frac{1}{n+1} x^{n+1} \tag{10}$$

So

$$\begin{aligned}\int_0^1 x^2 dx &= \frac{1}{3} x^3 |_0^1 \\ &= \frac{1}{3}(1^3 - 0^3) \\ &= \frac{1}{3}\end{aligned}$$

will also need the following formula (let c be a constant)

$$\int e^{cx} dx = \frac{1}{c} e^{cx} \tag{11}$$

2.1 Rules for Using Integration $\int$

$$\int_a^b c dx = cx|_a^b = c(b) - c(a) = c(b - a) \tag{12}$$

$$\int c \cdot f(x) dx = c \int f(x) dx \tag{13}$$

$$\int [f(x) + g(x)] dx = \int f(x) dx + \int g(x) dx \tag{14}$$

Exercise:

1. Sigma Notation

$(a) \sum_{i=1}^{5} k$ $\quad (b) \sum_{i=1}^{6} 2(i-1)$ $\quad (c) \sum_{k=0}^{5} (2k+1)$

$(d) \sum_{i=1}^{4} (-1)^i$ $\quad (e) \sum_{k=1}^{4} \frac{(k+1)!}{2^k}$ $\quad (f) \sum_{n=1}^{50} 1$

$(g) \sum_{k=0}^{3} \frac{k^2}{(k+1)!}$ $\quad (h) \sum_{n=0}^{7} 2^n$ $\quad (i) \sum_{n=1}^{100} n$

$(j) \sum_{x=1}^{3} (x^2 + ax + 5)$ $\quad (k) \sum_{i=1}^{4} (x^2 + 3i)$ $\quad (l) \sum_{x=2}^{5}\sum_{y=5}^{6} (x + y)$

2. Integration

$(a) \int_2^5 x dx$ $(b) \int_0^2 x^2 dx$ $(c) \int_{-5}^{-3} 6x^2 dx$

$(d) \int_{-3}^{2} (x+2) dx$ $(e) \int_{-7}^{-3} 5 dx$ $(f) \int_0^1 \frac{1}{2} e^{-2x} dx$

$(g) \int_{-1}^{4} (x^2+4x) dx$

(h) Find the value of c that makes the following true

$$\int_0^1 (x^2 + x + c) dx = 1.$$

3 Sets and Functions

A *set* S is made up of *elements*, and if x is one of these elements, we shall denote this fact by $x \in S$. There is exactly one set with no elements. It is the *empty set* $\emptyset$. We may describe a set either by giving a characterizing property of the elements, such as "the set of even whole positive numbers that are less than 9",

$\{ x : x$ is an even whole positive number less than $9 \}$

or by listing all the elements. The standard way to describe a set by listing elements is to enclose the designations of the elements , separated by commas, in braces, from example, $\{ 2, 4, 6, 8 \}$.

$$\begin{aligned} \{ 2, 4, 6, 8 \} &= \{ x : x \text{ is an even whole positive number less than } 9 \} \\ &= \{ 2x : x = 1, 2, 3, 4 \} \end{aligned}$$

A set is *well defined*, meaning that if S is a set and a is some object, then either a is definitely in S, denoted by $a \in S$, or a is definitely not in S, denoted by $a \notin S$. Throughout of this class, we will be working with familiar sets of numbers. The notation for these sets: $\mathbb{Z}$ is the set of all integers (that is, whole number: positive, negative, and zero). $\mathbb{Q}$ is the set of all rational numbers (that is, numbers that can be expressed as quotients m/n of integers, where $n \neq 0$). $\mathbb{R}$ is the set of all real numbers. $\mathbb{Z}^+$, $\mathbb{Q}^+$, and $\mathbb{R}^+$ are the sets of positive members of $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$, respectively.

If every element of B is in set A, then set B is a *subset* of a set A, denoted by $B \subset A$ or $A \supset B$.

Example: Let $S = \{ 1, 2, 3 \}$. This set has a total of eight subsets, namely $\emptyset$, $\{ 1 \}$, $\{ 2 \}$, $\{ 3 \}$, $\{ 1, 2 \}$, $\{ 1, 3 \}$, $\{ 2, 3 \}$ and $\{ 1, 2, 3 \}$.

Example: $\mathbb{Z}^+ \subset \mathbb{Z}$, $\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.

A *function* f is a rule that assigns each input number x from set A to exactly one output number $f(x)$ in set B. The set A of all input numbers is called the domain. The set of all output numbers is called the range. Notation $f : A \to B$.

One way of geometrically representing a function is by graphing it on a *rectangular coordinate* system. We plot *ordered pairs*, or *points*. For example, $y = x^2 + 1$.

There are many kinds of functions with special structure. Some are:

Constant functions

$$f(x) = 19$$

Polynomial functions (including linear and quadratic)

$$f(x) = 3x^2 + 12x - 5$$

Compound functions

$$f(s) = \begin{cases} 1, & if \quad -1 \leq s < 1, \\ 0, & if \quad 1 \leq s < 2, \\ s - 3, & if \quad 2 < s \leq 8. \end{cases}$$

We can also *plug* one function inside of another. This is called a *composition* of functions. Notation:

$$f \circ g(x) = f(g(x)) \tag{15}$$

Example: Let $f(x) = \sqrt{x}$ and $g(x) = x + 1$. Then $f \circ g(x) = ?$ And $g \circ f(x) = ?$

Example: Let $f(x) = \frac{2x}{x^2+3}$, $g(x) = e^x$. Then $f \circ g(x) = ?$ And $g \circ f(x) = ?$

4 Review of Algebra

4.1 Basic Properties of Real Numbers

Let a, b, c and d be real numbers:

1. $a + b = b + a$
2. $ab = ba$
3. $a + (b + c) = (a + b) + c$
4. $a(bc) = (ab)c$
5. $a(b + c) = ab + ac$
6. $a(b - c) = ab - ac$
7. $(a + b)c = ac + bc$
8. $(a - c)c = ac - bc$
9. $a + 0 = a$
10. $a \cdot 0 = 0$
11. $a \cdot 1 = a$
12. $a + (-a) = 0$
13. $-(-a) = a$
14. $(-1)a = -a$
15. $a - b = a + (-b)$
16. $a - (-b) = a + b$
17. $a\left(\frac{1}{a}\right) = 1$
18. $\frac{a}{b} = a \cdot \frac{1}{b}$
19. $(-a)b = -(ab) = a(-b)$
20. $(-a)(-b) = ab$
21. $\frac{-a}{-b} = \frac{a}{b}$
22. $\frac{-a}{b} = -\frac{a}{b} = \frac{a}{-b}$
23. $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$
24. $\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$
25. $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$
26. $\frac{a/b}{c/d} = \frac{ad}{bc}$
27. $\frac{a}{b} = \frac{ac}{bc}$

4.2 Basic Laws of Exponents

let a and b be real numbers and let m and n be integers:

1. $a^0 = 1 \qquad (a \neq 0)$
2. $a^{-n} = \frac{1}{a^n} \qquad (a \neq 0)$
3. $a^m a^n = a^{m+n}$
4. $(a^m)^n = a^{mn}$
5. $(ab)^n = a^n b^n$
6. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
7. $\frac{a^m}{a^n} = a^{m-n}$

4.3 Basic Laws of Radicals

let a and b be real numbers and let m and n be integers:

1. $\sqrt[n]{a} = a^{1/n}$
2. $(\sqrt[n]{a})^n = a$, $\left(\sqrt[n]{a^n}\right) = a \ (a > 0)$
3. $\sqrt[n]{a^m} = (\sqrt[n]{a})^m = a^{m/n}$
4. $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$
5. $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$
6. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$

4.4 Algebraic Expressions

A *variable* is a symbol representing a quantity which is capable of assuming any of a set of values. For example, in the expression $x^2 + y^2 = z^2$, x, y and z are variables.

Here is an example of an *algebraic expression*:

$$10y^3 - 3\sqrt{y} + \frac{5}{7 + y^2}.$$

In this expression variable is y. It has 3 terms: $10y^3$, $3\sqrt{y}$ and $\frac{5}{7+y^2}$. Some of the *factors* of the first term are 2, 5, y, $2y$, $5y$, $10y^2$ and $10y^3$.

4.5 Special Products

Let a and b be real numbers and let x be a variable:

1. $(x+a)(x+b) = x^2 + (a+b)x + ab$
2. $(x+a)^2 = x^2 + 2ax + a^2$
3. $(x-a)^2 = x^2 - 2ax + a^2$
4. $(x+a)(x-a) = x^2 - a^2$
5. $(x+a)^3 = x^3 + 3ax^2 + 3a^2x + a^3$
6. $(x-a)^3 = x^3 - 3ax^2 + 3a^2x - a^3$

4.6 Factoring Formulas

Let a, b and c be real numbers:

1. $ab + ac = a(b+c)$
2. $a^2 - b^2 = (a+b)(a-b)$
3. $a^2 + 2ab + b^2 = (a+b)^2$
4. $a^2 - 2ab + b^2 = (a-b)^2$
5. $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
6. $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

4.7 Equations and Inequalities

A *linear equation* in the variable x is an equation that can be written in the form

$$ax + b = 0 \tag{16}$$

where a and b are constants and $a \neq 0$. The solution is $x = -\frac{b}{a}$.

A *quadratic equation* in the variable x is an equation that can be written in the form

$$ax^2 + bx + c = 0 \tag{17}$$

where a, b and c are constants and $a \neq 0$. The solution is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{18}$$

Another method for solving a quadratic equation is based on factoring and on the fact that whenever the product of two (or more) quantities is zero, at least one of the quantities must be zero.

Example: $x^2 + x - 12 = 0$. Solve it in two ways. Answer: $x_1 = -4$, $x_2 = 3$.

A *linear inequality* in the variable x is an inequality that can be written in the form $ax + b < 0$ where a and b constants and $a \neq 0$.

We solve a linear inequality with the same techniques as solving a linear equality. Note: when multiplying both sides of a linear inequality by a negative number, the *direction* of the inequality changes.

Example: $\frac{3}{2}(s-2) + 1 > -2(s-4)$

4.8 Exercise

If you have difficulty in solving the following problems, you need to do a thorough review of basic algebra as **a prerequisite for MGT 201**.

1. Simplify: $3^2 + 2^0 + 27^{-2/3}$
2. Expand: $(4z^4 - 3z^2 + 1) - z(z^3 + 4z^2 - 4)$
3. Multiply: $(3a^2b)^2(2ab^3)$
4. Simplify: $\left(\frac{a^{2/3}b^{3/2}}{a^2b}\right)^6$
5. Simplify: $\frac{9p^4q^3 - 6p^2q^4 + 5p^3q^2}{3p^2q^2}$
6. Factor: $2r^4s^3 - 8r^2s^5$
7. Factor: $x^4 - 8x^3 + 16x^2$
8. Subtract: $\frac{1}{x} - \frac{2}{x^2+x} - \frac{3}{x+3}$
9. Simplify $\frac{x+2-\frac{3}{x+4}}{\frac{x}{x+4}+\frac{1}{x+4}}$
10. Solve: $\frac{3}{2}(s-2) + 1 > -2(s-4)$
11. Solve: $x^2 + x - 12 = 0$
12. Solve for μ: $(x-\mu)/s \geq b$