HISTORY AND FUTURE OF AVERAGING TECHNIQUES IN FINITE ELEMENT ERROR ANALYSIS

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Given a flux or stress approximation p_h from a low-order finite element simulation of an elliptic PDE, for instance, averaging techniques aim to compute an improved approximation $\mathcal{A}p_h$ by a (simple) post-processing of p_h. One example, occasionally named after Zienkiewicz & Zhu, computes $\mathcal{A}p_h(z)$ as the integral mean over the patch of the node z (this is the union of all finite elements which share the vertex z in a regular triangulation) and then linearly interpolates $\mathcal{A}p_h$ on each element. Motivated by heuristic assumptions this estimator appeared to work very well in practice — engineers seemed to be extremely happy with this tool.

The beginning of a mathematical justification of the error estimator $\eta_{\mathcal{A}} := \|p_h - \mathcal{A}p_h\|$ as a computable approximation of the (unknown) error $\|p - p_h\|$ involved the concept of super-convergence points. For highly structured meshes and a very smooth exact solution p, the error $\|p - \mathcal{A}p_h\|$ of the post-processed approximation $\mathcal{A}p_h$ may be (much) smaller than $\|p - p_h\|$ of the given p_h. Under the assumption that $\|p - \mathcal{A}p_h\| =$ h.o.t. is relatively sufficiently small, the triangle inequality immediately verifies reliability, i.e.,

$$\|p - p_h\| \leq C_{rel}\, \eta_{\mathcal{A}} + \text{ h.o.t.},$$

and efficiency, i.e.,

$$\eta_{\mathcal{A}} \leq C_{eff}\, \|p - p_h\| + \text{ h.o.t.},$$

of the averaging error estimator $\eta_{\mathcal{A}}$. However, the underlying assumptions essentially contradict the notion of adaptive grid refining for optimal experimental convergence rates when p is singular. Moreover, the proper treatment of boundary conditions lacks a serious inside.

The presentation reports on old and new arguments for reliability and efficiency in the above sense with multiplicative constants C_{rel} and C_{eff} and higher order terms h.o.t. Hi-lighted are the general class of meshes, averaging techniques, or finite element methods (conforming, nonconforming, and mixed elements) for elliptic PDEs. Numerical examples illustrate the amazing accuracy of $\eta_{\mathcal{A}}$. The presentation closes with a discussion on current developments and the limitations as well as the perspectives of averaging techniques.

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