Mock theta functions and quantum modular forms

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In 1920, Ramanujan gave a 17 “Eulerian series”, such as

\[ f(q) := \sum_{n \geq 0} \frac{q^{n^2}}{(-q)^n}, \]

where \((a; q)_n := (a)_n = \prod_{j=0}^{n-1} (1 - aq^j)\).
Ramanujan’s “Deathbed” letter

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Question

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What is the relation of functions like \(f(q)\) to modular forms?

**Example:** The Rogers-Ramanujan identities

\[ G(q) := \sum_{n \geq 0} \frac{q^{n^2}}{(q)_n} = \frac{1}{(q; q^5)_\infty(q^4; q^5)_\infty}, \]

\[ H(q) := \sum_{n \geq 0} \frac{q^{n^2+n}}{(q)_n} = \frac{1}{(q^2; q^5)_\infty(q^3; q^5)_\infty}. \]
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*How does one “detect” modularity of an Eulerian series?*

- Modular forms have very strong properties in their asymptotic expansions!
- For example, a modular form must have an asymptotic expansion as $t \to 0^+$ of the shape

$$e^{\frac{a}{t}} F(e^{-t}) \sim bt^{-k} + O(t^N) \text{ for all } N \geq 0,$$

but most Eulerian series have “unclosed” expansions.
Ramanujan noticed that all of his mock theta functions “look like” modular forms near roots of unity, and he defined mock theta functions by this property.

\[ b(q) := \frac{q^{\infty}}{(-q)^{\infty}}. \]

If \( \zeta \) is a primitive \( 2k \)-th order root of unity, then
\[ \lim_{q \to \zeta} (f(q) - (-1)^k b(q)) = O(1). \]

Hence, at even order roots of unity, the singularities of \( f(q) \) are “cut out” by \( \pm b(q) \). Mock theta functions are defined to be functions which have their singularities cut out by modular forms, but not in a trivial way.
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Example (Ramanujan-Watson)

Let \( b(q) := (q)_\infty / (-q)^2_\infty \). Then if \( \zeta \) is a primitive \( 2k \)-th order root of unity,

\[
\lim_{q \to \zeta} \left( f(q) - (-1)^k b(q) \right) = O(1).
\]
The mock theta functions

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**Example (Ramanujan-Watson)**

Let $b(q) := (q)_{\infty}/(-q)^2_{\infty}$. Then if $\zeta$ is a primitive $2k$-th order root of unity,

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- Hence, at even order roots of unity, the singularities of $f(q)$ are “cut out” by $\pm b(q)$. Mock theta functions are defined to be functions which have their singularities cut out by modular forms, but not in a trivial way.
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However, as noted by Berndt, no one had proven that Ramanujan’s function satisfied his own definition.
From Ramanujan to the future, and back

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Theorem (Griffin-Ono-R.)

Ramanujan’s original formulation of the mock theta functions was correct.
Looking further into Ramanujan’s definition

Question

Can we understand Ramanujan’s question more explicitly? Namely, can we provide an algorithm to systematically compute the modular forms to cut out the singularities, along with the “leftover constants”?

Example

Folsom, Ono, and Rhoades proved that if \( \zeta \) is a primitive \( 2^k \)-th order root of unity, then

\[
\lim_{q \to \zeta} (f(q) - (-1)^k b(q)) = -4k - 1 \sum_{n=0}^{\infty} (-\zeta; \zeta) 2^n \zeta^n + 1.
\]

Moreover, they fit this into an infinite family of relations beautifully connecting the rank, crank, and unimodal generating functions.
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“Universal” families

Idea (Rhoades)

Study Ramanujan’s definition for:

\[ g_2(\zeta; q) := \sum_{n \geq 0} \frac{(-q)_n q^{n(n+1)/2}}{(\zeta)_{n+1}(\zeta^{-1} q; q)_{n+1}} \]
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Goal

Find \( f_{a,b,A,B,h,k}(q) \in M_{1/2}^! \) and finite sums \( U_{a,b,A,B,h,k} \) such that

\[
\lim_{q \to e^{2\pi i h/k}} \left( g_2 \left( \zeta^a q^A; q^B \right) - f_{a,b,A,B,h,k}(q) \right) = U_{a,b,A,B,h,k}.
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Theorem (Bringmann-R.)

There is a canonical, finite procedure to solve this problem. At most three functions \( f_{a,b,A,B,h,k} \) are needed for fixed \( a, b, A, B \).
“Definition”

A quantum modular form is a function $f : \mathbb{P}^1(\mathbb{Q}) \to \mathbb{C}$ such that for all $\gamma \in \Gamma$, $f|_k(1 - \gamma)$ is “nice”.

Example

Kontsevich defined:

$$F(q) := \sum_{n \geq 0} (q)_n.$$  

This converges at roots of unity, and its values equal the radial limits of a “half-derivative” of the Dedekind eta function. Zagier used this to show that $F$ is a QMF of weight $\frac{3}{2}$.

Theorem (Choi-Lim-Rhoades)

If $f$ is a mock theta function, then the “leftover constants” in Ramanujan’s definition give a quantum modular form.
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Further examples of quantum modular forms

Theorem (Bringmann-R.)

The “Eichler integral” (the formal $k-1$st antiderivative) of any half-integral weight cusp form is a quantum modular form.
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- These quantum modular forms show up all over the place! Call them Eichler quantum modular forms.
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Theorem (Bringmann-Creutzig-R., Bringmann-R.-Zwegers)

The Fourier coefficients in $z$ of a negative index Jacobi form have “theta-type” expansions in terms of quasimodular forms and Eichler quantum modular forms.
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Goal (Bringmann-R.)

Understand the general framework of quantum modular forms, for example by starting with a well-defined subspace such as the Eichler quantum modular forms.
The function $F$ has a Taylor expansion at $q = 1$ given by

$$
\sum_{n \geq 0} (1 - q; 1 - q)_n =: \sum_{n \geq 0} \xi(n)q^n = 1 + q + 2q^2 + 5q^3 + \ldots.
$$
Arithmetic properties of quantum modular forms

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**Theorem (Andrews-Sellers, Straub)**

There are infinitely many primes $p$ for which there is a $B \in \mathbb{N}$ such that for all $A$,

$$\xi \left( p^A n - B \right) \equiv 0 \pmod{p^A}.$$
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**Theorem (Guerzhoy-Kent-R.)**

For any weight 1/2 theta function, there are analogous sequences defined by Taylor expansions of the associated Eichler quantum modular form. Moreover, these (almost always) satisfy congruences like those for $\xi(n)$ for a positive proportion of primes.
Motivating example from knot theory

Hikami considered

\[ F_m^{(\alpha)}(q) := \sum_{k_1, k_2, \ldots, k_m = 0}^\infty (q; q)_{k_m} q^{k_1^2 + \ldots + k_{m-1}^2 + k_\alpha + \ldots + k_{m-1}} \]

\[ \times \left( \prod_{i=1}^{m-1} \left[ \begin{array}{c} k_i + 1 \\ k_i \end{array} \right]_q \right) \cdot \left[ \begin{array}{c} k_{\alpha+1} + 1 \\ k_{\alpha} \end{array} \right]_q, \]

where

\[ \left[ \begin{array}{c} n \\ k \end{array} \right]_q := \begin{cases} \frac{(q)_n}{(q)_k(q)_{n-k}} & 0 \leq k \leq n \\ 0 & \text{otherwise}. \end{cases} \]
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\times \left( \prod_{i=1 \atop i \neq \alpha}^{m-1} \left[ k_i + 1 \right]_q \right) \cdot \left[ k_{\alpha+1} + 1 \right]_q \cdot \left[ k_{\alpha} \right]_q,
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- These are limits of the “half-derivative” of Andrews-Gordon functions, and related to Kashaev’s invariant for torus knots.
Define numbers $\xi_m^{(a)}$ as the Taylor coefficients of $F_m^{(a)}$ at $q = 1$. 

**Theorem (Guerzhoy-Kent-R.)**

Choose $\alpha, m \in \mathbb{N}$ with $\alpha < m$ such that

$$(2m - 2\alpha - 1)^2 - 8(2m + 1)$$

is not a square. Then

$$\xi_m^{(\alpha)} \equiv 0 \pmod{p^A}$$

for all $n, A \in \mathbb{N}$ for at least $50\%$ of primes $p$. 

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