Putnam Practice Set #: 5, 11/19/2012

**Problem 1.** For every integer $n \geq 2$, prove that

$$\sum_{k=1}^{n} (-1)^k k \binom{n}{k} = 0,$$

where $\binom{n}{k}$ is the usual binomial coefficient.

**Problem 2.** Show that if

$$u(x) = 1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \frac{x^9}{9!} + \cdots,$$

$$v(x) = x + \frac{x^4}{4!} + \frac{x^7}{7!} + \frac{x^{10}}{10!} + \cdots,$$

$$w(x) = \frac{x^2}{2!} + \frac{x^5}{5!} + \frac{x^8}{8!} + \frac{x^{11}}{11!} + \cdots,$$

then $u^3 + v^3 + w^3 - 3uvw = 1$.

**Problem 3.** Do there exist two different (that is, non-isomorphic) ellipses having the same area and circumference?

**Problem 4.** There are $n$ men in a warehouse, with not three in a straight line, and so that the distances between pairs of men are distinct.

Each man has a loaded pistol. At a signal, each shoots the man closest to him. Show that if $n$ is odd, then at least one man remains alive. Show also that if $n$ is even, then it is possible that every man dies.

**Problem 5.** Are there positive irrational numbers $a$ and $b$ such that $a^b$ is rational?