Problem 1. For every integer $n \geq 2$, prove that

$$\sum_{k=1}^{n} (-1)^k \binom{n}{k} = 0,$$

where $\binom{n}{k}$ is the usual binomial coefficient.

Solution. Let's consider

$$\varphi(x) = \sum_{k=0}^{n} \binom{n}{k} x^k = (1 + x)^n.$$

Then we have

$$\varphi'(x) = \sum_{k=0}^{n} k \binom{n}{k} x^{k-1} = n(1 + x)^{n-1},$$

which implies that

$$\sum_{k=1}^{n} (-1)^k \binom{n}{k} = \sum_{k=0}^{n} (-1)^{k-1} \binom{n}{k} = \varphi'(-1) = 0.$$

\[\square\]

Problem 2. Show that if

$$u(x) = 1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \frac{x^9}{9!} + \cdots,$$
$$v(x) = x + \frac{x^4}{4!} + \frac{x^7}{7!} + \frac{x^{10}}{10!} + \cdots,$$
$$w(x) = \frac{x^2}{2!} + \frac{x^5}{5!} + \frac{x^8}{8!} + \frac{x^{11}}{11!} + \cdots,$$

then $u^3 + v^3 + w^3 - 3uvw = 1$.

Solution. First, we will show that $(u^3 + v^3 + w^3 - 3uvw)' = 0$. Note that

$$u'(x) = \frac{x^2}{2!} + \frac{x^5}{5!} + \frac{x^8}{8!} + \frac{x^{11}}{11!} + \cdots = w(x),$$
$$v'(x) = 1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \frac{x^9}{9!} + \cdots = u(x),$$
$$w'(x) = x + \frac{x^4}{4!} + \frac{x^7}{7!} + \frac{x^{10}}{10!} + \cdots = v(x),$$

implies that

$$(u^3 + v^3 + w^3 - 3uvw)' = 3u^2u' + 3v^2v' + 3w^2w' - 3[u'vw + uv'w + uwv']$$
$$= 3u^2w + 3v^2u + 3w^2v - 3[w^2v + u^2w + uv^2] = 0$$

Now, since the derivative of $u^3 + v^3 + w^3 - 3uvw$ is zero, we must have, as a consequence of the mean value theorem, $u^3 + v^3 + w^3 - 3uvw = C$ for some constant $C$. Since $u(0) = 1$, $v(0) = 0$ and $w(0) = 0$, we see that $C = 1$ as desired. \[\square\]
Problem 3. Do there exist two different (that is, non-isomorphic) ellipses having the same area and circumference?

Solution. There do not exist two different (not-isomorphic) ellipses having the same area and circumference. We can prove this by showing that two non-isomorphic ellipses with the same area cannot have the same circumference. Since the area of any ellipse is \( \pi ab \) (\( a, b \) are lengths of semi-major and semi-minor axes respectively), if we consider a set of ellipses with the same area \( \pi A \) and write the length of semi-major axis as \( c\sqrt{A} \), then the length of the semi-minor axis should be \( \sqrt{A/c} \). Since the semi-major axis cannot be short than the semi-minor axis, we must have \( c \geq 1 \); in addition, each value of \( c \) corresponds to a set of ellipses that are isomorphic to each other. Let’s place the ellipse in a Cartesian coordinate system with semi-major axis along the \( x \)-axis and semi-minor axis along the \( y \)-axis, then it can be represented as

\[
\frac{x^2}{c^2A} + \frac{y^2}{A} = 1
\]

and any point on the ellipse has coordinate:

\[
x = c\sqrt{A}\cos(\theta), \quad y = \frac{\sqrt{A}}{c}\sin(\theta) \quad (0 \leq \theta < 2\pi).
\]

The circumference of the ellipse can then be written as the following integral:

\[
C = \int_0^{2\pi} \sqrt{(dx)^2 + (dy)^2}
\]

\[
= \sqrt{A} \int_0^\pi \sqrt{c^2\sin^2 \theta + \frac{1}{c^2}\cos^2 \theta} \, d\theta
\]

\[
= 4\sqrt{A} \left( \int_0^{\pi/4} \sqrt{c^2\sin^2 \theta + \frac{1}{c^2}\cos^2 \theta} \, d\theta + \int_{\pi/4}^{\pi/2} \sqrt{c^2\sin^2 \theta + \frac{1}{c^2}\cos^2 \theta} \, d\theta \right)
\]

\[
= 4\sqrt{A} \int_0^{\pi/4} \left( \sqrt{c^2\sin^2 \theta + \frac{1}{c^2}\cos^2 \theta} + \sqrt{c^2\cos^2 \theta + \frac{1}{c^2}\sin^2 \theta} \right) \, d\theta
\]

Now, let’s just focus on \( f(c, \theta) \) and consider the dependence on \( c \). Since \( f(c, \theta) \) is always positive, we can consider \((f(c, \theta))^2\) instead:

\[
(f(c, \theta))^2 = \left( \sqrt{c^2\sin^2 \theta + \frac{1}{c^2}\cos^2 \theta} + \sqrt{c^2\cos^2 \theta + \frac{1}{c^2}\sin^2 \theta} \right)^2
\]

\[
= c^2 + \frac{1}{c^2} + 2\sqrt{\left( c^2\sin^2 \theta + \frac{1}{c^2}\cos^2 \theta \right) \left( c^2\cos^2 \theta + \frac{1}{c^2}\sin^2 \theta \right)}
\]

Since the function \( g(x) = x + \frac{1}{x} \) is monotone increasing with \( x \) as \( x \geq 1 \), and \( x^2 \) and \( x^4 \) are monotone increasing with \( x \) as \( x \geq 0 \), \( f(c, \theta) \) is monotone increasing with \( c \) as \( c \geq 1 \) for any fixed value of \( \theta \).
If we consider the circumference $C$ as a function of $c$, it should be monotone increasing with $c$ as $c \geq 1$, so two ellipses with difference values of $c$ must have difference circumferences if they have the same area $\pi A$. So there does not exist two non-isomorphic ellipses having the same area and circumference.

**Problem 4.** There are $n$ men in a warehouse, with not three in a straight line, and so that the distances between pairs of men are distinct.

Each man has a loaded pistol. At a signal, each shoots the man closest to him. Show that if $n$ is odd, then at least one man remains alive. Show also that if $n$ is even, then it is possible that every man dies.

**Solution.** Since the distances between pairs of men are distinct, there must be a single minimum distance. The two men $A$ and $B$ separated by this distance must shoot each other, since no other man is closer to either of them. If one of the other $n - 2$ men shoot either $A$ or $B$, then $A$ or $B$ is shot twice. Since a total of $n$ shots are fired, this implies that at least one man survives by the pigeonhole principle. If, however, none of the $n - 2$ remaining men shoot $A$ or $B$, then $A$ and $B$ can be removed from consideration without affecting the parity of the number of men or the outcome of the shootout among the $n - 2$. Considering only the men other than $A$ and $B$, there must again be a single least distance and thus two men who shoot each other, and if either of them is shot twice then one man survives and if not then then may also be ignored. Repeat this process until it is found that one man survives or until all the men have been eliminated. Suppose $n$ is odd and the cases where $2, 4, 6, \ldots, n - 3$ men are ignored all fail to show the survival of at least one man. Of the remaining three, two must shoot each other and the third must shoot one of those two, leaving the third man alive. Therefore, if $n$ is odd, at least one man survives.

If $n$ is even, then it may be that there exist $n/2$ pairs of men who shoot each other, in which case every man dies. Such a situation could be constructed by consider the set $D_k$ of distance between the two men is less than the minimum of $D_k$ and so that the minimum distance between either of the two and any of the preceding $2k$ men exceeds the maximum of $D_k$. Since there are finitely-many men previously placed, it is also possible to place the $(k + 1)^{th}$ pair so that the distance between the men are all distance and that no three lie on a line.

**Problem 5.** Are there positive irrational numbers $a$ and $b$ such that $a^b$ is rational?

**Solution.** Yes. Consider the number $\sqrt{2}^{\sqrt{2}}$. Certainly $\sqrt{2}$ is irrational, so if $\sqrt{2}^{\sqrt{2}}$ is rational, then take $a = b = \sqrt{2}$. On the other hand, if $\sqrt{2}^{\sqrt{2}}$ is irrational, then take $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$ to get that $a^b = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^2 = 2$, which is an integer and therefore rational.