Infinitely Many Composite NSW Numbers:

An Inductive Proof

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1 Motivation

The NSW numbers were introduced approximately 20 years ago \cite{3} in connection with the order of certain simple groups. These are the numbers $f_n$ which satisfy the recurrence

$$f_{n+1} = 6f_n - f_{n-1}$$  \hspace{1cm} (1)

with initial conditions $f_1 = 1$ and $f_2 = 7$.

In recent years, these numbers have been studied from a variety of perspectives \cite{1}, \cite{2}. Moreover, the author, in collaboration with Hugh Williams, has proven that there are infinitely many composite NSW numbers \cite{4} as requested in \cite{1}. The goal of this note is to provide a purely inductive proof of the main theorem in \cite{4}. We restate it here.
Theorem 1.1. For all $m \geq 1$ and all $n \geq 0$, $f_m \mid f_{(2m-1)n+m}$.

2 The Necessary Tools

To prove Theorem 1.1 we need to develop a few key tools.

Proposition 2.1. For all integers $a, b \geq 0$, and for all $1 \leq j \leq a+b-2$, we have

$$f_{a+b} = s_{j+1}f_{a+b-j} - s_jf_{a+b-j-1}$$  \hspace{1cm} (2)

where $s_j = \sum_{i=1}^{j}(-1)^{i+j}f_i$.

Proof. We prove this proposition using induction on $j$. First, when $j = 1$, the right hand side of (2) is $(f_2 - f_1)f_{a+b-1} - f_1f_{a+b-2}$ or $6f_{a+b-1} - f_{a+b-2}$, which equals $f_{a+b}$ thanks to (1).

Next, we assume

$$f_{a+b} = s_{j+1}f_{a+b-j} - s_jf_{a+b-j-1}$$

for $j < a + b - 2$. Thus, since $f_{a+b-j} = 6f_{a+b-j-1} - f_{a+b-j-2}$, we have

$$f_{a+b} = s_{j+1}(6f_{a+b-j-1} - f_{a+b-j-2}) - s_jf_{a+b-j-1}$$

$$= (6s_{j+1} - s_j)f_{a+b-j-1} - s_{j+1}f_{a+b-j-2}.$$

Then we note that

$$6s_{j+1} - s_j = 6 \sum_{i=1}^{j+1}(-1)^{i+j+1}f_i - \sum_{i=1}^{j}(-1)^{i+j}f_i$$

$$= 6(-1)^{j+2}f_1 + 6 \sum_{i=2}^{j+1}(-1)^{i+j+1}f_i - \sum_{i=1}^{j}(-1)^{i+j}f_i$$

$$= 6(-1)^{j+2}f_1 + 6 \sum_{i=2}^{j+1}(-1)^{i+j+1}f_i - \sum_{i=1}^{j}(-1)^{i+j}f_i$$
\begin{align*}
&= 6(-1)^{j+2}f_1 + 6 \sum_{i=1}^{j} (-1)^{i+j+2}f_{i+1} - \sum_{i=1}^{j} (-1)^{i+j}f_i \\
&= (f_2 - f_1)(-1)^{j+2} + \sum_{i=1}^{j} (-1)^{i+j+2}(6f_{i+1} - f_i) \\
&= (f_2 - f_1)(-1)^{j+2} + \sum_{i=1}^{j} (-1)^{i+j+2}f_{i+2} \text{ by (1)} \\
&= (f_2 - f_1)(-1)^{j+2} + \sum_{i=3}^{j+2} (-1)^{i+j}f_i \\
&= \sum_{i=1}^{j+2} (-1)^{i+j}f_i \\
&= s_{j+2}.
\end{align*}

Therefore, we have

\begin{align*}
f_{a+b} &= (6s_{j+1} - s_j)f_{a+b-j-1} - s_{j+1}f_{a+b-j-2} \\
&= s_{j+2}f_{a+b-j-1} - s_{j+1}f_{a+b-j-2},
\end{align*}

which completes the proof of Proposition 2.1 \hfill \Box

**Proposition 2.2.** For all \(m \geq 1\) and for all \(1 \leq c \leq m-1\), \(f_m \mid f_{m+c} + f_{m-c}\).

**Proof.** For \(c = 1\), we know from (1) that \(6f_m = f_{m+1} + f_{m-1}\), so that \(f_m \mid f_{m+1} + f_{m-1}\). Next, we assume \(f_m \mid f_{m+c} + f_{m-c}\) for \(1 \leq c \leq d\) for some value \(d < m - 1\). Since

\[f_{m+d+1} = 6f_{m+d} - f_{m+d-1}\] and \(f_{m-d-1} = 6f_{m-d} - f_{m-d+1}\),

we know

\[f_{m+d+1} + f_{m-(d+1)} = 6f_{m+d} - f_{m+d-1} + 6f_{m-d} - f_{m-d+1}\]
\[ 6(f_{m+d} + f_{m-d}) - (f_{m+d-1} + f_{m-(d-1)}) . \]

By the induction hypothesis, the result follows. \qed

**Proposition 2.3.** For all \( m \geq 1 \), \( f_m \mid s_{2m-1} \).

**Proof.** We see that

\[
s_{2m-1} = \sum_{i=1}^{2m-1} (-1)^{i-1}f_i = f_1 - f_2 + f_3 - \ldots + (-1)^{m-1}f_m + \ldots + f_{2m-1}.
\]

Notice that this sum is centered about \( f_m \), which divides itself, and that the rest of the terms can be paired in such a way that Proposition 2.2 can be applied easily. \qed

We are now ready to prove Theorem 1.1

**Proof.** When \( n = 0 \), the result is clear. Next, assume \( f_m \mid f_{(2m-1)n+m} \) or \( f_m \mid f_{2mn-n+m} \). We want to prove

\[ f_m \mid f_{(2m-1)(n+1)+m} \text{ or } f_m \mid f_{2mn-n+m+(2m-1)}. \]

Using Proposition 2.1 with \( a = 2mn-n+m, b = 2m-1 \), and \( j = 2m-2 \), we have

\[ f_{2mn-n+m+(2m-1)} = s_{2m-1}f_{2mn-n+m+1} - s_{2m-2}f_{2mn-n+m}. \]

From Proposition 2.3 we know \( f_m \mid s_{2m-1} \), and from the induction hypothesis, \( f_m \mid f_{2mn-n+m} \). The result follows. \qed

4
References

[1] Barcucci, E., Brunetti, S., Del Lungo, A., and Del Ristoro, F., A combinatorial interpretation of the recurrence $f_{n+1} = 6f_n - f_{n-1}$, 


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