On the Infinitude of Composite NSW Numbers

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February 28, 2001

1 Motivation

The NSW numbers (named in honor of Newman, Shanks, and Williams [3]) were studied approximately 20 years ago in connection with the order of certain simple
groups. These are the numbers $f_n$ which satisfy the recurrence

$$f_{n+1} = 6f_n - f_{n-1}$$

(1)

with initial conditions $f_1 = 1$ and $f_2 = 7$.

These numbers have also been studied in other contexts. For example, Bonin, Shapiro, and Simion [2] discuss them in relation to Schröder numbers and combinatorial statistics on lattice paths.

Recently, Barcucci et al. [1] provided a combinatorial interpretation for the NSW numbers by defining a certain regular language $L$ and studying particular properties of $L$. They close their note by asking two questions:

1. Do there exist infinitely many $f_n$ prime?

2. Do there exist infinitely many $f_n$ composite?

The goal of this paper is to affirmatively answer the second question above, but in a much broader context. Fix an integer $k \geq 2$ and consider the sequence of values satisfying $f_{n+1} = kf_n - f_{n-1}, f_1 = 1, f_2 = k + 1$. Then we have the following:

**Theorem 1.1.** For all $m \geq 1$ and all $n \geq 0$, $f_m \mid f_{(2m-1)n+m}$.

## 2 The Necessary Tools

To prove Theorem [1.1], we need to develop a few key tools. First, let $\alpha$ be a zero of $x^2 - kx + 1$, the characteristic polynomial of the recurrence. If $\alpha \in \mathbb{Q}$ (the rational numbers), then we may assume that $\alpha = \frac{m}{n}$, where $m, n \in \mathbb{Z}$ and $(m, n) = 1$. Hence,
\[ m^2 - kmn + n^2 = 0 \text{ or } m^2 = kmn - n^2. \] It is clear then that \( m \mid n^2 \) and \( n \mid m^2 \), so that \( \frac{m}{n} = \pm 1 \) because \((m, n) = 1 \). Thus, \( \mathbb{Z}[\alpha] \cap \mathbb{Q} = \mathbb{Z} \).

Now define congruence in \( \mathbb{Z}[\alpha] \) by writing \( \lambda \equiv \mu \pmod{\nu} \) for \( \lambda, \mu, \nu \in \mathbb{Z}[\alpha] \) to mean that \( \frac{\lambda - \mu}{\nu} \in \mathbb{Z}[\alpha] \) (where \( \nu \neq 0 \). Note that if \( \lambda, \mu, \nu \in \mathbb{Z} \) and \( \lambda \equiv \mu \pmod{\nu} \) by this definition, then \( \frac{\lambda - \mu}{\nu} \in \mathbb{Z}[\alpha] \cap \mathbb{Q} \), which implies \( \frac{\lambda - \mu}{\nu} \in \mathbb{Z} \), so that \( \lambda \equiv \mu \pmod{\nu} \) by the conventional definition of congruence.

Also, note that if \( \gamma \in \mathbb{Q}(\alpha) \) and \( \lambda, \mu, \nu, \gamma \in \mathbb{Z}[\alpha] \), then \( \lambda \equiv \mu \pmod{\nu} \) implies \( \gamma \lambda \equiv \gamma \mu \pmod{\gamma \nu} \).

Now we are ready to complete the proof of Theorem 1.1.

**Proof.** We first handle the case \( k = 2 \) separately. In this case, it is easy to show that \( f_n = 2n - 1 \) for \( n \geq 1 \). Then \( f_{(2m-1)n+m} = 2((2m - 1)n + m) - 1 = (2m - 1)(2n + 1) \), and \( f_m \mid f_{(2m-1)n+m} \) clearly.

Next, we assume \( k > 2 \). Since \( \alpha \) is a zero of \( x^2 - kx + 1 \), \( \alpha \) is neither 0 nor 1. Also, \( \alpha^2 + 1 = k\alpha \). Note that \( \beta = \frac{1}{\alpha} \) is the other zero of \( x^2 - kx + 1 \), and \( \alpha + \beta = k \).

Since \( \alpha \) and \( \beta \) are distinct, we know \( f_n = A\alpha^n + B\beta^n \) for some constants \( A \) and \( B \).

Since \( f_0 = -1 \) and \( f_1 = 1 \), we have \( A + B = -1 \) and \( A\alpha + B\beta = 1 \). Solving these two equations yields

\[
A = \frac{1 + \beta}{\alpha - \beta} \quad \text{and} \quad B = -\frac{1 + \alpha}{\alpha - \beta}.
\]

Therefore,

\[
f_m = \frac{1}{\alpha - \beta}((1 + \beta)\alpha^m - (1 + \alpha)\beta^m)
= \frac{1}{\alpha - \beta} \left((1 + \frac{1}{\alpha})\alpha^m - (1 + \alpha)\beta^m\right)
\]
\[
\begin{align*}
&= \frac{1}{\alpha - \beta} \left( (1 + \alpha) \alpha^{m-1} - (1 + \alpha) \beta^m \right) \\
&= \frac{1 + \alpha}{\alpha - \beta} (\alpha^{m-1} - \beta^m).
\end{align*}
\]

Now let \( U_m = \alpha^{m-1} - \beta^m \in \mathbb{Z}[\alpha] \) \((\beta = k - \alpha)\) where \( m \geq 1 \). Then \( \alpha^{m-1} \equiv \beta^m \pmod{U_m} \) implies

\[ \alpha^{2m-1} = \alpha^m \alpha^{m-1} \equiv \alpha^m \beta^m \equiv 1 \pmod{U_m} \]

and

\[ \beta^{2m-1} = \beta^m \beta^{m-1} \equiv \alpha^{m-1} \beta^{m-1} \equiv 1 \pmod{U_m}. \]

Hence,

\[ U_{(2m-1)n+m} = \alpha^{(2m-1)n+m-1} - \beta^{(2m-1)n+m} \]

\[ \equiv \beta^m (\alpha^{(2m-1)n} - \beta^{(2m-1)n}) \pmod{U_m} \]

\[ \equiv 0 \pmod{U_m}. \]

Therefore,

\[ \left( \frac{1 + \alpha}{\alpha - \beta} \right) U_{(2m-1)n+m} \equiv 0 \pmod{\left( \frac{1 + \alpha}{\alpha - \beta} \right) U_m} \]

or \( f_m | f_{(2m-1)n+m} \). \( \square \)

### 3 Closing Thoughts

We close by noting that this theorem proves \( f_m | f_{(2m-1)n+m} \) for a variety of well-known sequences \( \{f_m\}_{m=1}^{\infty} \) other than the NSW numbers, including the odd numbers \((k = 2)\), the Lucas numbers \( L_{2n} \) \((k = 3)\), and the Fibonacci numbers \( F_{4n+2} \) \((k = 7)\).
References


2000 *Mathematics Subject Classification.* 11B37, 11B83

*keywords: NSW numbers, recurrence relation*