A DIFFERENT VIEW OF M–ARY PARTITIONS

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In 1969, R. F. Churchhouse [2] studied the number of binary partitions of an integer n. That is, Churchhouse proved various properties of the partition function $b_2(n)$, which counts the number of partitions of n into parts which are powers of 2.

Soon after, Andrews [1], Gupta [4–6], and Rodseth [7] extended Churchhouse's results. They considered a generalization of $b_2(n)$, which we will denote $b_m(n)$, which is the number of m-ary partitions of n, or the number of ways to write n as a sum of powers of m (for a fixed $m \geq 2$).

In more recent years, new properties of $b_m(n)$ and related restricted m-ary partition functions have been discovered and proven by a number of authors [3, 8, 9], revitalizing an interest in the topic.

In this note we consider a second family of partitions enumerated by the function $a_m(n)$. For a fixed value of $m \geq 2$, this function counts the number of partitions of n of the form

$$n = p_1 + p_2 + \ \dots \ + p_k$$

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where $k \geq 1$ and the parts satisfy the following system of inequalities:

$$(*) \qquad \begin{aligned} p_1 &\geq (m-1)(p_2 + \ \dots \ + p_k), \\ p_2 &\geq (m-1)(p_3 + \ \dots \ + p_k), \\ &\vdots \\ p_{k-2} &\geq (m-1)(p_{k-1} + p_k), \\ p_{k-1} &\geq (m-1)p_k \end{aligned}$$

Our goal is to prove the following theorem.

Theorem: For all $n \geq 0$,

$$a_m(n) = b_m(n)$$

where $a_m(n)$ and $b_m(n)$ are defined above.

We prove this theorem by showing that the generating functions for $a_m(n)$ and $b_m(n)$ are identical. The tools employed are elementary and reminiscent of rudiments of MacMahon's partition analysis.

Before proving our theorem, we note one key lemma which is easily proven by induction.

Lemma: Let $x_1 < x_2 < \cdots < x_j$ be positive integers. Then

$$1 + \sum_{k=1}^{j} \frac{q^{x_k}}{(1-q^{x_1})(1-q^{x_2}) \ \dots \ (1-q^{x_k})} = \prod_{k=1}^{j} \frac{1}{(1-q^{x_k})}.$$

Letting j tend to infinity in this lemma we obtain

$$(**) \qquad 1 + \sum_{k=1}^{\infty} \frac{q^{x_k}}{(1-q^{x_1})(1-q^{x_2}) \ \dots \ (1-q^{x_k})} = \prod_{k=1}^{\infty} \frac{1}{(1-q^{x_k})}.$$

Now we turn to the proof of our theorem.

Proof: Assume $n = p_1 + p_2 + \cdots + p_k$ is a partition of n satisfying (*). For $\epsilon_1, \ \epsilon_2, \ \dots \ , \epsilon_k \geq 0$, we can write

$$\begin{aligned} p_k &= 1 + \epsilon_k, \\ p_{k-1} &= (m-1)p_k + \epsilon_{k-1} = (m-1) + (m-1)\epsilon_k + \epsilon_{k-1}, \\ p_{k-2} &= (m-1)(p_{k-1} + p_k) + \epsilon_{k-2} = (m-1)m + (m-1)m\epsilon_k + (m-1)\epsilon_{k-1} + \epsilon_{k-2}, \end{aligned}$$

and for $r = 3,\ 4,\ \ldots\ , k-1$ by induction,

$$p_{k-r} = (m-1)m^{r-1} + (m-1)m^{r-1}\epsilon_k + (m-1)m^{r-2}\epsilon_{k-1} + \ \ldots\ + (m-1)\epsilon_{k-r+1} + \epsilon_{k-r}$$

so that for $r = k-1$,

$$p_1 = (m-1)m^{k-2} + (m-1)m^{k-2}\epsilon_k + \ \ldots\ (m-1)\epsilon_2 + \epsilon_1.$$

Then, for $k \geq 2$,

$$\begin{aligned} n &= p_1 + \ \ldots\ + p_k \\ &= \quad [1 + (m-1)(1 + m + \ \ldots\ + m^{k-2})] \\ &\quad + [1 + (m-1)(1 + m + \ \ldots\ + m^{k-2})]\epsilon_k \\ &\quad + [1 + (m-1)(1 + m + \ \ldots\ + m^{k-3})]\epsilon_{k-1} \\ &\quad + \ \ldots \\ &\quad + [1 + (m-1)]\epsilon_2 \\ &\quad + \epsilon_1 \\ &= m^{k-1} + m^{k-1}\epsilon_k + m^{k-2}\epsilon_{k-1} + \ \ldots\ + m\epsilon_2 + \epsilon_1 \end{aligned}$$

after simplification.

Now let $a_{m,k}(n)$ be the number of partitions of type (*) with exactly k parts. Then we have

$$\sum_{n\geq 1} a_{m,1}(n)q^n = \sum_{\epsilon_1 \geq 0} q^{p_1} = \sum_{\epsilon_1 \geq 0} q^{1+\epsilon_1} = \frac{q}{1-q}$$

and, for $k \geq 2$,

$$\begin{aligned} \sum_{n\geq 1} a_{m,k}(n)q^n &= \sum_{\epsilon_1,\ldots,\epsilon_k \geq 0} q^{p_1 + \ \ldots\ + p_k} \\ &= \sum_{\epsilon_1,\ldots,\epsilon_k \geq 0} q^{m^{k-1} + m^{k-1}\epsilon_k + \ \ldots\ + m\epsilon_2 + \epsilon_1} \\ &= \frac{q^{m^{k-1}}}{(1-q)(1-q^m)\ \ldots\ (1-q^{m^{k-1}})}. \end{aligned}$$

Thus,

$$1 + \sum_{n\geq 1} a_m(n)q^n = 1 + \sum_{k\geq 1}\sum_{n\geq 1} a_{m,k}(n)q^n$$

$$= 1 + \sum_{k \geq 1} \frac{q^{m^{k-1}}}{(1-q)(1-q^m)\ \ldots\ (1-q^{m^{k-1}})}$$

$$= \frac{1}{(1-q)(1-q^m)(1-q^{m^2})\ \ldots} \quad \text{by (**)}$$

$$= 1 + \sum_{n \geq 1} b_m(n) q^n.$$

This yields the desired result. $\square$

References

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