A NOTE ON INFINITELY MANY ODD NONUNITARY ABUNDANT NUMBERS

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Abstract. In response to a recent article by K. R. S. Sastry, we exhibit infinitely many odd nonunitary abundant numbers.

In a recent article in Mathematics and Computer Education, K. R. S. Sastry [4] discussed a variety of results and problems involving nonunitary numbers. (A number of articles dealing with unitary and nonunitary numbers have appeared in the last several years. See, for example, [1], [2], and [3].) In particular, Sastry asked for an example of an odd nonunitary abundant number. The goal of this short note is to exhibit an infinite family of such integers.

A brief review of some key terms is in order. An integer $N$ is said to be nonunitary abundant if the sum of its nonunitary divisors is bigger than $N$. A nonunitary divisor $d$ of $N$ is a divisor which satisfies $(d,N/d) > 1$, while a divisor $d$ of $N$ is called a unitary divisor of $N$ if $(d,N/d) = 1$. Using Sastry’s notation, we will denote the sum of the divisors of $N$ by $\sigma(N)$, while the sum of the unitary divisors of $N$ will be denoted by $\sigma^*(N)$.

Sastry [4] notes that if $n = p_1^{a_1}p_2^{a_2}\cdots p_r^{a_r}$, then

$$\sigma(n) = \prod_{i=1}^{r} \frac{p_i^{a_i+1}}{p_i - 1}$$

and

$$\sigma^*(n) = \prod_{i=1}^{r} (p_i^{a_i} + 1).$$

We will use these facts below.

Thanks to a brief Maple search, the following was discovered.

Theorem 1. The number $N = 3^3 \cdot 5^2 \cdot 7^2 = 33075$ is an odd nonunitary abundant number.

Indeed, it is the case that 33075 is the smallest odd nonunitary abundant number.
Proof. The proof involves a simple set of calculations.

\[
\sigma(N) - \sigma^*(N) = \frac{3^4 - 1}{2} \cdot \frac{5^3 - 1}{4} \cdot \frac{7^3 - 1}{6} - (3^3 + 1)(5^2 + 1)(7^2 + 1) \\
= 70680 - 36400 \\
= 34280 > 33075 \\
= N \quad \blacksquare
\]

Hence, we have found one odd nonunitary abundant number. It is the case that a fairly large infinite family of such numbers can be exhibited.

**Theorem 2.** Let \( N = 3^m \cdot 5^l \cdot 7^k \) with \( m \geq 3, l \geq 2, \) and \( k \geq 2 \). Then \( N \) is an odd nonunitary abundant number.

**Proof.** We begin by noting that for any prime \( q \),

\[
\frac{q^{s+1} - 1}{q - 1} \geq \frac{q^{s-a}(q^{a+1} - 1)}{q - 1} \quad (1)
\]

and

\[
q^a + 1 \leq q^{s-a}(q^a + 1) \quad (2)
\]

provided that \( s \) and \( a \) are natural numbers and \( s - a \geq 0 \). Thus,

\[
\sigma(3^m 5^l 7^k) - \sigma^*(3^m 5^l 7^k) \\
= \frac{3^{m+1} - 1}{2} \cdot \frac{5^{l+1} - 1}{4} \cdot \frac{7^{k+1} - 1}{6} - (3^m + 1)(5^l + 1)(7^k + 1) \\
\geq \frac{3^{m-3}(3^4 - 1)}{2} \cdot \frac{5^{l-2}(5^3 - 1)}{4} \cdot \frac{7^{k-2}(7^3 - 1)}{6} \\
\quad - 3^{m-3}(3^4 + 1) \cdot 5^{l-2}(5^2 + 1) \cdot 7^{k-2}(7^2 + 1) \\
\quad \text{using (1) and (2) repeatedly} \\
= 3^{m-3} 5^{l-2} 7^{k-2} \left[ \sigma(3^3 5^2 7^2) - \sigma^*(3^3 5^2 7^2) \right] \\
> 3^{m-3} 5^{l-2} 7^{k-2} [3^3 5^2 7^2] \quad \text{via Theorem 1} \\
= 3^m 5^l 7^k. \quad \blacksquare
\]

One final generalization is worth noting.

**Theorem 3.** Let \( N = 3^m \cdot 5^l \cdot 7^k \cdot p_1^{a_1} \cdot p_2^{a_2} \cdots p_r^{a_r} \) where the \( p_i \)'s are distinct, odd primes greater than 7, \( m \geq 3, l \geq 2, k \geq 2, \) and \( a_i \geq 1 \) for \( i = 1, 2, \ldots, r \) with \( r \in \mathbb{N} \). Then \( N \) is an odd nonunitary abundant number.

**Proof.** We need to show \( \sigma(N) - \sigma^*(N) > N \).
We see that
\[ \sigma(N) = \frac{3^{m+1} - 1}{2} \cdot \frac{5^{l+1} - 1}{4} \cdot \frac{7^{k+1} - 1}{6} \cdot \prod_{i=1}^{r} \frac{p_i^{a_i+1} - 1}{p_i - 1} \]
and
\[ \sigma^*(N) = (3^m + 1)(5^l + 1)(7^k + 1) \prod_{i=1}^{r} (p_i^{a_i} + 1). \]
Next we have
\[ \prod_{i=1}^{r} \frac{p_i^{a_i+1} - 1}{p_i - 1} = \prod_{i=1}^{r} (p_i^{a_i} + p_i^{a_i-1} + \cdots + p_i + 1) \geq \prod_{i=1}^{r} (p_i^{a_i} + 1) \text{ for each prime } p_i. \]
Hence,
\[ \sigma(N) - \sigma^*(N) \geq \left[ \prod_{i=1}^{r} (p_i^{a_i} + 1) \right] \left[ \sigma(3^m5^l7^k) - \sigma^*(3^m5^l7^k) \right] \]
\[ > \left[ \prod_{i=1}^{r} p_i^{a_i} \right] \left[ \sigma(3^m5^l7^k) - \sigma^*(3^m5^l7^k) \right]. \]  (**)
Therefore,
\[ \sigma(N) - \sigma^*(N) > \left[ \prod_{i=1}^{r} p_i^{a_i} \right] \left[ \sigma(3^m5^l7^k) - \sigma^*(3^m5^l7^k) \right] \text{ by (**)} \]
\[ > \left[ \prod_{i=1}^{r} p_i^{a_i} \right] [3^m5^l7^k] \text{ by (*)} \]
\[ = N. \]
This completes the proof of Theorem 3. ■

These three theorems clearly satisfy the request of Sastry concerning the existence of odd nonunitary abundant numbers. Certainly, a classification of all odd nonunitary abundant numbers is desirable. Theorem 3 may be a beginning to such a task.

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References
