On the Dimension of the Space of Magic Squares Over a Field

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Abstract

We study various spaces of magic squares over a field, and determine their dimensions. These results generalize the main result of Small from 1988.

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1. Introduction

A magic square of order \( n \) over a field \( F \) is an \( n \times n \) matrix with entries in \( F \) with the property that every row, every column, and the two main diagonals all have the same sum, called the magic sum. The set of all such magic squares over a given field \( F \) forms a vector space. In [4], Small proved that for \( n \geq 5 \), the dimension of this space of magic squares of order \( n \) is \( n^2 - 2n \), independent of the field \( F \). For \( n < 5 \), the results depend upon the characteristic of the field and are summarized in [4, page 622].

This definition of a magic square differs from the more traditional definition in which a magic square of order \( n \) is an \( n \times n \) square which uses each of the numbers \( 0, 1, \ldots, n^2 - 1 \) exactly once and for which each row, each column, and each of the two main diagonals has the constant sum \( n(n^2 - 1)/2 \), called the magic sum. Sometimes the square is based on the symbols \( 1, 2, \ldots, n^2 \), in which case the magic sum is \( n(n^2 + 1)/2 \); see [1, pages 524-528] for properties and methods of construction for various kinds magic squares. Thompson [5] considers multiplicative properties of sets of magic squares (under normal matrix multiplication).

Let \( F \) be a field, \( t \in F \), \( n \) a positive integer, and \( 0 \leq k \leq n \). Let \( \mathcal{M}_{n,k}(t) \) be the set of all \( n \times n \) matrices \( [a_{ij}]_{0 \leq i,j \leq n-1} \) over \( F \) satisfying the following...
conditions:
\[
\begin{align*}
\sum_{j=0}^{n-1} a_{ij} &= t \quad \text{for all } 0 \leq i \leq n-1 \quad \text{(row sums)} \\
\sum_{i=0}^{n-1} a_{ij} &= t \quad \text{for all } 0 \leq j \leq n-1 \quad \text{(column sums)} \\
\sum_{j+i=2t-1} a_{ij} &= t \quad \text{for all } 0 \leq l \leq k-1 \quad \text{(antidiagonal sums)} \\
\sum_{j-i=2l} a_{ij} &= t \quad \text{for all } 0 \leq l \leq k-1 \quad \text{(diagonal sums)}
\end{align*}
\]

where the subscripts are taken modulo \( n \). Define \( \mathcal{M}_{n,k} = \bigcup_{t \in F} \mathcal{M}_{n,k}(t) \).

Thus when \( k = 0 \), we have a square which is magic for all rows and columns but not necessarily for the two main diagonals. When \( k = 1 \), we have a magic square in the sense of Small [4]; and when \( k = n \), we have a square in which each row, each column, and each of the \( 2n \) wrap around diagonals has the same sum. Following terminology from Latin squares, such a square might be called a pandiagonal magic square.

The problem which we wish to address in this paper is easily stated – for any field \( F \) and any \( 0 \leq k \leq n \), determine \( \dim_F \mathcal{M}_{n,k} \). It is clear that

\[
\dim_F \mathcal{M}_{n,k} = \dim_F \mathcal{M}_{n,k}(0) + \begin{cases} 
1 & \text{if } \mathcal{M}_{n,k}(1) \neq \emptyset, \\
0 & \text{if } \mathcal{M}_{n,k}(1) = \emptyset.
\end{cases}
\]

Our main results provide the determination of \( \dim_F \mathcal{M}_{n,k}(0) \). The main theorems are stated in Section 2 and proved in Section 3. The remaining question is when \( \mathcal{M}_{n,k}(1) \neq \emptyset \). We do not have the complete answer to the question. In Section 4, we include some sufficient conditions for \( \mathcal{M}_{n,n}(1) \) to be nonempty.

2. Statement of main results

In this section, we give the main results of our paper. We break the results into two theorems based on whether the characteristic of \( F \), which we denote by \( \text{char } F \), divides \( n \) or does not divide \( n \).

**Theorem 2.1.** Let \( n \geq 0 \) and let \( F \) be a field such that \( \text{char } F \nmid n \). Also, define \( \alpha(n) \) by
\[
\alpha(n) := \begin{cases} 
3 & \text{if } 2 \nmid n, \\
4 & \text{if } 2 \mid n.
\end{cases}
\]

Then
\[
\dim \mathcal{M}_{n,k}(0) = \begin{cases} 
n^2 - 4n + \alpha(n) & \text{if } n-1 \leq k \leq n, \\
n^2 - 2n - 2k + 1 & \text{if } 0 \leq k \leq n-2.
\end{cases}
\]
Theorem 2.2. Let \( n \geq 0 \) and let \( F \) be a field such that \( \text{char } F \mid n \). Also, define \( \beta_1(n) \) and \( \beta_2(n) \) by

\[
\beta_1(n) := \begin{cases} 
6 & \text{if } 2 \nmid n, \\
5 & \text{if } 2 \mid n \text{ and } \text{char } F \nmid 2, \\
7 & \text{if } 2 \mid n \text{ and } \text{char } F \mid 2 
\end{cases}
\]

and

\[
\beta_2(n) := \begin{cases} 
7 & \text{if } 2 \nmid n \text{ or } \text{char } F \nmid 2, \\
8 & \text{if } 2 \mid n \text{ and } \text{char } F \mid 2 
\end{cases}
\]

Then

\[
\dim \mathcal{M}_{n,k}(0) = \begin{cases} 
n^2 - 4n + \beta_1(n) & \text{if } n - 2 \leq k \leq n, \\
n^2 - 4n + \beta_2(n) & \text{if } k = n - 3, \\
n^2 - 2n - 2k + 1 & \text{if } 0 \leq k \leq n - 4.
\end{cases}
\]

3. Proofs of Theorems 2.1 and 2.2

We begin with the proof of Theorem 2.1.

Proof of Theorem 2.1

Throughout this proof, let \( F \) be a field such that \( \text{char } F \nmid n \). Let \( E_{ij} \in M_{n \times n}(F) \) be the matrix whose \((i,j)\) entry is 1 and whose other entries are 0. Put

\[
A_i = \sum_{j=0}^{n-1} E_{ij}, \quad B_j = \sum_{i=0}^{n-1} E_{ij}, \quad C_l = \sum_{j+i=l-1} E_{ij}, \quad D_l = \sum_{j=i-l} E_{ij}.
\]

Endow \( M_{n \times n}(F) \) with the standard inner product

\[
\langle [a_{ij}], [b_{ij}] \rangle = \sum_{i,j} a_{ij}b_{ij}.
\]

Then clearly,

\[
\mathcal{M}_{n,k}(0) = \{A_0, \ldots, A_{n-1}, B_0, \ldots, B_{n-1}, C_0, \ldots, C_{k-1}, D_0, \ldots, D_{k-1}\}^\perp.
\]

Thus

\[
\dim \mathcal{M}_{n,k}(0) = n^2 - \dim(A_0, \ldots, A_{n-1}, B_0, \ldots, B_{n-1}, C_0, \ldots, C_{k-1}, D_0, \ldots, D_{k-1}),
\]

where \( \langle \cdots \rangle \) denotes linear span.

Consider the equation

\[
\sum_{i=0}^{n-1} [a(i)A_i + b(i)B_i + c(i)C_i + d(i)D_i] = 0,
\]

(2)
where \(a, b, c, d\) are functions from \(\mathbb{Z}_n\) to \(F\) to be determined. Entry wise, equation (2) is equivalent to
\[
a(i) + b(j) + c(-j - i - 1) + d(j - i) = 0, \quad i, j \in \mathbb{Z}_n. \tag{3}
\]
Let
\[
S_n = \{(a, b, c, d) : a, b, c, d : \mathbb{Z}_n \to F \text{ satisfy (3)}\},
\]
and
\[
S_{n,k} = \{(a, b, c, d) \in S_n : c(i) = d(i) = 0 \text{ for } k \leq i < n\}.
\]
Then
\[
\text{dim}(A_0, \ldots, A_{n-1}, B_0, \ldots, B_{n-1}, C_0, \ldots, C_{k-1}, D_0, \ldots, D_{k-1}) = 2n + 2k - \text{dim} S_{n,k}. \tag{4}
\]
By (1) and (4),
\[
\text{dim} M_{n,k}(0) = n^2 - 2n - 2k + \text{dim} S_{n,k}. \tag{5}
\]
Now we see that the essential question is to solve the functional equation (3).

**Lemma 3.1.** For \((a, b, c, d) \in S_n\), the functions \(a, b, c,\) and \(d\) satisfy the following: \(a(i) = \alpha\) for all \(i \in \mathbb{Z}_n\), where \(\alpha \in F\), \(b(j) = \beta\) for all \(j \in \mathbb{Z}_n\), where \(\beta \in F\), and
\[
c(-j - 2i - 1) + d(j) = -\alpha - \beta. \tag{6}
\]

**Proof of Lemma 3.1.** Let \((a, b, c, d) \in S_n\). By (3),
\[
a(i) = - \sum_{j \in \mathbb{Z}_n} [b(j) + c(-j - i - 1) + d(j - i)]
\]
\[
= - \sum_{j \in \mathbb{Z}_n} [b(j) + c(j) + d(j)].
\]
So \(a\) is a constant function: \(a(i) = \alpha\) for all \(i \in \mathbb{Z}_n\), where \(\alpha \in F\). In the same way, \(b(j) = \beta\) for all \(j \in \mathbb{Z}_n\), where \(\beta \in F\). By (3),
\[
c(-j - 2i - 1) + d(j) = -\alpha - \beta.
\]
\(\square\)

At this stage, we break our proof into two lemmas which depend on the parity of \(n\).

**Lemma 3.2.** If \(2 \nmid n\), then
\[
\text{dim} M_{n,k}(0) = \begin{cases} n^2 - 4n + 3 & \text{if } k = n, \\ n^2 - 2n - 2k + 1 & \text{if } 0 \leq k \leq n - 1. \end{cases}
\]
Proof of Lemma 3.2. By (6),
\[ nd(j) = -n(\alpha + \beta) - \sum_{i \in \mathbb{Z}_n} c(-j - 2i - 1) = -(\alpha + \beta) - \sum_{i \in \mathbb{Z}_n} c(i). \]
Thus \( d(j) = \delta \) for all \( j \in \mathbb{Z}_n \), where \( \delta \in F \). Hence the solutions of (3) are
\[
\begin{align*}
  a(i) &= \alpha, \\
  b(i) &= \beta, \\
  c(i) &= \gamma, \\
  d(i) &= \delta,
\end{align*}
\]
where \( \alpha, \beta, \gamma, \delta \in F \) and \( \alpha + \beta + \gamma + \delta = 0 \). When \( 0 \leq k \leq n - 1 \), the solutions of (3) in \( S_{n,k} \) are
\[
\begin{align*}
  a(i) &= \alpha, \\
  b(i) &= \beta, \\
  c(i) &= 0, \\
  d(i) &= 0,
\end{align*}
\]
where \( \alpha + \beta = 0 \). It is clear that
\[
\dim S_{n,k} = \begin{cases} 
  3 & \text{if } k = n, \\
  1 & \text{if } 0 \leq k \leq n - 1.
\end{cases}
\]
Thus equation (5) completes the proof.

\[
\square
\]

Lemma 3.3. If \( 2 \mid n \), then
\[
\dim M_{n,k}(0) = \begin{cases} 
  n^2 - 4n + 4 & \text{if } k = n \text{ or } n - 1, \\
  n^2 - 2n - 2k + 1 & \text{if } 0 \leq k \leq n - 2.
\end{cases}
\]

Proof of Lemma 3.3. By (6),
\[ nd(j) = -n(\alpha + \beta) - \sum_{i \in \mathbb{Z}_n} c(-j - 2i - 1) = -(\alpha + \beta) - 2 \sum_{i \equiv j - 1 \pmod 2} c(i). \]
So we have
\[
\begin{align*}
  d(j) &= \delta_0 \quad \text{if } j \equiv 0 \pmod 2, \\
  d(j) &= \delta_1 \quad \text{if } j \equiv 1 \pmod 2,
\end{align*}
\]
where \( \delta_0, \delta_1 \in F \). Therefore the solutions of (3) are
\[
\begin{align*}
  a(i) &= \alpha, \\
  b(i) &= \beta, \\
  c(i) &= \begin{cases} \gamma_0 & \text{if } i \equiv 0 \pmod 2, \\
  \gamma_1 & \text{if } i \equiv 1 \pmod 2, \end{cases} \\
  d(i) &= \begin{cases} \delta_0 & \text{if } i \equiv 0 \pmod 2, \\
  \delta_1 & \text{if } i \equiv 1 \pmod 2, \end{cases}
\end{align*}
\]
where \( \alpha, \beta, \gamma_0, \gamma_1, \delta_0, \delta_1 \in F \) and \( \alpha + \beta + \gamma_0 + \delta_1 = 0, \alpha + \beta + \gamma_1 + \delta_0 = 0 \).

When \( k = n - 1 \), the solutions of (3) in \( S_{n,k} = S_{n,n-1} \) are

\[
\begin{align*}
a(i) &= \alpha, \\
b(i) &= \beta, \\
c(i) &= \begin{cases} \\
g_0 & \text{if } i \equiv 0 \pmod{2}, \\ 0 & \text{if } i \equiv 1 \pmod{2}, \end{cases} \\
d(i) &= \begin{cases} \\
d_0 & \text{if } i \equiv 0 \pmod{2}, \\ 0 & \text{if } i \equiv 1 \pmod{2}, \end{cases}
\end{align*}
\]

where \( \alpha + \beta + \gamma_0 = 0, \alpha + \beta + \delta_0 = 0 \). When \( 0 \leq k \leq n - 2 \), the solutions of (3) in \( S_{n,k} \) are

\[
\begin{align*}
a(i) &= \alpha, \\
b(i) &= \beta, \\
c(i) &= 0, \\
d(i) &= 0,
\end{align*}
\]

where \( \alpha + \beta = 0 \). It is easy to see that

\[
\dim S_{n,k} = \begin{cases} \\
4 & \text{if } k = n, \\ 2 & \text{if } k = n - 1, \\ 1 & \text{if } 0 \leq k = n - 2. \end{cases}
\]

Thus (5) gives

\[
\dim \mathcal{M}_{n,k}(0) = \begin{cases} \\
n^2 - 4n + 4 & \text{if } k = n \text{ or } n - 1, \\ n^2 - 2n - 2k + 1 & \text{if } 0 \leq k \leq n - 2. \end{cases}
\]

Combining Lemmas 3.2 and 3.3 completes the proof of Theorem 2.1.

We now proceed to a proof of Theorem 2.2. Thus, we let \( F \) be a field such that \( \text{char } F \mid n \).

**Lemma 3.4.** For \( (a,b,c,d) \in S_n \), we have

\[
(\Delta^2 d)(j) = (\Delta^2 c)(-j + 2i), \quad i,j \in \mathbb{Z}_n,
\]

where \( (\Delta c)(i) = c(i+1) - c(i) \).

**Proof of Lemma 3.4.** Let \( f : \mathbb{Z}_n \times \mathbb{Z}_n \to F \) be a function. The equation

\[
a(i) + b(j) = f(i,j), \quad i,j \in \mathbb{Z}_n
\]

has a solution \( a,b : \mathbb{Z}_n \to F \) if and only if

\[
f(i,j) - f(i+1,j) - f(i,j+1) + f(i+1,j+1) = 0 \quad \text{for all } i,j \in \mathbb{Z}_n.
\]
When (9) is satisfied, the solutions of (8) are
\[
\begin{align*}
  a(i) &= f(i, 0) + \sigma, \\
  b(i) &= f(0, i) + \tau,
\end{align*}
\]
\[i \in \mathbb{Z}_n,\]
where \(\sigma + \tau = -f(0, 0)\).

Assume that \((a, b, c, d) \in S_n\). Put
\[f(i, j) = -c(-j - i - 1) - d(j - i).\]
Then (9) becomes
\[
0 = f(i, j) - f(i + 1, j) - f(i, j + 1) + f(i + 1, j + 1)
= -c(-j - i - 1) + 2c(-j - i - 2) - c(-j - i - 3)
+ d(j - i + 1) - 2d(j - i) + d(j - i - 1)
= (\Delta^2 d)(j - i - 1) - (\Delta^2 c)(-j - i - 3),
\]
where \((\Delta c)(i) = c(i + 1) - c(i)\). The above equation is equivalent to
\[
(\Delta^2 d)(j) = (\Delta^2 c)(-j + 2i), \quad i, j \in \mathbb{Z}_n.
\]
\[\square\]

As with the proof of Theorem 2.1, we now break our proof into two lemmas which depend on the parity of \(n\).

**Lemma 3.5.** If \(2 \nmid n\), then
\[
\dim M_{n,k}(0) = \begin{cases} 
  n^2 - 4n + 6 & \text{if } n - 2 \leq k \leq n, \\
  n^2 - 2n - 2k + 1 & \text{if } 0 \leq k \leq n - 3.
\end{cases}
\]

**Proof of Lemma 3.5.** By (7) we have
\[(\Delta^2 d)(j) = (\Delta^2 c)(j) = \alpha \quad \text{for all } j \in \mathbb{Z}_n,\]
where \(\alpha \in F\). Thus
\[
\begin{align*}
  c(j) &= \frac{\alpha}{2} j^2 + \beta j + \gamma, \\
  d(j) &= \frac{\alpha}{2} j^2 + \beta' j + \gamma',
\end{align*}
\]
where \(\beta, \beta', \gamma, \gamma' \in F\). By (10), (11) and (13),
\[
\begin{align*}
  a(i) &= f(i, 0) + \sigma = -\alpha^2 + (-\alpha + \beta + \beta')i - \frac{\alpha}{2} + \beta - \gamma - \gamma' + \sigma, \\
  b(i) &= f(0, i) + \tau = -\alpha^2 + (-\alpha + \beta - \beta')i - \frac{\alpha}{2} + \beta - \gamma - \gamma' + \tau,
\end{align*}
\]
where
\[
\sigma + \tau = -f(0, 0) = \frac{\alpha}{2} - \beta + \gamma + \gamma'.
\]
Thus the solutions of (3) are
\[
\begin{align*}
a(i) &= -\alpha i^2 + (-\alpha + \beta + \beta')i + \sigma', \\
b(i) &= -\alpha i^2 + (-\alpha + \beta - \beta')i + \tau', \\
c(i) &= \frac{\alpha}{2} i^2 + \beta i + \gamma, \\
d(i) &= \frac{\alpha}{2} i^2 + \beta' i + \gamma',
\end{align*}
\]
i ∈ ℤₙ, (14)
where \(\alpha, \beta, \beta', \gamma, \gamma' ∈ ℱ\) and \(\sigma' + \tau' = -\frac{\alpha}{2} + \beta - \gamma - \gamma'\).

The \(ℱ\)-map
\[
φ : \{ (\alpha, \beta, \beta', \gamma, \gamma', \sigma', \tau') ∈ ℱ^7 : \sigma' + \tau' = -\frac{\alpha}{2} + \beta - \gamma - \gamma' \} \rightarrow S_n
\]
(15)
is onto with \(\ker φ = 0\). Thus
\[
\dim S_n = 6.
\]

When \(k = n - 1\), the solutions of (3) in \(S_{n,k} = S_{n,n-1}\) are given by (14) subject to the conditions
\[
\begin{align*}
\sigma' + \tau' &= -\frac{\alpha}{2} + \beta - \gamma - \gamma', \\
\frac{\alpha}{2} - \beta + \gamma &= 0, \\
\frac{\alpha}{2} - \beta' + \gamma' &= 0.
\end{align*}
\]
An argument similar to (15) gives
\[
\dim S_{n,n-1} = 4.
\]

When \(k = n - 2\), the solutions of (3) in \(S_{n,k} = S_{n,n-2}\) are given by (14) subject to the conditions
\[
\begin{align*}
\sigma' + \tau' &= -\frac{\alpha}{2} + \beta - \gamma - \gamma', \\
\frac{\alpha}{2} - \beta + \gamma &= 0, \\
\frac{\alpha}{2} - \beta' + \gamma' &= 0, \\
2\alpha - 2\beta + \gamma &= 0, \\
2\alpha - 2\beta' + \gamma' &= 0.
\end{align*}
\]
(16)
The system (16) is equivalent to
\[
\begin{align*}
\sigma' + \tau' &= -\frac{\alpha}{2} + \beta - \gamma - \gamma', \\
\beta &= \beta' = \frac{3}{2} \alpha, \\
\gamma &= \gamma' = \alpha.
\end{align*}
\]
Then it is easy to see that
\[
\dim S_{n,n-2} = 2.
\]
When \(0 \leq k \leq n - 3\), the solutions of (3) in \(S_{n,k}\) are given by (14) subject to the conditions
\[
\begin{aligned}
\sigma' + \tau' &= 0, \\
\alpha = \beta = \beta' = \gamma = \gamma' &= 0.
\end{aligned}
\]
Thus
\[
\dim S_{n,k} = 1.
\]
Therefore we have
\[
\dim M_{n,k}(0) = \begin{cases} 
  n^2 - 4n + 6 & \text{if } n - 2 \leq k \leq n, \\
  n^2 - 2n - 2k + 1 & \text{if } 0 \leq k \leq n - 3.
\end{cases}
\]
\(\blacksquare\)

**Lemma 3.6.** If \(2 \mid n\), then
\[
\dim M_{n,k}(0) = \begin{cases} 
  n^2 - 4n + 7 & \text{if } n - 2 \leq k \leq n, \\
  n^2 - 4n + 8 & \text{if } k = n - 3, \\
  n^2 - 2n - 2k + 1 & \text{if } 0 \leq k \leq n - 4.
\end{cases}
\]

**Proof of Lemma 3.6.** By (7) we have
\[
(\Delta^2 c)(j) = (\Delta^2 d)(j) = \begin{cases} 
  \alpha_0 & \text{if } j \equiv 0 \pmod{2}, \\
  \alpha_1 & \text{if } j \equiv 1 \pmod{2},
\end{cases}
\]
where \(\alpha_0, \alpha_1 \in F\). Thus
\[
(\Delta c)(2j) = (\Delta^2 c)(0) + \cdots + (\Delta^2 c)(2j - 1) + \beta \quad (\beta = (\Delta c)(0))
\]
\[
= \alpha_0 + \alpha_1 + \cdots + \alpha_0 + \alpha_1 + \beta
\]
\[
= j\alpha_0 + j\alpha_1 + \beta,
\]
and
\[
(\Delta c)(2j + 1) = (j + 1)\alpha_0 + j\alpha_1 + \beta.
\]
Consequently,
\[
c(2j) = (\Delta c)(0) + \cdots + (\Delta c)(2j - 1) + \gamma \quad (\gamma = c(0))
\]
\[
= \begin{bmatrix}
  0\alpha_0 + 0\alpha_1 + \beta \\
  + 1\alpha_0 + 0\alpha_1 + \beta \\
  + 1\alpha_0 + 1\alpha_1 + \beta \\
  + 2\alpha_0 + 1\alpha_1 + \beta \\
  \vdots \\
  + (j - 1)\alpha_0 + (j - 1)\alpha_1 + \beta \\
  + j\alpha_0 + (j - 1)\alpha_1 + \beta + \gamma \\
\end{bmatrix}
\]
\[
= \begin{bmatrix}
  j^2\alpha_0 + j(j - 1)\alpha_1 + 2j\beta + \gamma,
\end{bmatrix}
\]
(17)
and
\[ c(2j + 1) = j(j + 1)\alpha_0 + j^2\alpha_1 + (2j + 1)\beta + \gamma. \] (18)

In the same way,
\[
\begin{cases}
    d(2j) = j^2\alpha_0 + j(j - 1)\alpha_1 + 2j\beta' + \gamma', \\
    d(2j + 1) = j(j + 1)\alpha_0 + j^2\alpha_1 + (2j + 1)\beta' + \gamma'.
\end{cases}
\] (19)

From (10), (11), (17) – (19), we have
\[
a(2i) = f(2i, 0) + \sigma \\
= -2(\alpha_0 + \alpha_1)i^2 + (-\alpha_0 - 3\alpha_1 + 2\beta + 2\beta')i - \alpha_1 + \beta - \gamma - \gamma' + \sigma,
\]
\[
a(2i + 1) = f(2i + 1, 0) + \sigma \\
= -2(\alpha_0 + \alpha_1)i^2 + (-3\alpha_0 - 5\alpha_1 + 2\beta + 2\beta')i - \alpha_0 - 3\alpha_1 + 2\beta + \beta' - \gamma - \gamma' + \sigma,
\]
\[
b(2i) = f(0, 2i) + \tau \\
= -2(\alpha_0 + \alpha_1)i^2 + (-\alpha_0 - \alpha_1 + 2\beta - 2\beta')i - \alpha_1 + \beta - \gamma - \gamma' + \tau,
\]
\[
b(2i + 1) = f(0, 2i + 1) + \tau \\
= -2(\alpha_0 + \alpha_1)i^2 + (-3\alpha_0 - 3\alpha_1 + 2\beta - 2\beta')i - \alpha_0 - 2\alpha_1 + 2\beta - \beta' - \gamma - \gamma' + \tau,
\]

where
\[ \sigma + \tau = -f(0, 0) = \alpha_1 - \beta + \gamma + \gamma'. \]

It is important to note that in order for the functions \(a, b, c, d\) obtained above to be well defined on \(\mathbb{Z}_n\), it is necessary and sufficient that
\[
\begin{align*}
    \left\{ \frac{n}{2} \left( \frac{n + 1}{2} \right) \alpha_0 + \frac{n}{2} \alpha_1 \right\} &= 0, \\
    \frac{n(n + 1)}{2} (\alpha_0 + \alpha_1) &= 0.
\end{align*}
\] (20)

Therefore the solutions of (3) are given by
\[
\begin{align*}
    a(2i) &= -2(\alpha_0 + \alpha_1)i^2 + (-\alpha_0 - 3\alpha_1 + 2\beta + 2\beta')i + \sigma', \\
    a(2i + 1) &= -2(\alpha_0 + \alpha_1)i^2 + (-3\alpha_0 - 5\alpha_1 + 2\beta + 2\beta')i - \alpha_0 - 2\alpha_1 + \beta + \beta' + \sigma', \\
    b(2i) &= -2(\alpha_0 + \alpha_1)i^2 + (-\alpha_0 - \alpha_1 + 2\beta - 2\beta')i + \tau', \\
    b(2i + 1) &= -2(\alpha_0 + \alpha_1)i^2 + (-3\alpha_0 - 3\alpha_1 + 2\beta - 2\beta')i - \alpha_0 - \alpha_1 + \beta - \beta' + \tau', \\
    c(2i) &= i^2\alpha_0 + i(i - 1)\alpha_1 + 2i\beta + \gamma, \\
    c(2i + 1) &= i(i + 1)\alpha_0 + i^2\alpha_1 + (2i + 1)\beta + \gamma, \\
    d(2i) &= i^2\alpha_0 + i(i - 1)\alpha_1 + 2i\beta' + \gamma', \\
    d(2i + 1) &= i(i + 1)\alpha_0 + i^2\alpha_1 + (2i + 1)\beta' + \gamma'.
\end{align*}
\] (21)
where \( \alpha_0, \alpha_1, \beta, \beta', \gamma, \gamma', \sigma', \tau' \in F \), \( \sigma' + \tau' = -\alpha_1 + \beta - \gamma - \gamma' \), and \( \alpha_0, \alpha_1 \) satisfy (20).

**Case 1.** Assume \( \text{char } F \not\equiv \frac{n}{2} \). Then \( \text{char } F = 2 \). From (20) we have \( \alpha_0 = \alpha_1 = 0 \). Thus (21) becomes

\[
\begin{aligned}
\begin{cases}
    a(2i) = \sigma', \\
    a(2i + 1) = \beta + \beta' + \sigma', \\
    b(2i) = \tau', \\
    b(2i + 1) = \beta - \beta' + \tau', \\
    c(2i) = \gamma, \\
    c(2i + 1) = \beta + \gamma, \\
    d(2i) = \gamma', \\
    d(2i + 1) = \beta' + \gamma', \\
\end{cases}
\end{aligned}
\]

where \( \beta, \beta', \gamma, \gamma', \sigma', \tau' \in F \) and \( \sigma' + \tau' = \beta - \gamma - \gamma' \).

The \( F \)-map

\[
\psi : \left\{ \begin{aligned}
    (\beta, \beta', \gamma, \gamma', \sigma', \tau') \in F^8 \\
    : \sigma' + \tau' = \beta - \gamma - \gamma'
\end{aligned} \right\} \longrightarrow \mathcal{S}_n
\]

is onto with \( \ker \psi = 0 \). Thus

\[ \dim \mathcal{S}_n = 5. \]

When \( k = n - 1 \), the solutions of (3) in \( \mathcal{S}_{n,k} = \mathcal{S}_{n,n-1} \) are given by (22) subject to the conditions \( \sigma' + \tau' = \beta - \gamma - \gamma' \), \( \beta + \gamma = 0 \), \( \beta' + \gamma' = 0 \). Thus

\[ \dim \mathcal{S}_{n,n-1} = 3. \]

When \( 0 \leq k \leq n - 2 \), the solutions of (3) in \( \mathcal{S}_{n,k} \) are given by (22) subject to the conditions \( \sigma' + \tau' = 0 \), \( \beta = \beta' = \gamma = \gamma' = 0 \). Thus

\[ \dim \mathcal{S}_{n,k} = 1. \]

It follows from (5) that

\[ \dim \mathcal{M}_{n,k}(0) = \begin{cases} n^2 - 4n + 5 & \text{if } k = n \text{ or } n - 1, \\
    n^2 - 2n - 2k + 1 & \text{if } 0 \leq k \leq n - 2. \end{cases} \]

**Case 2.** Assume \( \text{char } F \mid \frac{n}{2} \). Note that (20) is automatically satisfied. The \( F \)-map

\[
\Phi : \left\{ \begin{aligned}
    (\alpha_0, \alpha_1, \beta, \beta', \gamma, \gamma', \sigma', \tau') \in F^8 \\
    : \sigma' + \tau' = -\alpha_1 + \beta - \gamma - \gamma'
\end{aligned} \right\} \longrightarrow \mathcal{S}_n
\]

is onto with \( \ker \Phi = 0 \). Thus

\[ \dim \mathcal{S}_n = 5. \]
is onto with \( \ker \Phi = 0 \). So
\[
\dim \mathcal{S}_n = 7.
\]

When \( k = n - 1 \), the solutions of (3) in \( \mathcal{S}_{n,k} = \mathcal{S}_{n,n-1} \) are given by (21) subject to the conditions
\[
\begin{align*}
\sigma' + \tau' &= -\alpha_1 + \beta - \gamma - \gamma', \\
\alpha_1 - \beta + \gamma &= 0, \\
\alpha_1 - \beta' + \gamma' &= 0.
\end{align*}
\]

Therefore
\[
\dim \mathcal{S}_{n,n-1} = 5.
\]

When \( k = n - 2 \), the solutions of (3) in \( \mathcal{S}_{n,k} = \mathcal{S}_{n,n-2} \) are given by (21) subject to the conditions
\[
\begin{align*}
\sigma' + \tau' &= -\alpha_1 + \beta - \gamma - \gamma', \\
\alpha_0 &= \gamma = \gamma', \\
\alpha_1 &= \beta - \gamma, \\
\beta &= \beta'.
\end{align*}
\]

The system (23) is equivalent to
\[
\begin{align*}
\sigma' + \tau' &= -\alpha_1 + \beta - \gamma - \gamma', \\
\alpha_0 &= \gamma = \gamma', \\
\alpha_1 &= \beta - \gamma, \\
\beta &= \beta'.
\end{align*}
\]

Thus it is easy to see that
\[
\dim \mathcal{S}_{n,n-2} = 3.
\]

When \( k = n - 3 \), the solutions of (3) in \( \mathcal{S}_{n,k} = \mathcal{S}_{n,n-3} \) are given by (21) subject to the conditions
\[
\begin{align*}
\sigma' + \tau' &= -\alpha_1 + \beta - \gamma - \gamma', \\
\alpha_0 &= \beta = \beta' = \gamma = \gamma', \\
\alpha_1 &= 0.
\end{align*}
\]

Hence
\[
\dim \mathcal{S}_{n,n-3} = 2.
\]

When \( 0 \leq k \leq n - 4 \), the solutions of (3) in \( \mathcal{S}_{n,k} \) are given by (21) subject to the conditions
\[
\begin{align*}
\sigma' + \tau' &= -\alpha_1 + \beta - \gamma - \gamma', \\
\alpha_0 &= \alpha_1 = \beta = \beta' = \gamma = \gamma' = 0.
\end{align*}
\]
Thus \[ \dim S_{n,k} = 1. \]

Now (5) gives

\[
\dim \mathcal{M}_{n,k}(0) = \begin{cases} 
  n^2 - 4n + 7 & \text{if } n - 2 \leq k \leq n, \\
  n^2 - 4n + 8 & \text{if } k = n - 3, \\
  n^2 - 2n - 2k + 1 & \text{if } 0 \leq k \leq n - 4.
\end{cases}
\]

\[ \square \]

Combining Lemmas 3.5 and 3.6 yields Theorem 2.2.

4. Sufficient conditions for \( \mathcal{M}_{n,n}(1) \) to be nonempty

In [4], Small proved that

\[ \mathcal{M}_{n,1}(1) = \begin{cases} 
  \emptyset & \text{if char } F = n = 2 \text{ or char } F = n = 3, \\
  \not\emptyset & \text{otherwise}.
\end{cases} \]

It follows that \( \mathcal{M}_{n,n}(1) = \emptyset \) if char \( F = n = 2 \) or char \( F = n = 3 \). In this section, we collect some sufficient conditions for \( \mathcal{M}_{n,n}(1) \) to be nonempty.

**Fact 4.1.** If char \( F \nmid n \), then \( \mathcal{M}_{n,n}(1) \not= \emptyset \).

In fact, \[
\frac{1}{n} \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} \in \mathcal{M}_{n,n}(1).
\]

**Fact 4.2.** If \( 2 \nmid n \) and \( 3 \nmid n \), then \( \mathcal{M}_{n,n}(1) \not= \emptyset \).

To see this fact, let \( \sigma(i) = 2i \), \( i \in \mathbb{Z}_n \). Then \( \sigma, \sigma - \text{id}, \sigma + \text{id} \) are all permutations of \( \mathbb{Z}_n \). It is easy to see that the permutation matrix \( [a_{ij}] \) of \( \sigma \) belongs to \( \mathcal{M}_{n,n}(1) \), where

\[
a_{ij} = \begin{cases} 
  1 & \text{if } j = 2i, \\
  0 & \text{otherwise}.
\end{cases}
\]

**Fact 4.3.** If \( A \in \mathcal{M}_{m,m}(\alpha), B \in \mathcal{M}_{n,n}(\beta) \), where \( m, n \) are positive integers and \( \alpha, \beta \in F \), then \( A \otimes B \in \mathcal{M}_{mn,mn}(\alpha\beta) \). In particular, if \( \mathcal{M}_{m,m}(1) \not= \emptyset \) and \( \mathcal{M}_{n,n}(1) \not= \emptyset \), then \( \mathcal{M}_{mn,mn}(1) \not= \emptyset \).

We leave the proof of Fact 4.3 to the reader.

By Fact 4.1 and 4.2, \( \mathcal{M}_{n,n}(1) \) can possibly be empty only when char \( F = 2 \) or \( 3 \) and char \( F \mid n \).
Example 4.4. Assume char $F = 2$. We have $\mathcal{M}_{4,4}(1) \neq \emptyset$ since
\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\in \mathcal{M}_{4,4}(1).
\]
Interested readers may compare this example with Example (a) in [4, §2]. The matrix there belongs to $\mathcal{M}_{4,1}(1)$ but not to $\mathcal{M}_{4,2}(1)$.

References


