We’ve spent a lot of time talking about perfect numbers and their “cousins” the abundant and deficient numbers. In today’s lesson, we generalize the concept of perfect numbers by talking about amicable numbers.

Definition: Two different positive integers $m$ and $n$ are called amicable numbers if the sum of the proper divisors of $m$ equals $n$ and the sum of the proper divisors of $n$ equals $m$.

Example: Let’s look at the case $m = 220$ and $n = 284$. Let’s look at the sum of the proper divisors of 220. One way to calculate that is to find $\sigma(220) - 220$. (Remember: The sigma function adds in the number itself, so if we want to the sum of the proper divisors of 220, we have to subtract 220 away from $\sigma(220)$.) Notice that

$$\sigma(220) = \sigma(2^2 \times 5 \times 11) = \sigma(2^2) \times \sigma(5) \times \sigma(11) = 7 \times 6 \times 12 = 504$$

So $\sigma(220) - 220 = 504 - 220 = 284$ (which is the other number I wanted us to look at).

Next, let’s calculate the sum of the proper divisors of 284. That would be $\sigma(284) - 284$.

$$\sigma(284) = \sigma(2^2 \times 71) = \sigma(2^2) \times \sigma(71) = 7 \times 72 = 504$$

So $\sigma(284) - 284 = 504 - 284 = 220$.

This proves that 220 and 284 are amicable numbers! (And, by the way, this is the smallest pair of amicable numbers!)

History
The history of amicable numbers is a bit of a mystery. It is not obvious that the early Greek mathematicians had considered amicable numbers. They do not seem to appear in Euclid’s *Elements*, for example. Iamblichus (circa 250-330), a Syrian philosopher who wrote extensively about the Pythagoreans, states that Pythagoras and his followers defined amicable numbers and identified the pair (220, 284) as the first pair of amicable numbers.
In the 9th century, Arab mathematician Thabit ibn Qurra (826 – 901 AD) apparently discovered the next amicable pair, (17296, 18416). He also developed a formula which, under certain assumptions, produces other amicable pairs. In the 1600’s, Pierre Fermat rediscovered this pair, and his mathematical rival René Descartes discovered another pair, (9363584, 9437056). Then came Leonhard Euler. In 1747, he published a paper in which he spoke of the three examples above, as well as 27 more pairs!

So how did these mathematicians “find” these pairs? After all, there were no computers or calculators around at that time. To see this, we return to Thabit.

**Thabit’s Rule for Computing Amicable Numbers**

Let \( p = 3 \times 2^{n-1} - 1, \quad q = 3 \times 2^n - 1, \quad r = 9 \times 2^{2n-1} - 1 \) where \( n \) must be a number larger than 1. If \( p, \ q, \) and \( r \) are all primes (and that’s a BIG if), then the numbers \( 2^n \times p \times q \) and \( 2^n \times r \) are amicable numbers. It turns out that Fermat and Descartes “rediscovered” Thabit’s rule in 1636 and 1638 (respectively), and used it to find the new amicables mentioned above. Euler seriously generalized this rule to get his new amicable pairs.

So how does the rule work? Well, let’s plug in \( n = 2 \) and see what happens.

\[
\begin{align*}
p &= 3 \times 2^{2-1} - 1 = 3 \times 2^1 - 1 = 6 - 1 = 5 \\
q &= 3 \times 2^2 - 1 = 3 \times 4 - 1 = 12 - 1 = 11 \\
r &= 9 \times 2^{2(2)-1} - 1 = 9 \times 2^3 - 1 = 72 - 1 = 71
\end{align*}
\]

All three of these numbers are prime, so \( 2^n \times p \times q \) and \( 2^n \times r \) must be amicable. These numbers are \( 2^2 \times 5 \times 11 \) and \( 2^2 \times 71 \), the prime factorizations we saw above! These are 220 and 284 respectively.

What about trying \( n = 3 \)?
What about trying $n = 4$?

What Else Can We Say?

First of all, TONS of amicable pairs of numbers are now known, especially thanks to computer searches. Check out the following website for more information along these lines:

http://amicable.homepage.dk/knwnc2.htm

While in the above website, notice the names of the people who discovered the amicable number pairs, and notice that some of the “smaller” pairs were actually found much more recently than others.

Next, here are some other facts and conjectures we should highlight.

- In every known case, the numbers of an amicable pair are either both even or both odd. It is not known whether an even-odd pair of amicable numbers exists.

- Also, every known amicable pair shares at least one common factor greater than 1. It is not known whether a pair of relatively prime amicable numbers exists.