REFINING OVERLINED PARTS IN OVERPARTITIONS VIA RESIDUE CLASSES: BIJECTIONS, GENERATING FUNCTIONS, AND CONGRUENCES

AUGUSTINE O. MUNAGI AND JAMES A. SELLERS

Abstract. Over the past several years, numerous authors have studied properties of the combinatorial objects known as overpartitions (which are natural generalizations of integer partitions). In this paper, we consider various classes of overpartitions where the "overlined parts" belong to certain residue classes modulo a positive integer m. We state new identities between such restricted overpartitions and standard partition functions. Finally, we prove a number of Ramanujan–like congruences for many of the restricted overpartition functions using elementary generating function manipulations.

2010 Mathematics Subject Classification: 05A17, 11P83

Keywords: partition, overpartition, residue class, generating function, congruence

1. Introduction

An overpartition of a positive integer n is a partition of n in which the *first* occurrence of each part may be overlined. Overpartitions generalize ordinary partitions. For example, there are 14 overpartitions of 4:

$$(4), (\overline{4}), (3,1), (\overline{3},1), (3,\overline{1}), (\overline{3},\overline{1}), (2,2), (\overline{2},2),$$
$$(2,1,1), (\overline{2},1,1), (2,\overline{1},1), (\overline{2},\overline{1},1), (1,1,1,1), (\overline{1},1,1,1)$$

The five overpartitions with no overlined parts are the ordinary partitions of 4.

In a seminal paper, Corteel and Lovejoy [5] explored numerous aspects of the overpartition function, denoted by $\overline{p}(n)$. Further works on overpartitions have since followed quite rapidly (see, for example, [7, 8, 9, 10, 11, 14]). Some papers have placed various restrictions on the parts of overpartitions, such as limiting the number of all parts or the number of overlined

Date: November 14, 2013.

J. A. Sellers thanks the John Knopfmacher Centre for Applicable Analysis and Number Theory for their generous support which allowed him to visit the University of Witswatersrand in May 2013. It was at this time that this work was initiated.

parts [10, 11], besides other arithmetic properties. Chen, Sang and Shi [4] have noted a connection between the generating function for the number of anti-lecture hall compositions in [6] and the number of overpartitions with non-overlined parts greater than 1. This led them to study the class of overpartitions in which the non-overlined parts are not congruent to $0, \pm 1 \pmod{m}$.

In this paper we consider a systematic refinement of overpartitions by focusing on the overlined parts that belong to specified residue classes modulo a positive integer. We will state and prove certain identities between the latter and certain ordinary partition functions. We will also reveal a wealth of congruences satisfied by these special overpartitions.

We began the investigation with classes of overpartitions of n in which the first occurrence of each part $\equiv r \pmod{m}$ may be overlined, for different choices of r and m. When $r = 1$ and $m = 2$, we have the following identity.

Theorem 1.1. *The number of overpartitions of n in which the first occurrence of each odd part may be overlined equals the number of partitions of $2n$ in which odd parts occur with multiplicity 2 and even parts are unrestricted.*

A generating function proof of Theorem 1.1 will be deducible from a more general result shortly. For now we give a bijective proof. Let the enumerators of the two classes of objects in the theorem be denoted by $A_2(n)$ and $B_2(2n)$, respectively.

Start with a partition counted by $B_2(2n)$; then obtain the corresponding overpartition counted by $A_2(n)$ as follows:

- replace each even part, say $2j$, by the part j; and
- replace each pair of odd parts, say $2j+1, 2j+1$, by one overlined copy of $2j+1$.

Conversely, starting with an overpartition of weight n, any non-overlined part is doubled, and any overlined part is replaced by two copies of itself. Since the overlined parts are unique (for each part size), one is guaranteed to have odd parts appear in pairs.

The bijection is illustrated in Table 1 when $n = 4$, so that $B_2(8) = 10 = A_2(4)$.

Theorem 1.1 will be generalized to an arbitrary modulus $m > 1$ in Section 2, so that the correspondence of weights is $n \to mn$, followed with an extension to a third set of ordinary partitions of $2n$. In Section 3, we prove a similar identity for overpartitions in which the multiples of a fixed integer may be overlined. The final section, Section 4, is devoted to the exploration of numerous congruences satisfied by the overpartition functions.

(8)	$\longrightarrow$	(4)
$(6,2)$	$\longrightarrow$	$(3,1)$
$(6,1,1)$	$\longrightarrow$	$(3,\overline{1})$
$(4,4)$	$\longrightarrow$	$(2,2)$
$(4,2,2)$	$\longrightarrow$	$(2,1,1)$
$(4,2,1,1)$	$\longrightarrow$	$(2,1,\overline{1})$
$(3,3,2)$	$\longrightarrow$	$(\overline{3},1)$
$(3,3,1,1)$	$\longrightarrow$	$(\overline{3},\overline{1})$
$(2,2,2,2)$	$\longrightarrow$	$(1,1,1,1)$
$(2,2,2,1,1)$	$\longrightarrow$	$(1,1,1,\overline{1})$

TABLE 1. The bijection of Theorem 1.1 for $n = 4$

2. General Identities

In this section we prove a generalization of Theorem 1.1 and give a general composite mapping when the modulus is an odd integer.

Theorem 2.1. *Let $A_m(n)$ be the number of overpartitions of n where only parts not divisible by m may be overlined. Let $B_m(mn)$ be the number of partitions of mn where parts which are not multiples of m appear 0 or m times. Then, for all $m > 1$ and all $n > 0$, $A_m(n) = B_m(mn)$.*

Clearly, Theorem 2.1 becomes Theorem 1.1 when $m = 2$.

Proof. We first give a generating function proof. If λ is counted by $A_m(n)$, then the overlined parts of λ form a partition into distinct elements of the set $\{d \mid 0 < d \not\equiv 0 \pmod{m}\}$, and the non-overlined parts form an ordinary partition. Therefore we have the generating function

$$\sum_{n=0}^{\infty} A_m(n)q^n = \prod_{n=1}^{\infty} \frac{1+q^n}{(1-q^n)(1+q^{mn})} = \prod_{n=1}^{\infty} \frac{(1-q^{2n})(1-q^{mn})}{(1-q^n)^2(1-q^{2mn})}. \tag{1}$$

Similarly,

$$\begin{aligned}\sum_{n=0}^{\infty} B_m(mn)q^{mn} &= \prod_{n=1}^{\infty} \frac{1+q^{mn}}{(1-q^{mn})(1+q^{m(mn)})} \\ &= \prod_{n=1}^{\infty} \frac{(1-q^{2mn})(1-q^{m^2n})}{(1-q^{mn})(1-q^{mn})(1-q^{2m^2n})}.\end{aligned}$$

Lastly, replace q^m by q to obtain

$$\sum_{n=0}^{\infty} B_m(mn)q^n = \prod_{n=1}^{\infty} \frac{(1-q^{2n})(1-q^{mn})}{(1-q^n)^2(1-q^{2mn})} = \sum_{n=0}^{\infty} A_m(n)q^n.$$

Next, we define a bijection. Given a partition λ counted by $B_m(mn)$, obtain the corresponding overpartition counted by $A_m(n)$, by applying the following procedure in succession to λ:

- replace each multiple of m, say $M > 0$, by M/m; and
- replace any other part (which occurs exactly m times) with one copy of that part and overline it.

Clearly the new object has weight n. By construction, no overlined part is divisible by m. So the resulting overpartition is counted by $A_m(n)$.

To reverse the mapping, if one encounters an overlined part in a given partition, say $\overline{j}$, then replace it with $\underbrace{j, j, j, \ldots, j}_{m \text{ times}}$; and if one sees a non-overlined part, say k, then just replace it with the part km. This returns a unique partition counted by $B_m(mn)$. ∎

The identity asserted in Theorem 2.1 associates certain overpartitions of n with certain ordinary partitions of mn. We now state an extension of the theorem to certain ordinary partitions of a smaller weight $2n$, achieved at the price of using only odd moduli.

Theorem 2.2. *Let d be an odd positive integer.*

- *Let $A_d(n)$ be the number of overpartitions of n where only parts not divisible by d may be overlined.*
- *Let $B_d(dn)$ be the number of partitions of dn where parts which are not multiples of d appear 0 or d times.*
- *Let $C_d(2n)$ be the number of partitions of $2n$ in which odd parts and parts that are multiples of d occur with even multiplicities, with the remaining even parts unrestricted.*

Then

$$A_d(n) = B_d(dn) = C_d(2n)$$

for all positive integers n.

Proof. In view of Theorem 2.1, it will suffice to show that $A_d(n) = C_d(2n)$. We do this in two ways - using generating functions and providing an explicit bijection. Firstly, we note that the generating function for $C_d(2n)$ is

$$\prod_{i=1}^{\infty} \frac{(1 + q^{2(2i-1)} + q^{4(2i-1)} + \cdots)(1 + q^{2(2di)} + q^{4(2di)} + \cdots)(1 - q^{2di})}{1 - q^{2i}}.$$

That is,

$$\begin{aligned}\sum_{n=0}^{\infty} C_d(2n)q^{2n} &= \prod_{i=1}^{\infty} \frac{(1-q^{2di})}{(1-q^{2i})(1-q^{4i-2})(1-q^{4di})} \\ &= \prod_{i=1}^{\infty} \frac{(1-q^{2di})(1-q^{4i})}{(1-q^{2i})^2(1-q^{4di})}.\end{aligned}$$

Replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} C_d(2n)q^n = \prod_{i=1}^{\infty} \frac{(1-q^{2i})(1-q^{di})}{(1-q^i)^2(1-q^{2di})} = \sum_{n=0}^{\infty} A_d(n)q^n.$$

Secondly, for an odd integer $d > 0$, we define a bijection between the sets of objects counted by $A_d(n)$ and $C_d(2n)$. The image of an overpartition λ is obtained by replacing each overlined part $\bar{t}$ with the single part $2t$; and then replacing each non-overlined part t with the two copies, t, t (this insures that odd parts occur with even multiplicities).

Conversely, given a partition counted by $C_d(2n)$, replace each sequence, of even length h of consecutive odd parts with $h/2$ copies of the part; and replace each even part with half its size and overline it.

Hence a (composite) bijection also exists from the set of partitions counted by $B_d(dn)$ to the set of partitions counted by $C_d(2n)$.

These bijections are illustrated in Table 2 when $n = 4$ and $d = 3$; the lists under respective enumerators correspond one-to-one under the bijections.

■

$A_d(n)$	$B_d(dn)$	$C_d(2n)$
(1, 1, 1, 1)	(3, 3, 3, 3)	(1, 1, 1, 1, 1, 1, 1, 1)
($\bar{1}$,1, 1, 1)	(1, 1, 1, 3, 3, 3)	(2, 1, 1, 1, 1, 1, 1)
(2, 1, 1)	(6, 3, 3)	(2, 2, 1, 1, 1, 1)
($\bar{2}$, 1, 1)	(2, 2, 2, 3, 3)	(4, 1, 1, 1, 1)
(2, $\bar{1}$, 1)	(6, 3, 1, 1, 1)	(2, 2, 2, 1, 1)
($\bar{2}$, $\bar{1}$, 1)	(3, 2, 2, 2, 1, 1, 1)	(4, 2, 1, 1)
(2, 2)	(6, 6)	(2, 2, 2, 2)
($\bar{2}$, 2)	(6, 2, 2, 2)	(4, 2, 2)
(3, 1)	(9, 3	(3, 3, 1, 1)
(3, $\bar{1}$)	(9, 1, 1, 1)	(3, 3, 3)
(4)	(12)	(4, 4)
($\bar{4}$)	(4, 4, 4)	(8)

Table 2. The bijections of Theorem 2.2 for $n = 4$, $d = 3$

3. Overpartitions when multiples of m may be overlined

We give a brief complementary treatment to the previous viewpoint and consider overpartitions in which only multiples (rather than non-multiples) of a given integer may be overlined.

A general identity can also be stated for such overpartitions.

Theorem 3.1. *The number of partitions of n where only multiples of $m > 1$ can be overlined at most $m-1$ times equals the number of partitions of mn in which non-multiples of m^2 occur with multiplicity divisible by m.*

Proof. Let the two classes of partitions in Theorem 3.1 be denoted by $D_m(n)$ and $E_m(n)$ respectively. Since up to $m-1$ parts of a partition counted by $D_m(n)$ may be overlined, we obtain the generating function

$$\begin{aligned}\sum_{n=0}^{\infty} D_m(n)q^n &= \prod_{n=1}^{\infty} \frac{1+q^{mn}+q^{2mn}+\cdots+q^{(m-1)mn}}{1-q^n} \\ &= \prod_{n=1}^{\infty} \frac{1-q^{m^2n}}{(1-q^n)(1-q^{mn})} \\ &= \sum_{n=0}^{\infty} E_m(n)q^n,\end{aligned}$$

which completes a generating function proof of the identity.

Lastly, we define a bijection f between the sets of objects enumerated by $D_m(n)$ and $E_m(n)$. If λ is counted by $D_m(n)$, then $f(\lambda)$ is obtained, much as in previous bijections, by replacing an overlined part $\bar{t}$ by the part mt (note that t is a multiple of m), and then by replacing a non-overlined part t by $\underbrace{t, t, \ldots, t}_{m \text{ times}}$. Then the resulting partition $f(\lambda)$ is counted by $E_m(n)$. Thus a multiple of m^2 can occur as a part of the image only from an overlined part of the pre-image.

The reverse mapping is clear. ∎

4. Congruence Properties

We can use the generating function $\sum_{n\geq 0} A_m(n)q^n$ to prove a number of congruence properties satisfied by $A_m(n)$. We first consider the generating function modulo 2. From (1), we have

$$\begin{aligned}\sum_{n\geq 0} A_m(n)q^n &= \prod_{n\geq 1}\frac{(1-q^{2n})(1-q^{mn})}{(1-q^{2mn})(1-q^n)^2}\\ &\equiv \prod_{n\geq 1}\frac{(1-q^{2n})(1-q^{mn})}{(1-q^{mn})^2(1-q^{2n})} \pmod 2\\ &= \prod_{n\geq 1}\frac{1}{(1-q^{mn})} \pmod 2.\end{aligned}$$

But this is a function of q^m, which means that if n is not a multiple of m, then the coefficient of q^n on the right-hand side of this congruence is zero. Hence we have proved our first theorem related to the parity of $A_m(n)$:

Theorem 4.1. *Let $m \geq 2$ be fixed. For any $n \geq 1$, if $m \nmid n$, then $A_m(n) \equiv 0 \pmod 2$.*

Numerous corollaries which involve Ramanujan–like congruences satisfied by $A_m(n)$ now follow. We state just a few.

Corollary 4.2. *For any $k \geq 1$ and all $n \geq 0$, $A_{2k}(2n+1) \equiv 0 \pmod 2$.*

Corollary 4.3. *For any $j \geq 1$ and all $n \geq 0$, each of the following congruences holds:*

$$\begin{aligned}A_{2^j}(2n+1) &\equiv 0 \pmod 2,\\ A_{2^j}(4n+2) &\equiv 0 \pmod 2,\\ &\vdots\\ A_{2^j}(2^jn+2^{j-1}) &\equiv 0 \pmod 2.\end{aligned}$$

Corollary 4.4. *Let p be an odd prime which divides m. Then, for all $n \geq 0$ and r such that $1 \leq r \leq p-1$, we have*

$$A_m(pn+r) \equiv 0 \pmod 2.$$

While the above parity results are satisfying, we desire to prove additional congruences for moduli larger than 2. In order to do so, we return to the generating function for $A_m(n)$ and analyze it further.

Note that

$$\begin{aligned}\sum_{n\geq 0} A_m(n)q^n &= \prod_{n\geq 1}\frac{(1-q^{mn})}{(1-q^{2mn})} \times \prod_{n\geq 1}\frac{(1-q^{2n})}{(1-q^n)^2}\\ &= \prod_{n\geq 1}\frac{(1-q^{mn})}{(1-q^{2mn})} \times \frac{1}{D(q)}\end{aligned}$$

where

$$D(q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \prod_{n\geq 1} \frac{(1-q^n)^2}{(1-q^{2n})}. \tag{2}$$

We now consider congruences for the function $A_m(pn+r)$ where p is an odd prime and r is a quadratic nonresidue modulo p. If $p \mid m$, then the function

$$\prod_{n\geq 1} \frac{(1-q^{mn})}{(1-q^{2mn})}$$

can be ignored because it is a function of q^m and we are considering only what happens on the arithmetic progression $pn+r$ where $p \mid m$. Thus, we really only need to focus our attention on $\dfrac{1}{D(q)}$.

As noted in [9, Lemma 2.11], we know

$$\frac{1}{D(q)} = \frac{\varphi(q)}{D(q^2)^2}, \tag{3}$$

where

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + 2\sum_{n=1}^{\infty} q^{n^2}$$

is one of Ramanujan's famous theta functions. Iteration of (3) ad infinitum implies

$$\frac{1}{D(q)} = \varphi(q) \times \varphi(q^2)^2 \times \varphi(q^4)^4 \dots \tag{4}$$

Thus, in order to understand $A_m(pn+r)$ modulo powers of 2 where p is an odd prime, r is a quadratic nonresidue modulo p, and $p \mid m$, we simply need to analyze (4) modulo powers of 2.

Since

$$\varphi(q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2},$$

it is clear that $\varphi(q^{2^i})^{2^i} \equiv 0 \pmod{4}$ for all $i \geq 1$. Taking into account all that has been discussed so far, we see that

$$\begin{aligned} \sum_{n\geq 0} A_m(pn+r) q^{pn+r} &\equiv \varphi(q) \pmod{4} \\ &\equiv 1 + 2\sum_{n\geq 1} q^{n^2} \pmod{4}. \end{aligned}$$

Since r is a quadratic nonresidue modulo p, we know that $pn+r \neq m^2$ for any n (by simply considering such an equality modulo p together with the definition of quadratic nonresidues). Therefore, the coefficient of q^{pn+r} on the left–hand side of the above, namely $A(pn + r, m)$, must be congruent to 0 modulo 4.

We can summarize the above in the following theorem.

Theorem 4.5. *Let p be an odd prime dividing m and let r be a quadratic nonresidue modulo p. Then, for all $n \geq 0$, $A_m(pn + r) \equiv 0 \pmod 4$.*

For each prime p, Theorem 4.5 provides $(p - 1)/2$ congruences modulo 4. Interestingly the above analysis also leads to a family of congruences modulo 8.

Theorem 4.6. *Let $p \equiv \pm 1 \pmod 8$ be an odd prime dividing m and let r be a quadratic nonresidue modulo p. Then, for all $n \geq 0$, $A_m(pn + r) \equiv 0 \pmod 8$.*

Proof. Since $\varphi(q^{2^i})^{2^i} \equiv 0 \pmod 8$ for all $i \geq 2$, we have

$$\begin{aligned}
&\sum_{n\geq 0} A_m(pn + r)q^{pn+r} \\
\equiv\ & \varphi(q)\varphi(q^2)^2 \pmod 8 \\
=\ & (1 + 2\sum_{n\geq 1} q^{n^2})(1 + 2\sum_{n\geq 1} q^{2n^2})^2 \\
\equiv\ & 1 + 2\sum_{n\geq 1} q^{n^2} + 4\sum_{n\geq 1} q^{2n^2} + 4\sum_{n\geq 1} q^{4n^2} \pmod 8 \\
\equiv\ & 1 + 2\sum_{n \text{ odd}} q^{n^2} + 6\sum_{n \text{ even}} q^{n^2} + 4\sum_{n\geq 1} q^{2n^2} \pmod 8.
\end{aligned}$$

We already know that $pn + r \neq m^2$ for any m given that r is a quadratic nonresidue modulo the prime p. But we also know that $pn + r \neq 2m^2$ for any m because the Legendre symbol $\left(\frac{2}{p}\right)$ equals 1 precisely when $p \equiv \pm 1 \pmod 8$. Thus, no terms of the form q^{pn+r} will appear on the right–hand side of the congruence above. Therefore, $A_m(pn + r) \equiv 0 \pmod 8$ under the hypotheses of the theorem. ∎

We next consider congruences modulo 3 which are satisfied by specific functions in this family.

Theorem 4.7. *For all $n \geq 0$, $A_3(27n + 26) \equiv 0 \pmod 3$.*

Proof. We begin with the following generating function manipulations:

$$\begin{aligned}
\sum_{n\geq 0} A_3(n)q^n &= \prod_{n\geq 1}\frac{(1-q^{2n})(1-q^{3n})}{(1-q^{6n})(1-q^n)^2} \\
&\equiv \prod_{n\geq 1}\frac{(1-q^{2n})(1-q^{n})^3}{(1-q^{2n})^3(1-q^n)^2} \pmod 3 \\
&= \prod_{n\geq 1}\frac{(1-q^{n})}{(1-q^{2n})^2} \\
&= \frac{1}{\psi(q)}
\end{aligned}$$

where

$$\psi(q) = \sum_{n\geq 0} q^{n(n+1)/2} = \prod_{n\geq 1}\frac{(1-q^{2n})^2}{(1-q^n)}$$

is another of Ramanujan's theta functions. Thanks to properties of the triangular numbers, it is clear that

$$\psi(q) = H(q^3) + q\psi(q^9) \tag{5}$$

where

$$H(q) = 1 + q^1 + q^2 + q^5 + q^7 + q^{12} + q^{15} + \dots, \tag{6}$$

where the exponents on q in $H(q)$ are the pentagonal numbers $n(3n-1)/2$ where n is any integer. We now exploit this representation of $\psi(q)$ to prove our result.

Continuing the work above, we have the following:

$$\begin{aligned}
\sum_{n\geq 0} A_3(n)q^n &\equiv \frac{1}{\psi(q)} \pmod 3 \\
&= \frac{\psi(q)^2}{\psi(q)^3} \\
&\equiv \frac{\psi(q)^2}{\psi(q^3)} \pmod 3 \\
&= \frac{(H(q^3)+q\psi(q^9))^2}{\psi(q^3)} \\
&= \frac{H(q^3)^2 + 2qH(q^3)\psi(q^9) + q^2\psi(q^9)^2}{\psi(q^3)}.
\end{aligned}$$

Thus,

$$\sum_{n\geq 0} A_3(3n+2)q^{3n+2} \equiv \frac{q^2\psi(q^9)^2}{\psi(q^3)} \pmod 3$$

or

$$\sum_{n\geq 0} A_3(3n+2)q^n \equiv \frac{\psi(q^3)^2}{\psi(q)} \pmod 3.$$

We now further dissect this generating function modulo 3.

$$\begin{aligned}\sum_{n\geq 0} A_3(3n+2)q^n &\equiv \frac{\psi(q^3)^2}{\psi(q)} \pmod 3\\ &= \frac{\psi(q^3)^2\psi(q)^2}{\psi(q)^3}\\ &\equiv \frac{\psi(q^3)^2\psi(q)^2}{\psi(q^3)} \pmod 3\\ &= \psi(q^3)(H(q^3)+q\psi(q^9))^2 \text{ from comments above.}\end{aligned}$$

Therefore, selecting only those powers of the form q^{3n+2} from both sides, we obtain

$$\sum_{n\geq 0} A_3(9n+8)q^{3n+2} \equiv q^2\psi(q^3)\psi(q^9)^2 \pmod 3$$

or

$$\begin{aligned}\sum_{n\geq 0} A_3(9n+8)q^n &\equiv \psi(q)\psi(q^3)^2 \pmod 3\\ &\equiv (H(q^3)+q\psi(q^9))\psi(q^3)^2 \pmod 3.\end{aligned}$$

To complete the proof, note that there are no terms of the form q^{3n+2} on the right–hand side of the last congruence. Hence for all $n \geq 0$,

$$A_3(9(3n+2)+8) = A_3(27n+26) \equiv 0 \pmod 3.$$

∎

Before we move to our next congruence modulo 3, we note the following lemma which recently appeared as equation (1.6) in a recent paper of Yao and Xia [13] and is easily proven using Jacobi's Triple Product Identity [1, Theorem 2.8].

Lemma 4.8. *Let* $H(q) = \sum_{n=-\infty}^{\infty} q^{n(3n-1)/2}$, *as defined in (6). Then*

$$H(q) = \prod_{n\geq 1} \frac{(1-q^{3n})^2(1-q^{2n})}{(1-q^n)(1-q^{6n})}.$$

With Lemma 4.8 in hand, we can now prove the following congruence modulo 3.

Theorem 4.9. *For all* $n \geq 0$, $A_9(27n+24) \equiv 0 \pmod 3$.

Proof. We begin with some basic manipulations of the generating function for $A_9(n)$:

$$\begin{aligned}
\sum_{n\geq 0} A_9(n)q^n &= \prod_{n\geq 1} \frac{(1-q^{2n})(1-q^{9n})}{(1-q^{18n})(1-q^n)^2} \\
&\equiv \prod_{n\geq 1} \frac{(1-q^{2n})(1-q^{n})^9}{(1-q^{2n})^9(1-q^n)^2} \pmod 3 \\
&= \prod_{n\geq 1} \frac{(1-q^{n})^7}{(1-q^{2n})^8} \\
&= \prod_{n\geq 1} \frac{(1-q^{n})}{(1-q^{2n})^2} \times \prod_{n\geq 1} \frac{(1-q^{n})^6}{(1-q^{2n})^6} \\
&\equiv \frac{1}{\psi(q)} \prod_{n\geq 1} \frac{(1-q^{3n})^2}{(1-q^{6n})^2} \pmod 3 \\
&\equiv \frac{\psi(q)^2}{\psi(q^3)} \prod_{n\geq 1} \frac{(1-q^{3n})^2}{(1-q^{6n})^2} \pmod 3 \\
&= \frac{(H(q^3)+q\psi(q^9))^2}{\psi(q^3)} \prod_{n\geq 1} \frac{(1-q^{3n})^2}{(1-q^{6n})^2} \quad \text{using (5) above}
\end{aligned}$$

Extracting powers of the form q^{3n} from both sides, we have

$$\sum_{n\geq 0} A_9(3n)q^{3n} \equiv \frac{H(q^3)^2}{\psi(q^3)} \prod_{n\geq 1} \frac{(1-q^{3n})^2}{(1-q^{6n})^2} \pmod 3$$

or

$$\sum_{n\geq 0} A_9(3n)q^{n} \equiv \frac{H(q)^2}{\psi(q)} \prod_{n\geq 1} \frac{(1-q^{n})^2}{(1-q^{2n})^2} \pmod 3.$$

From Lemma 4.8, we see that

$$\begin{aligned}
\sum_{n\geq 0} A_9(3n)q^n &\equiv \frac{1}{\psi(q)} \prod_{n\geq 1} \frac{(1-q^{2n})^2(1-q^{3n})^4(1-q^n)^2}{(1-q^{n})^2(1-q^{6n})^2(1-q^{2n})^2} \pmod 3 \\
&= \frac{1}{\psi(q)} \prod_{n\geq 1} \frac{(1-q^{3n})^4}{(1-q^{6n})^2} \\
&\equiv \frac{\psi(q)^2}{\psi(q^3)} \prod_{n\geq 1} \frac{(1-q^{3n})^4}{(1-q^{6n})^2} \pmod 3 \\
&= \frac{(H(q^3)+q\psi(q^9))^2}{\psi(q^3)} \prod_{n\geq 1} \frac{(1-q^{3n})^4}{(1-q^{6n})^2}.
\end{aligned}$$

Extracting powers of the form q^{3n+2} yields

$$\sum_{n\geq 0} A_9(9n+6)q^{3n+2} \equiv \frac{q^2\psi(q^9)^2}{\psi(q^3)} \prod_{n\geq 1} \frac{(1-q^{3n})^4}{(1-q^{6n})^2} \pmod 3$$

or

$$\sum_{n\geq 0} A_9(9n+6)q^{n} \equiv \frac{\psi(q^3)^2}{\psi(q)} \prod_{n\geq 1} \frac{(1-q^{n})^4}{(1-q^{2n})^2} \pmod 3.$$

But

$$\begin{aligned}
\frac{\psi(q^3)^2}{\psi(q)} \prod_{n\geq 1} \frac{(1-q^{n})^4}{(1-q^{2n})^2} &= \frac{\psi(q^3)^2}{\psi(q)^2} \prod_{n\geq 1}(1-q^n)^3 \\
&\equiv \frac{\psi(q^3)^2}{\psi(q)^2} \prod_{n\geq 1}(1-q^{3n}) \pmod 3 \\
&\equiv \frac{\psi(q^3)^2\psi(q)}{\psi(q^3)} \prod_{n\geq 1}(1-q^{3n}) \pmod 3 \\
&= \psi(q^3)\psi(q) \prod_{n\geq 1}(1-q^{3n}) \\
&= \psi(q^3)(H(q^3)+q\psi(q^9)) \prod_{n\geq 1}(1-q^{3n}).
\end{aligned}$$

Note that there are no terms of the form q^{3n+2} in the resulting term. Therefore,

$$A_9(9(3n+2)+6) = A_9(27n+24) \equiv 0 \pmod 3.$$

∎

We conclude our discussion of congruences modulo 3 by proving an infinite family of congruences. Our proof utilizes the same tools already used to prove Theorems 4.7 and 4.9.

Theorem 4.10. *For all $n \geq 0$ and all $j \geq 3$,* $A_{3^j}(27n+18) \equiv 0 \pmod 3$.

Proof. We have the following:

$$\begin{aligned}
\sum_{n\geq 0} A_{3^j}(n)q^n &= \prod_{n\geq 1} \frac{(1-q^{2n})(1-q^{3^j n})}{(1-q^{2\cdot 3^j n})(1-q^n)^2} \\
&\equiv \prod_{n\geq 1} \frac{(1-q^n)}{(1-q^{2n})^2} \times \prod_{n\geq 1} \frac{(1-q^n)^{3^j-3}}{(1-q^{2n})^{3^j-3}} \pmod 3 \\
&\equiv \frac{1}{\psi(q)} \times \prod_{n\geq 1} \frac{(1-q^{3n})^{3^{j-1}-1}}{(1-q^{6n})^{3^{j-1}-1}} \pmod 3 \\
&\equiv \frac{\psi(q)^2}{\psi(q^3)} \times \prod_{n\geq 1} \frac{(1-q^{3n})^{3^{j-1}-1}}{(1-q^{6n})^{3^{j-1}-1}} \pmod 3 \\
&= \frac{(H(q^3)+q\psi(q^9))^2}{\psi(q^3)} \prod_{n\geq 1} \frac{(1-q^{3n})^{3^{j-1}-1}}{(1-q^{6n})^{3^{j-1}-1}}
\end{aligned}$$

Extracting powers of the form q^{3n}, we have

$$\sum_{n\geq 0} A_{3^j}(3n)q^n \equiv \frac{H(q)^2}{\psi(q)} \prod_{n\geq 1} \frac{(1-q^n)^{3^{j-1}-1}}{(1-q^{2n})^{3^{j-1}-1}} \pmod 3.$$

From Lemma 4.8, we see that

$$\begin{aligned}
\sum_{n\geq 0} A_{3^j}(3n)q^n &\equiv \frac{1}{\psi(q)} \prod_{n\geq 1} \frac{(1-q^{2n})^2(1-q^{3n})^4(1-q^n)^{3^{j-1}-1}}{(1-q^n)^2(1-q^{6n})^2(1-q^{2n})^{3^{j-1}-1}} \pmod 3 \\
&= \frac{1}{\psi(q)} \prod_{n\geq 1} \frac{(1-q^{3n})^4(1-q^n)^{3^{j-1}-3}}{(1-q^{6n})^2(1-q^{2n})^{3^{j-1}-3}} \\
&\equiv \frac{\psi(q)^2}{\psi(q^3)} \prod_{n\geq 1} \frac{(1-q^{3n})^{3^{j-2}+3}}{(1-q^{6n})^{3^{j-2}+1}} \pmod 3 \\
&= \frac{(H(q^3)+q\psi(q^9))^2}{\psi(q^3)} \prod_{n\geq 1} \frac{(1-q^{3n})^{3^{j-2}+3}}{(1-q^{6n})^{3^{j-2}+1}},
\end{aligned}$$

and extracting powers of the form q^{3n} again yields

$$\begin{aligned}
\sum_{n\geq 0} A_{3^j}(9n)q^n &\equiv \frac{H(q)^2}{\psi(q)} \prod_{n\geq 1} \frac{(1-q^n)^{3^{j-2}+3}}{(1-q^{2n})^{3^{j-2}+1}} \pmod 3 \\
&= \frac{1}{\psi(q)} \prod_{n\geq 1} \frac{(1-q^{2n})^2(1-q^{3n})^4}{(1-q^n)^2(1-q^{6n})^2} \prod_{n\geq 1} \frac{(1-q^n)^{3^{j-2}+3}}{(1-q^{2n})^{3^{j-2}+1}} \\
&\equiv \frac{\psi(q)^2}{\psi(q^3)} \prod_{n\geq 1} \frac{(1-q^{2n})(1-q^{3n})^{3^{j-3}+5}}{(1-q^n)^2(1-q^{6n})^{3^{j-3}+2}} \pmod 3 \\
&= \prod_{n\geq 1} \frac{(1-q^{2n})^5(1-q^{3n})^{3^{j-3}+6}}{(1-q^n)^4(1-q^{6n})^{3^{j-3}+4}} \\
&= \psi(q) \prod_{n\geq 1} \frac{(1-q^{2n})^3(1-q^{3n})^{3^{j-3}+6}}{(1-q^n)^3(1-q^{6n})^{3^{j-3}+4}} \\
&\equiv \psi(q) \prod_{n\geq 1} \frac{(1-q^{3n})^{3^{j-3}+5}}{(1-q^{6n})^{3^{j-3}+3}} \pmod 3 \\
&= (H(q^3)+q\psi(q^9)) \prod_{n\geq 1} \frac{(1-q^{3n})^{3^{j-3}+5}}{(1-q^{6n})^{3^{j-3}+3}}.
\end{aligned}$$

Note that there are no terms of the form q^{3n+2} in the resulting term. So we obtain

$$A_{3^j}(9(3n+2)) = A_{3^j}(27n+18) \equiv 0 \pmod 3.$$

∎

Lastly, we examine some congruences for the counting functions mentioned in Theorem 3.1, starting with the generating function

$$\sum_{n\geq 0} D_m(n)q^n = \prod_{n\geq 1} \frac{(1-q^{m^2n})}{(1-q^n)(1-q^{mn})}, \quad m>1.$$

We establish two sets of congruences for specific members of this family of overpartition functions.

Theorem 4.11. *Let $p \geq 5$ be prime and let r be such that $24r+1$ is a quadratic nonresidue modulo p. Then, for all $n \geq 0$, $D_2(pn+r) \equiv 0$* (mod 2).

Proof. We have

$$\begin{aligned}\sum_{n\geq 0} D_2(n)q^n &= \prod_{n\geq 1}\frac{(1-q^{4n})}{(1-q^n)(1-q^{2n})}\\ &\equiv \prod_{n\geq 1}\frac{(1-q^{n})^4}{(1-q^n)(1-q^{n})^2} \pmod 2\\ &= \prod_{n\geq 1}(1-q^n)\\ &\equiv \sum_{m=-\infty}^{\infty} q^{m(3m-1)/2} \pmod 2,\end{aligned}$$

where the last equality follows from Euler's Pentagonal Number Theorem [1, Corollary 1.7]. Thus to find the coefficient of q^{pn+r} on the right–hand side, we need to find all solutions to the equation

$$pn+r = \frac{m(3m-1)}{2},$$

where m is any integer. That is,

$$r \equiv \frac{m(3m-1)}{2} \pmod p,$$

which can be expressed as

$$24r+1 \equiv (6m-1)^2 \pmod p.$$

But since $24r+1$ is a quadratic nonresidue modulo p, we know that this last congruence has no solutions. The theorem follows. ∎

Theorem 4.12. *For all* $n \geq 0$*,*

$$D_p(pn+r) \equiv 0 \pmod p$$

for each ordered pair $(p,r) = (5,4), (7,5), (11,6)$.

Proof. For any prime p, we have

$$\begin{aligned}\sum_{n\geq 0} D_p(n)q^n &= \prod_{n\geq 1}\frac{(1-q^{p^2n})}{(1-q^n)(1-q^{pn})}\\ &= \prod_{n\geq 1}\frac{(1-q^{p^2n})}{(1-q^{pn})}\prod_{n\geq 1}\frac{1}{(1-q^n)}.\end{aligned}$$

Now

$$\prod_{n\geq 1}\frac{(1-q^{p^2n})}{(1-q^{pn})}$$

may be ignored since it is function of q^p. Thus, we only need to concentrate on

$$\prod_{n\geq 1} \frac{1}{(1-q^n)}$$

which is the generating function for $p(n)$, the unrestricted partition function. Our theorem then follows from well–known congruences modulo 5, 7, and 11 satisfied by $p(n)$ (see, for example, [3, Chapter 2]). ∎

References

1. G. E. Andrews, *The Theory of Partitions*, Addison-Wesley, Reading 1976; reprinted, Cambridge University Press, Cambridge, 1984, 1998
2. G. E. Andrews, M. D. Hirschhorn, and J. A. Sellers, Arithmetic properties of partitions with even parts distinct, Ramanujan Journal 23 (2010), 169–181
3. B. C. Berndt, *Number Theory in the Spirit of Ramanujan*, American Mathematical Society, Providence, 2004
4. W. Y. C. Chen, D. D. M. Sang, and D. Y. H. Shi, Anti-lecture hall compositions and overpartitions, J. Comb. Theory Ser. A. 118 (2003), 1451–1464.
5. S. Corteel and J. Lovejoy, Overpartitions, Trans. Amer. Math. Soc. 356 (2004), 1623–1635
6. S. Corteel and C.D. Savage, Anti-lecture hall compositions, Discrete Math., 263 (2003), pp. 275–280.
7. M. D. Hirschhorn and J. A. Sellers, Arithmetic Properties for Overpartitions, Journal of Combinatorial Mathematics and Combinatorial Computing (JCMCC) 53 (2005), 65–73
8. M. D. Hirschhorn and J. A. Sellers, An Infinite Family of Overpartition Congruences Modulo 12, INTEGERS 5 (2005), Article A20
9. M. D. Hirschhorn and J. A. Sellers, Arithmetic Properties of Overpartitions into Odd Parts, Annals of Combinatorics 10, no. 3 (2006), 353–367
10. J. Lovejoy, Gordon's theorem for overpartitions, J. Comb. Theory Ser. A. 103 (2003), 393–401
11. J. Lovejoy, Overpartition theorems of the Rogers-Ramanujan type, J. London Math. Soc. 69 (2004), 562–574
12. *The On-Line Encyclopedia of Integer Sequences*, published electronically at www.oeis.org (2013)
13. O. X. M. Yao and E. X. W. Xia, The number of representations of an integer as a sum involving generalized pentagonal numbers, International J. Number Theory 8, no. 4 (2012), 1041–1056
14. O. X. M. Yao and E. X. W. Xia, New Ramanujan-like congruences modulo powers of 2 and 3 for overpartitions, J. Number Theory 133 (2013), no. 6, 1932–1949

The John Knopfmacher Centre for Applicable Analysis and Number Theory, University of the Witwatersrand, Johannesburg, South Africa, Augustine.Munagi@wits.ac.za

Department of Mathematics, Penn State University, University Park, PA 16802, USA, sellersj@psu.edu