Index theory and coarse geometry
Lecture 2

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Outline

1. Elliptic theory on noncompact manifolds
2. Coarse structures
3. Operator algebras
Coarse index theory begins from the idea of generalizing the index theorem to Riemannian manifolds that are \textit{complete} but \textit{noncompact}. A helpful example to bear in mind is the unit disk in $\mathbb{C}$, equipped with its Poincaré (hyperbolic) metric. It is easy to see that

- The kernel of an elliptic operator $D$ now depends on what global growth conditions we impose (e.g. there exist nonzero bounded harmonic functions on the disk, but no nonzero $L^2$ harmonic functions).
- The kernel may be infinite dimensional, even with the ‘nicest’ growth conditions (e.g. $L^2$ harmonic 1-forms on the disk.)
We will always work with $L^2$ growth conditions. Let $M$ be a complete Riemannian manifold and let $D$ be a first order differential operator (acting on sections of a hermitian vector bundle $E$). If $\langle Ds, s' \rangle = \langle s, Ds' \rangle$ for all smooth, compactly supported sections $s, s'$ we say that $D$ is formally self-adjoint.

**Definition**

The *propagation speed* of $D$ is $\sup \| \sigma_D(\xi) \|$, where the supremum is taken over all unit (co)vectors $\xi$ in $T^*M$.

The natural geometric operators (e.g. the de Rham operator $d + d^*$) have unit propagation speed.

**Theorem**

*Let $M$ be complete and $D$ be formally self-adjoint with finite propagation speed. Then $D$ is essentially self-adjoint.*
Let $D$ be as above. The essential self-adjointness of $D$ means that we can apply the spectral theorem to form an operator $f(D)$ for every $f \in C_0(\mathbb{R})$, or indeed for every bounded Borel function $f$ on $\mathbb{R}$.

**Theorem**

*The operator $D$ has locally compact resolvent, in other words the operator $f(D)$ is locally compact for all $f \in C_0(\mathbb{R})$.*

Here an operator $T$ on $L^2(M)$ is said to be *locally compact* if $TM_g$ and $M_g T$ are compact whenever $M$ is the operator of multiplication by some $g \in C_0(M)$. 


The resolvents $f(D)$ have another property which has more geometric content. To state it, it is convenient first to study the wave operators $e^{itD}$.

**Theorem**

Let $D$ be a self-adjoint elliptic operator on a complete Riemannian manifold $M$, having finite propagation speed $c$. Then the wave operator $e^{itD}$ has the property

$$\text{Support}(e^{itD}s) \subseteq N(\text{Support}(s); c|t|).$$

**Proof.**

Energy estimates.
This result gives information about a general operator of the form $f(D)$ because of the Fourier representation

$$f(D) = \frac{1}{2\pi} \int \hat{f}(t)e^{itD} \, dt.$$ 

Thus if $\hat{f}$ is compactly supported, $f(D)$ is an operator of \textit{finite propagation}: there is a constant $R$ such that

$$\text{Support}(f(D)s) \subseteq N(\text{Support}(s); cR).$$

A general $f(D)$ is a limit (in norm) of finite propagation operators.
Let $M$ be a complete Riemannian manifold and let $D$ be a self-adjoint first order differential operator on a bundle $E$ over $M$, having finite propagation speed. Let $A$ denote the $C^*$-algebra which is the norm closure of the finite propagation, locally compact operators on $H = L^2(M; E)$. The results that we have stated prove that $f(D) \in A$ for all $f \in C_0(\mathbb{R})$.

Thus, by the constructions of the previous lecture, there is a coarse index $\text{Index}(D) \in K_i(A)$, where $i = 0$ in the graded case and 1 in the ungraded case. The project of coarse index theory is to understand this index, and the group to which it belongs.
Suppose for example that $M$ is a compact manifold. Then every operator on $L^2(M)$ is of finite propagation, and the locally compact operators are just the compact ones.

The algebra $A$ is then just the algebra $\mathcal{K}$ of compact operators, and $K_0(A) = \mathbb{Z}$.

The coarse index in this case is just the usual Atiyah-Singer index.
Continue to consider the example of a compact $M$. There is a proof of the index theorem (due to Atiyah, Bott and Patodi) which fits well with the ‘coarsening’ idea. This is the heat equation proof.

- It is based on the heat equation

$$\frac{\partial s}{\partial t} + D^2 s = 0.$$ 

- The operator $e^{-tD^2}$ can be defined by the functional calculus and provides the solution operator to the above equation with prescribed initial data.
The operator $e^{-tD^2}$ is represented by a smoothing kernel

$$e^{-tD^2} s(x) = \int k_t(x, y)s(y)dy$$

where $k_t$ is a smooth function that approaches the familiar Gaussian shape as $t \downarrow 0$.

**Definition**

The *trace* of a smoothing operator on a compact manifold can be defined by integrating its kernel over the diagonal.

**Lemma**

*(McKean-Singer)* For any $t > 0$ the ‘supertrace’

$$\text{Tr}(\epsilon e^{-tD^2})$$

exists and is equal to the index of $D$. 
Let us prove the McKean-Singer formula using $K$-theory.

1. The $K$-theory index is $[-S \epsilon S] \ominus [\epsilon]$, where
   \[ S = \chi(D) + \epsilon \sqrt{1 - \chi(D)^2}. \]
2. Choose $\chi$ so that $1 - \chi(\lambda)^2 = e^{-t\lambda^2}$.
3. The (integer) index is $\text{Tr}(-S \epsilon S - \epsilon)$, and this gives the McKean-Singer formula.
The McKean-Singer formula led to a proof of the index theorem on compact manifolds via an analysis of the asymptotic behavior of the heat kernel $k_t$ as $t \downarrow 0$. On non-compact manifolds, smoothing operators need not be traceable so the proof fails. However, it still leads to the following philosophy:

- For short times $t$, the heat operator $e^{-tD^2}$ represents a local (topological) invariant.
- For long times, it represents a global (analytical) invariant — the projection onto the kernel of $D$.
- Passing from the operator $D$ to its index is a process of passing from local to global — coarsening.
We now give axioms which qualitatively model the ‘large scale structure’ of a space. Compare *topology* which models the *small scale* structure.

**Definition**

A *coarse structure* on a set $X$ is a collection $\mathcal{E}$ of subsets of $X \times X$, called the *controlled sets* or *entourages* for the coarse structure, which contains the diagonal and is closed under the formation of subsets, inverses, products, and (finite) unions. A set equipped with a coarse structure is called a *coarse space*.

Here by the *product* of $E, E' \subseteq X \times X$ we mean the set

$$\{(x, z) : \exists y, (x, y) \in E, (y, z) \in E'\}.$$

These axioms are ‘increasingly directed’; contrast Weil’s axioms for a uniformity.
The basic example

Example

Let \((X, d)\) be a metric space and let \(E\) be the collection of all those subsets \(E \subseteq X \times X\) for which the coordinate projection maps \(\pi_1, \pi_2 : E \to X\) are close; otherwise put, the supremum

\[
\sup\{d(x, x') : (x, x') \in E\}
\]

is finite. Then \(E\) is a coarse structure. It is called the bounded coarse structure associated to the given metric.
Example

(N. Wright) Again, let \((X, d)\) be a metric space and let \(\mathcal{E}\) be the collection of all those subsets \(E \subseteq X \times X\) for which the distance function \(d\), when restricted to \(E\), tends to zero at infinity. Then \(\mathcal{E}\) is a coarse structure on \(X\), called the \(C_0\) coarse structure associated to the metric \(d\).

In the next result, let \(X\) be a locally compact space. We say that a subset \(E \subseteq X \times X\) is proper if, for every compact \(K \subseteq X\), the set

\[
\{x : \exists y, \ (x, y), (y, x) \cap E \neq \emptyset\}
\]

is compact.
Theorem

Let $X$ be a locally compact Hausdorff space, with a metrizable compactification $\overline{X}$. Let $E \subseteq X \times X$. The following conditions are equivalent.

1. The closure $\overline{E}$ of $E$ in $\overline{X} \times \overline{X}$ meets the complement of $X \times X$ only in the diagonal $\Delta_{\partial X} = \{(\omega, \omega) : \omega \in \partial X\}$.

2. $E$ is proper, and for every sequence $(x_n, y_n)$ in $E$, if $\{x_n\}$ converges to a point $\omega \in \partial X$, then $\{y_n\}$ also converges to $\omega$.

Moreover, the sets $E$ satisfying these equivalent conditions form the controlled sets for a coarse structure on $X$.

This is called the continuously controlled coarse structure associated to the given compactification.
Let $X$ be a coarse space.

**Definition**

Let $Y$ be any set. Two maps $f_1, f_2 : Y \to X$ are close if \( \{(f_1(y), f_2(y)) : y \in Y\} \) is a controlled subset of $X \times X$.

- It is easy to see that closeness is an equivalence relation. In fact, the closeness relation completely determines the coarse structure.
- We can say that two *points* of $X$ are close if the corresponding inclusion maps are close. If all pairs of points are close, $X$ is *coarsely connected*.

**Definition**

A subset $B$ of $X$ is *bounded* if the inclusion $B \to X$ is close to a constant map. Equivalently, $B$ is bounded if $B \times B$ is controlled.
Let $X$ and $Y$ be coarse spaces.

**Definition**

A map $f : X \to Y$ is **coarse** if

1. Whenever $E \subseteq X \times X$ is controlled, $(f \times f)(E) \subseteq Y \times Y$ is controlled;
2. Whenever $B \subseteq Y$ is bounded, $f^{-1}(B) \subseteq X$ is bounded also.

For metric (bounded) control, the first condition translates to: for every $R > 0$ there is $S > 0$ such that $d_X(x, x') < R$ implies $d_Y(f(x), f(x')) < S$. 
Examples

Example

The map from the plane to the line sending \((r, \theta)\) to \(r\) (polar coordinates) is a coarse map; the map sending \((x, y)\) to \(x\) (rectangular coordinates) is not.

Example

Let \(X\) be the plane with its bounded coarse structure and let \(Y\) be the plane with the cc structure coming from its radial compactification. The identity map \(X \to Y\) is coarse.

Example

Let \(X\) be complete Riemannian, 1-connected, \(K \leq 0\). The ‘logarithm’ map \(X \to T_{x_0}X\) is coarse (Cartan-Hadamard).
Coarse equivalence

A *coarse equivalence* is a coarse map that has an inverse up to closeness. More exactly, a coarse map \( f: X \to Y \) is a coarse equivalence if there is a coarse map \( g: Y \to X \) such that \( g \circ f \) and \( f \circ g \) are close to the identity maps on \( X \) and \( Y \) respectively.

**Example**

The inclusion \( \mathbb{Z}^n \to \mathbb{R}^n \) is a coarse equivalence. The inclusion of Example 15 (of the metric plane into the cc plane) is *not* a coarse equivalence (exercise: why not?)

There is a *coarse category* of coarse spaces and maps up to coarse equivalence.
**Definition**

Let $X$, $Y$ be coarse spaces. A map $i: X \to Y$ that is a coarse equivalence onto its image is called a *coarse embedding*.

(Here of course $i(X)$ inherits a coarse structure from $Y$.)

In the metric case, a popular alternative definition is the following.

**Lemma**

*If $X$ and $Y$ are metric spaces (with bounded coarse structure) then $i: X \to Y$ is a coarse embedding iff there are functions $\rho_1, \rho_2: \mathbb{R}^+ \to \mathbb{R}^+$ with $\rho_1(t) \to \infty$ as $t \to \infty$ and*

\[ \rho_1(d_X(x, x')) \leq d_Y(i(x), i(x')) \leq \rho_2(d_X(x, x')). \]
A non trivial example of a coarse embedding is the following.

**Example**

Let $T$ be a regular tree and let $H$ be the Hilbert space $L^2(T)$. Fix a base point $x_0 \in T$. Then the map which sends $x \in T$ to the characteristic function of the (unique) geodesic segment $[x_0, x]$ is a coarse embedding.
Let $D$ be a set. For a subset $E \subseteq D \times D$ we define the cross sections

$$E_x = \{ y : (x, y) \in E \}, \quad E^x = \{ y : (y, x) \in E \}.$$ 

A coarse structure on $D$ is a discrete bounded geometry structure on $X$ if for every controlled set $E$ there is a constant $C_E$ such that, for each $x \in D$, the cross sections $E^x$ and $E_x$ have at most $C_E$ elements.

**Definition**

A coarse space $X$ has bounded geometry if it is coarsely equivalent to some discrete bounded geometry coarse space.
Compatibilty with topology

**Definition**

Let $X$ be a Hausdorff space. We say that a coarse structure on $X$ is *proper* if

1. There is a controlled neighborhood of the diagonal, and
2. Every bounded subset of $X$ is relatively compact.

Notice that $X$ must be locally compact.

**Theorem**

*Let $X$ be a connected topological space provided with a proper coarse structure. Then $X$ is coarsely connected. A subset of $X$ is bounded if and only if it is relatively compact. Moreover, every controlled subset of $X \times X$ is proper.*
## Examples

### Example

The (bounded) coarse structure on a metric space is proper if the metric is a *proper metric* (closed bounded sets are compact).

### Example

The continuously controlled coarse structure defined by a metrizable compactification is always proper.

When dealing with proper coarse spaces it is natural to consider the subcategory whose morphisms are *continuous* coarse maps.
Let $X$ be a locally compact Hausdorff space.

**Definition**

A geometric $X$-module is a Hilbert space $H$, equipped with a representation $\rho : C_0(X) \to B(H)$.

For example, $L^2(X, \mu)$ (relative to a Borel measure $\mu$) is a geometric $X$-module.

**Definition**

Let $H$ be a geometric $X$-module. The support of $\xi \in H$ is the complement of the set of $x \in X$ having the following property: there is a neighborhood $U$ of $x$ in $X$ such that $\rho(C_0(U))$ annihilates $\xi$. 
Properties of supports

It is easy to see that the support is a closed set and

$$\text{Support}(\xi + \xi') \subseteq \text{Support}(\xi) \cup \text{Support}(\xi').$$

Moreover, \(\text{Support}(\rho(f)\xi) \subseteq \text{Support}(f) \cap \text{Support}(\xi)\).

Suppose that \(X\) is a proper coarse space and \(H\) is a geometric \(X\)-module. Let \(T \in \mathcal{B}(H)\).

**Definition**

We say that \(T\) is **controlled** if there is a controlled set \(E\) such that for all \(\xi \in M\),

$$\text{Support}(T\xi) \cup \text{Support}(T^*\xi) \subseteq E[\text{Support}(\xi)].$$

Here the notation \(E[K]\) refers to \(\{x : \exists y \in K, (x, y) \in E\}\).
Theorem

The controlled operators form a $\ast$-algebra of operators on $H$.

Definition

An operator $T$ on a geometric $X$-module $H$ is locally compact if $\rho(f)T$ and $T\rho(f)$ are compact whenever $f \in C_0(X)$.

Let $X$ be a proper coarse space and let $H$ be a geometric $X$-module.

Definition

The translation algebra $C^*(X; H)$ is the $C^*$-algebra generated by the controlled, locally compact operators on $H$. 
Let $f: X \to Y$ be a coarse and continuous map of proper coarse spaces. Then $f$ induces a map $C_0(Y) \to C_0(X)$, which in turn makes every $X$-module $H$ into a $Y$-module $f_*(H)$.

**Theorem**

*In the above situation there is a natural map of $C^*$-algebras*

$$f_* : C^*(X; H) \to C^*(Y; f_* H).$$

In fact, the map is the identity map!
Our discussion at the beginning of the lecture shows that (suitable) elliptic operators on a complete Riemannian manifold $M$ have indices in the $K$-theory of the translation algebra of $M$. This motivates us to develop techniques for computing this $K$-theory. We shall do that in the next lecture.