Syllabus for Math 597D, Fall 2008
Index Theory

This is a course about the Atiyah-Singer Index Theorem, one of the central results of 20th century mathematics. The index theorem concerns itself with counting the "number of independent solutions" of certain partial differential equations. To be precise, these are *homogeneous, linear* partial differential equations of *elliptic type* on *closed manifolds*. The solution space to such an equation is a finite dimensional vector space, and the index theorem tells us something about the dimension of this space.

What the index theorem supplies is a formula relating solution-space dimensions – quintessentially *analytic* objects – with other invariants belonging to the realm of algebraic topology, namely the *characteristic classes* of *vector bundles*. Special cases of the formula include many key results about manifold topology such as

- The *Gauss-Bonnet theorem* relating curvature to the Euler characteristic for surfaces, and its high-dimensional generalization the *Gauss-Bonnet-Chern theorem*.
- The *Hirzebruch signature formula* relating the intersection properties of middle-dimensional submanifolds of an oriented 4n-manifold to its characteristic classes. This is foundational for differential topology.
- The *Hirzebruch-Riemann-Roch* formula for complex manifolds.
- The *Lichnerowicz vanishing theorem* constraining the existence of metrics of positive scalar curvature on spin manifolds.

In this course we will develop the techniques necessary to state, appreciate, prove, and apply the index theorem. The overall outline is based on the arguments given in the Annals papers of Atiyah and Singer published in 1968. But we will deploy some more modern, geometric machinery which is inspired by Connes' noncommutative geometry. We will express the constructions that are needed to prove the index theorem in the language of *smooth groupoids* and their associated operator algebras. These techniques have other applications too and are important in various areas of geometry, analysis, and physics.

The text for the course is the book "Operator algebras and index theory" which is in final preparation by the two of us (Higson and Roe). A pre-publication version of the book will be made available to students. There will be a reward for each error or misprint spotted in the book! For a more extended introduction to the subject, please follow the link to the article that we wrote for the forthcoming *Princeton Companion to Mathematics*.

Prerequisites for the course include a familiarity with the basics of smooth manifolds (differential forms, de Rham cohomology, etc) and the analysis of Hilbert space.

Instructors	Nigel Higson (higson@math.psu.edu) and John Roe (roe@math.psu.edu)
Meetings	1:00 – 2:15 Tuesdays and Thursdays, 106 Ag Sci Ind
Office Hours	By appointment.
Mode of Assessment	Exercises and projects. There will be no final exam. Details will be announced.
Academic Integrity	All Penn State Policies regarding ethics and honorable behavior apply to this course.

Course Outline (Tentative)

Copies of the book chapters listed below will be provided to students

Lecture	Date	Instructor	Book Sections (link)	Subject
1	8/26	Roe	Chapter 1	De Rham cohomology
2	8/28	Roe		Hodge Theory, Fredholm index
3	9/2	Roe	Chapter 2	Basic C* Algebras and Spectral Theory
4	9/4	Higson		Differential Operators
5	9/9	Higson	Chapter 3	Elliptic Operators and Analysis
6	9/11	Higson		The Smooth Theory of Elliptic Operators
7	9/16	Higson		Elliptic Operators and Hilbert Space
8	9/18	Higson	Chapter 4	Introduction to K-Theory
9	9/23	Roe		Operator Algebras and K-Theory
10	9/25	Roe		Bundles and Morita Equivalence
11	9/30	Roe		Multiplicative Structure in K-Theory
12	10/2	Roe	Chapter 5	K-Theory and the Index
13	10/7	Roe		K-Theory and the Symbol
14	10/9	Roe	Chapter 6	Vector Bundles and Characteristic Classes
15	10/14	Higson		Calculations with Characteristic Classes
16	10/16	Higson	Chapter 7	The Statement of the Index Theorem
17	10/21	Higson		Complex Manifolds and the Dolbeault Operator
18	10/23	Higson		Dirac Symbols and Operators
19	10/28	Roe	Chapter 8	Smooth groupoids and their C*-algebras
20	10/30	Roe		Groupoids and operators
21	11/4	Higson		Groupoids and K-theory maps
22	11/6	Higson	Chapter 9	Bott Periodicity
23	11/11	Higson		The Thom Isomorphism
24	11/13	Higson		The Chern Character of the Thom Isomorphism
25	11/18	Higson	Chapter 10	The Tangent Groupoid
26	11/20	Higson		The Tangent Groupoid and the Index
27	12/2	Roe	Chapter 11	Deforming the Tangent Groupoid
28	12/4	Roe		The Main Diagram
29	12/9	Higson		Proof of the Index Theorem
30	12/11	Higson		Coda