Math 597A Homework 5 — Due December 10th, 2009

Exercise 1. Prove that a bounded holomorphic function f from $\mathbb{C}$ to a (complex) Banach space E is necessarily constant. (Hint: Apply the classical form of Liouville's theorem to $\phi \circ f$, for all linear functionals $\phi \in E^*$.)

Solution. If $f\colon \mathbb{C} \to E$ is holomorphic and $\phi \in E^*$, then the function

$$g_\phi\colon z \mapsto \phi(f(z) - f(0))$$

is holomorphic (by the chain rule). If f is bounded, then g_ϕ is a bounded entire function on $\mathbb{C}$, hence constant (by the ordinary version of Liouville's theorem). Since $g_\phi(0) = 0$ the constant is zero. Thus $\phi(f(z)) = \phi(f(0))$ for all $\phi \in E^*$, hence $f(z) = f(0)$ since E^* separates points of E by the Hahn-Banach theorem.

Exercise 2. Let $A = \ell^1(\mathbb{Z})$. Show that the operation of *convolution*

$$(f * g)(n) = \sum_k f(k)g(n-k)$$

makes A into a commutative Banach algebra. Show that there is a continuous homomorphism of Banach algebras $A \to C(S^1)$, given by sending $f \in A$ to the sum of the absolutely convergent Fourier series $\sum f(n)e^{inx}$. Show further that the induced map on Gelfand duals

$$S^1 \to \widehat{A}$$

is a homeomorphism.

Defining $f * g$ as given, we have

$$\|f * g\| = \sum_n \left| \sum_k f(k)g(n-k) \right| \leqslant \sum_n \sum_k |f(k)||g(n-k)| = \|f\|\|g\|.$$

The commutativity and associativity of convolution have standard proofs. Thus, A becomes a Banach algebra. The map $\Phi\colon A \to C(S^1)$ defined in the question is a homomorphism because

$$\Phi(f)\Phi(g) = \left(\sum_k f(k)e^{ikx} \right) \left(\sum_m g(m)e^{imx} \right) = \sum_{k,n} f(k)g(n-k)e^{inx}$$

and it is continuous because

$$\|\Phi(f)\| = \sup \left| \sum_n f(n) e^{inx} \right| \leqslant \sum_n |f(n)| = \|f\|.$$

The elements of the Gelfand dual of $C(S^1)$ are evaluations at points $z \in S^1$. All of these evaluations are distinct when considered as homomorphisms $A \to \mathbb{C}$, since the function e^{ix} separates them. Thus $S^1 \to \widehat{A}$ is injective. We must show that it is also surjective, i.e. that every homomorphism $A \to \mathbb{C}$ is an evaluation at some z.

Well, let $\phi\colon A \to \mathbb{C}$ be a complex homomorphism, and let $u \in A$ be the element $u(n) = 1$ if $n = 1$, $u(n) = 0$ otherwise. Then ϕ sends u to some complex number z. Because the powers (positive and negative) of u form a bounded set in A, the powers (positive and negative) of z form a bounded set in $\mathbb{C}$. Thus, z has modulus 1. ϕ sends u^n to z^n and so sends any finitely supported f to $\sum f(n) z^n$. By continuity, this formula extends to all f, and ϕ is evaluation at z.

We have proved that the induced map on Gelfand transforms is a bijection. But it is certainly continuous, and the Gelfand spaces are both compact Hausdorff spaces, so the map is a homeomorphism.

Exercise 3. Construct examples of bounded self-adjoint operators T_1, T_2 on a separable Hilbert space, both of which have spectrum $[0,1]$, such that T_1 has no eigenvalues whereas the eigenvalues of T_2 are dense in $[0,1]$. Is it possible also to construct such an operator T_3 such that *every* point of $[0,1]$ is an eigenvalue? Give reasons.

Let $H = L^2[0,1]$ and let T_1 be the operator of multiplication by the function $f(x) = x$. The spectrum of T_1 is the closure of the range of f, which is $[0,1]$. But, the equation $f(x)g(x) = \lambda g(x)$ has no nonzero solutions for any λ, so T_1 has no eigenvalues.

The functions $e_n(x) = e^{2\pi i n x}$ form an orthonormal basis for H. Let T_2 be defined by $T_2 e_n = q_n e_n$, where q_n is an enumeration of the rationals in $[0,1]$. Then T_2 is a positive operator and bounded by 1, so its spectrum is contained in $[0,1]$. Each rational in $[0,1]$ is an eigenvalue. Since the spectrum is closed, it is equal to $[0,1]$.

The eigenspaces corresponding to distinct eigenvalues of a self-adjoint operator T are orthogonal, by the same argument as in finite dimensions: if

u, v are eigenvectors for eigenvalues λ, μ then

$$\lambda\langle u, v\rangle = \langle Tu, v\rangle = \langle u, Tv\rangle = \mu\langle u, v\rangle.$$

A separable Hilbert space cannot contain an uncountable orthonormal set, so it is impossible that *every* point on $[0, 1]$ should be an eigenvalue.

Exercise 4. Show that for any normal operator T one can find a unitary U, commuting with T, such that $T^* = UT$. Deduce that T and T^* have the same kernel and the same image. Give an example of an operator T such that T and T^* have the same kernel and the same image, but are not normal.

Let $a\colon \mathbb{C} \to \mathbb{C}$ be the Borel function defined by $a(\lambda) = \bar{\lambda}/\lambda$ if $\lambda \neq 0$, $a(0) = 1$. Then $U = a(T)$ is defined by the functional calculus. The values of a lie in the unit circle, so U is unitary. Moreover, $UT = b(T) = TU$ where $b(\lambda) = a(\lambda)\lambda = \bar{\lambda}$, so $b(T) = T^*$. It follows that $Tx = 0$ implies $T^*x = 0$ and also that $y = T^*x$ implies $y = T(Ux)$; so $\ker T \subseteq \ker T^*$ and $\operatorname{Im} T^* \subseteq \operatorname{Im} T$. Interchanging T and T^* gives equality in both cases.

If T is *any* non-normal invertible operator, then T and T^* have the same kernel (zero) and the same image (the whole space).

Exercise 5. Show that if λ is an isolated point in the spectrum of a normal operator T, then it is an eigenvalue of T.

If λ is isolated in the spectrum of T, then the function $f(z)$ that is equal to 1 if $z = \lambda$ and 0 otherwise is continuous on the spectrum of T. Thus, by the functional calculus, there exists a nonzero self-adjoint projection $P = f(T)$ in the C^*-algebra generated by T. If ξ belongs to the range of P, then

$$T\xi = TP\xi = \lambda P\xi = \lambda\xi$$

so ξ is an eigenvector with eigenvalue λ.

Exercise 6. *Fuglede's theorem* states that if N is a normal operator on a Hilbert space and T is any operator that commutes with N, then T also commutes with any Borel function $f(N)$ — in particular it commutes with N^*. Fuglede's original proof of this fact (*Proceedings of the National Academy of Sciences*, Volume 36, 1950, 35–40) is outlined in this exercise.

(a) Suppose that H is an orthogonal direct sum of eigenspaces for N (one says that N has *pure point spectrum*). Prove that T maps each eigenspace to itself. Deduce the theorem in this case.

(b) Now let N be general. Let $f_k\colon \mathbb{C} \to \mathbb{C}$ be the complex-valued Borel function defined as follows: $f_k(x+iy) = (1/k)(\lfloor kx \rfloor + i\lfloor ky \rfloor)$. Show that $N_k := f_k(N)$ has pure point spectrum and that $\|N_k - N\| \leqslant (\sqrt{2})/k$.

(c) Let P_λ and P_μ be eigenprojections for N_k corresponding to eigenvalues λ, μ. Show that $P_\lambda T P_\mu = 0$ provided that $|\lambda - \mu|$ is sufficiently large.

(d) Let E denote the spectral measure of N. Show from the above that if K, L are disjoint closed subsets of $\mathbb{C}$ then $E(K)TE(L) = 0$.

(e) Deduce that T commutes with each $E(K)$, and hence prove Fuglede's theorem.

Solution. (a) If T commutes with N and $N\xi = \lambda\xi$ then $NT\xi = TN\xi = \lambda T\xi$. Thus T preserves the λ-eigenspace of N. Let P_i denote the orthogonal projection onto the eigenspace of N with eigenvalue λ_i. Then $N = \sum \lambda_i P_i$ and T commutes with each P_i. It follows that T commutes with $f(N) = \sum f(\lambda_i)P_i$, for any Borel function f.

(b) For all z, $|f_k(z) - z| \leqslant \sqrt{2}/k$, so $\|N_k - N\| \leqslant \sqrt{2}/k$ by the functional calculus. For each m, n let

$$S_{m,n,k} = \{x + iy : m/k \leqslant x < (m+1)/k,\ n/k \leqslant y < (n+1)/k\}.$$

Then $N_k = f_k(N) = \sum_{m,n}(m + in)/k\chi_{S_{m,n,k}}(N)$. The operators $\chi_{S_{m,n,k}}(N)$ are orthogonal projections, so N_k has pure point spectrum (with eigenspaces equal to the ranges of these projections).

(c) Let $M = N_k - N$, so that $\|M\| \leqslant \delta = \sqrt{2}/k$. The projections P_λ, P_μ commute with N and M, and $P_\lambda N_k = \lambda P_\lambda$. Thus

$$(\lambda - \mu)P_\lambda T P_\mu = P_\lambda[T, N_k]P_\mu = P_\lambda[T, M]P_\mu = [P_\lambda T P_\mu, M]$$

since T commutes with N. Thus

$$|\lambda - \mu|\|P_\lambda T P_\mu\| \leqslant 2\delta\|P_\lambda T P_\mu\|$$

by the triangle inequality. If $|\lambda - \mu| > 2\delta$ this implies $P_\lambda T P_\mu = 0$.

(d) If K and L are disjoint closed sets one can cover $\mathbb{C}$ by a grid of squares so small that the diagonal δ of the grid is less than one-fourth the distance between K and L. Let $S_{m,n}$ be the sets of the grid. Then

$$E(K)TE(L) = \sum_{m,n,m',n'} E(K)E(S_{m,n})TE(S_{m',n'})E(L).$$

The only nonzero terms that can appear here are when $S_{m,n}$ meets K and $S_{m',n'}$ meets L. But then $S_{m,n}$ and $S_{m',n'}$ are separated by at least 2δ, so these terms also vanish by (c) above.

(e) Now let K be any closed set in $\mathbb{C}$ and write the complement of K as the union of an increasing sequence of closed sets L_n. Since $E(K)TE(L_n) = 0$ for all n, countable additivity gives us $E(K)TE(\mathbb{C} \setminus K) = 0$, that is, T commutes with $E(K)$. Since T commutes with the spectral projection for every closed set, it commutes with the spectral projection for every Borel set, and therefore with every Borel function of T.