**Math 597A Homework 5 — Due December 10th, 2009**

*Exercise* 1. Prove that a bounded holomorphic function $f$ from $\mathbb{C}$ to a (complex) Banach space $E$ is necessarily constant. (Hint: Apply the classical form of Liouville's theorem to $\phi \circ f$, for all linear functionals $\phi \in E^*$.)

*Exercise* 2. Let $A = \ell^1(\mathbb{Z})$. Show that the operation of *convolution*

$$(f * g)(n) = \sum_k f(k)g(n-k)$$

makes $A$ into a commutative Banach algebra. Show that there is a continuous homomorphism of Banach algebras $A \to C(S^1)$, given by sending $f \in A$ to the sum of the absolutely convergent Fourier series $\sum f(n)e^{inx}$. Show further that the induced map on Gelfand duals

$$S^1 \to \widehat{A}$$

is a homeomorphism.

*Exercise* 3. Construct examples of bounded self-adjoint operators $T_1, T_2$ on a separable Hilbert space, both of which have spectrum $[0,1]$, such that $T_1$ has no eigenvalues whereas the eigenvalues of $T_2$ are dense in $[0,1]$. Is it possible also to construct such an operator $T_3$ such that *every* point of $[0,1]$ is an eigenvalue? Give reasons.

*Exercise* 4. Show that for any normal operator $T$ one can find a unitary $U$, commuting with $T$, such that $T^* = UT$. Deduce that $T$ and $T^*$ have the same kernel and the same image. Give an example of an operator $T$ such that $T$ and $T^*$ have the same kernel and the same image, but are not normal.

*Exercise* 5. Show that if $\lambda$ is an isolated point in the spectrum of a normal operator $T$, then it is an eigenvalue of $T$.

*Exercise* 6. *Fuglede's theorem* states that if $N$ is a normal operator on a Hilbert space and $T$ is any operator that commutes with $N$, then $T$ also commutes with any Borel function $f(N)$ — in particular it commutes with $N^*$. Fuglede's original proof of this fact (*Proceedings of the National Academy of Sciences*, Volume 36, 1950, 35–40) is outlined in this exercise.

(a) Suppose that $H$ is an orthogonal direct sum of eigenspaces for $N$ (one says that $N$ has *pure point spectrum*). Prove that $T$ maps each eigenspace to itself. Deduce the theorem in this case.

(b) Now let $N$ be general. Let $f_k \colon \mathbb{C} \to \mathbb{C}$ be the complex-valued Borel function defined as follows: $f_k(x+iy) = (1/k)(\lfloor kx \rfloor + i\lfloor ky \rfloor)$. Show that $N_k := f_k(N)$ has pure point spectrum and that $\|N_k - N\| \leqslant (\sqrt{2})/k$.

(c) Let $P_\lambda$ and $P_\mu$ be eigenprojections for $N_k$ corresponding to eigenvalues $\lambda, \mu$. Show that $P_\lambda T P_\mu = 0$ provided that $|\lambda - \mu|$ is sufficiently large.

(d) Let $E$ denote the spectral measure of $N$. Show from the above that if $K, L$ are disjoint closed subsets of $\mathbb{C}$ then $E(K)TE(L) = 0$.

(e) Deduce that $T$ commutes with each $E(K)$, and hence prove Fuglede's theorem.