Math 597A Homework 4 — Due November 17th, 2009

Exercise 1. Let E, F be Banach spaces. Consider the canonical map $(x, y) \mapsto x \otimes y$ of $E \times F$ to the (projective) tensor product $E \otimes_\pi F$. What is the derivative of this map at the point (x_0, y_0)?

Solution: All that matters here is that the tensor product is a bilinear map. Write

$$(x + h) \otimes (y + k) = x \otimes y + h \otimes y + x \otimes k + h \otimes k.$$

If $\|h\|, \|k\| \leqslant \epsilon$ then the last term is of order ϵ^2. Therefore by definition of the derivative, the derivative of the tensor product at (x, y) is the linear map $E \times F \to E \otimes F$ given by

$$(h, k) \mapsto h \otimes y + x \otimes k.$$

This is the "product rule for differentiation."

Exercise 2. Show that in a Hilbert space H the norm function $x \mapsto \|x\|$ is differentiable at every nonzero point, and find its derivative.

By contrast, show that the norm function on the Banach space ℓ^1 is not differentiable anywhere.

Solution: Let H be a real Hilbert space and consider the function $f \colon H \to \mathbb{R}$ given by $f(x) = \|x\|^2$. We have

$$f(x + h) - f(x) = 2\langle x, h\rangle + \|h\|^2$$

which shows that f is differentiable everywhere, with derivative the linear map $h \mapsto 2\langle x, h\rangle$. The norm function is the composite of f with the square root function $\mathbb{R} \to \mathbb{R}$, which is differentiable everywhere except at 0. So by the chain rule. $x \mapsto \|x\|$ is differentiable at every nonzero point, and its derivative at x is the linear map

$$h \mapsto 2\langle x, h\rangle \cdot \tfrac{1}{2}\|x\|^{-1} = \langle x/\|x\|, h\rangle.$$

Now consider instead the norm on $E = \ell^1$. Suppose that $f(x) = \|x\|$ is differentiable at some $x = (x_1, x_2, x_3, \ldots)$ with derivative $T \in E^*$. Then, given any $\epsilon > 0$, say $\epsilon = \frac{1}{4}$, there is $\delta > 0$ such that

$$|f(x + h) - f(x) - Th| \leqslant \epsilon\|h\|$$

whenever $\|h\| < \delta$. In particular we must have

$$|f(x+\lambda e_n) + f(x-\lambda e_n) - 2f(x)| \leqslant 2\epsilon|\lambda|$$

for all $|\lambda| < \delta$, where e_n is the basis vector with 1 in the n'th place and 0 elsewhere.

Choose n such that $|a_n| < \delta/3$. We have

$$f(x+\lambda e_n) = |\lambda - a_n| + c$$

where c is a constant. Thus if $\lambda = 2\delta/3$

$$|f(x+\lambda e_n) + f(x-\lambda e_n) - 2f(x)| \geqslant \delta/3 + \delta - 2\delta/3 = 2\delta/3 = |\lambda|$$

and this contradicts the last display of the previous paragraph. Hence the norm function is nowhere differentiable.

Exercise 3. Let Ω be an open subset of a real Banach space E. A function f from Ω to another real Banach space F is called *quasi-differentiable* at $x_0 \in \Omega$ if there is a continuous linear map $T\colon E \to F$ such that, for every continuous curve $\gamma\colon (-1,1) \to E$, differentiable at 0 with $\gamma(0) = x_0$, the curve $f\circ\gamma\colon (-1,1) \to F$ is also differentiable at 0, and $(f\circ\gamma)'(0) = T(\gamma'(0))$.

Show that every differentiable map is quasi-differentiable and that if E is finite-dimensional then every quasi-differentiable map is differentiable.

Solution. It follows from the chain rule that every differentiable function is quasi-differentiable.

Let $f\colon \Omega \to F$ be quasi-differentiable with derivative T and E finite-dimensional. We may assume that E has a Euclidean norm. Suppose that f is *not* differentiable at x_0. Then for some $\epsilon > 0$ there exists a sequence $h_n \to 0$ of nonzero vectors in E with

$$\|f(x_0+h_n) - f(x_0) - T\cdot h_n\| \geqslant \epsilon\|h_n\|.$$

We may assume without loss of generality that the norms $\|h_n\|$ form a strictly decreasing sequence. The unit vectors $h_n/\|h_n\|$ form a subset of the unit sphere in E, which is compact because E is finite-dimensional. By passing to a subsequence, we may therefore assume that $h_n/\|h_n\|$ converges to a unit vector u as $n \to \infty$. Define a path γ in Ω as follows: $\gamma(t) = x_0 + tu$ for $t \leqslant 0$, $\gamma(t) = x_0 + h_n$ if $t = t_n := \|h_n\|$, and for other positive values of

t, $t_{n+1} < t < t_n$, $\gamma(t)$ is the unique point on the line segment joining h_{n+1} and h_n that has $\|\gamma(t)\| = t$. Then γ is a continuous path. Moreover it is differentiable at $t = 0$ with derivative u, because if $t_{n+1} < t < t_n$, $\gamma(t)/t$ lies on the great circle arc joining $h_n/\|h_n\|$ and $h_{n+1}/\|h_{n+1}$; since $h_n/\|h_n\| \to u$ as $n \to \infty$, it follows that $\gamma(t)/t \to u$ as $t \to 0$. However, by construction, $f \circ \gamma$ is not differentiable. It follows that f is not quasi-differentiable, a contradiction.

Exercise 4. Suppose that E is the Banach space of continuous real-valued functions on $[-1, 1]$ (with the sup norm), that $f\colon E \to \mathbb{R}$ is the norm function, and that $h \in E$ is the function $h(t) = 1 - |t|$. Show that f is quasi-differentiable at h but not differentiable.

Solution. First, to show that the norm function is not differentiable at h, observe that given any $\epsilon > 0$ one can find a continuous function k : $[-1, 1] \to [0, 1]$, equal to 1 outside $(-2\epsilon, 2\epsilon)$ and vanishing on $(-\epsilon, \epsilon)$. Then $\|h + \lambda k\| = 1$ for $-4\epsilon < \lambda \leqslant 0$, and $\geqslant 1 + \lambda - 2\epsilon$ for $2\epsilon < \lambda < 4\epsilon$. This can be used to contradict differentiability exactly as in the case of the norm on ℓ^1 (problem 2 above).

Nevertheless the norm is quasi-differentiable at h, and its (quasi)-derivative is the linear functional $T \in E^*$ given by evaluation at 0. To see this, let $h_s(t)$, $s \in (-1, 1)$, be a path in E, differentiable at $s = 0$ with $h_0(t) = h(t)$. Let $k(t)$ be the derivative of $h_s(t)$, with respect to s, at $s = 0$. It is clear that for s small, the norm $\|h_s\|$ is in fact the maximum $\max h_s$, and that $\max h_s \geqslant h_s(0) = h(0) + sk(0) + o(s)$. We will obtain an inequality in the opposite direction.

Let $\epsilon > 0$ be given. There is $\delta > 0$ such that $|k(t) - k(0)| < \epsilon$ for all $t \in [-\delta, \delta]$. Moreover there is μ' such that $|h_s(t) - h_0(t) - sk(t)| < \epsilon|s|$ for all $|s| < \mu'$. Furthermore, there is $\mu'' > 0$ such that if $|s| < \mu''$ then $|h_s(t) - h_0(t)| < \delta/2$ for all t. It follows that if $|t| > \delta$ then

$$h_s(t) < h_0(t) + \delta < h_0(0) - \delta < h_s(0),$$

so h_s attains its maximum for some value or values of t in the interval $[-\delta, \delta]$. Let $\mu = \min\{\mu', \mu''\}$. If $|s| < \mu$ we have $\max h_s = h(t)$ for some $t \in [-\delta, \delta]$ and therefore

$$\max h_s \leqslant h(t)+sk(t)+\epsilon|s| \leqslant h(0)+sk(0)+s|k(t)-k(0)|+\epsilon|s| \leqslant h(0)+sk(0)+2\epsilon|s|.$$

Combined with the inequality in the previous paragraph this gives the result.

Exercise 5. Let f be a continuously differentiable map from an open subset Ω of a Banach space E to a Banach space F, and suppose that $Df(x)\colon E \to F$ is a surjective linear map for every $x \in \Omega$. Prove that f is open (that is, $f(U)$ is open in F for every open subset U of Ω.)

Solution. Let $x_0 \in \Omega$ and $f(x_0) = y_0$. It suffices to show that the image of f contains a neighborhood of x_0. Let $T = Df(x_0)$, which is a surjective linear map from E to F. By the open mapping theorem, there is a constant $k > 0$ such that for each $y \in F$ there exists $x \in E$ with $Tx = y$ and $\|x\| \leqslant k\|y\|$.

Choose $r > 0$ so small that $\|Df(x) - Df(x_0)\| < 1/(2k)$ for all $\|x - x_0\| < r$, and let $z \in B(y_0; r/2k)$. We will construct by induction a Cauchy sequence $x_0, x_1, \ldots \in B(x_0; r)$ such that $f(x_n) \to z$ as $n \to \infty$. We begin with x_0 and $y_0 = f(x_0)$. Notice that $\|z - y_0\| < r/2k$. Choose x_1 such that $T(x_1 - x_0) = z - y_0$ and $\|x_1 - x_0\| < r/2$. By the mean value theorem

$$\|f(x_1) - f(x_0) - T(x_1 - x_0)\| \leqslant \|x - x_0\| \sup\{\|Df(x) - Df(x_0)\| : x \in [x_0, x_1]\}.$$

The expression on the left here is $\|y_1 - z\|$, where $y_1 = f(x_1)$; the expression on the right is bounded by $1/(2k) \cdot r/2 = r/4k$. Thus $\|y_1 - z\| \leqslant r/4k$. Proceed now by induction defining a sequence $x_2, x_3, \ldots$ with $\|x_j - x_{j-1}\| < 2^{-j}r$ (so all the x_j belong to $B(x_0; r)$) and $\|y_j - z\| \leqslant r/(2^{j+1}k)$ (where $y_j = f(x_j)$). Now the $\{x_j\}$ form a Cauchy sequence and their limit is a point x_∞ with $f(x_\infty) = z$.

Exercise 6. Use the Schauder fixed point theorem to prove the existence of a continuous function $f\colon [0,1] \to \mathbb{R}$ that satisfies the integral equation

$$f(x) = \int_0^1 \sin\left(x + f(t)^2\right)\, dt.$$

Solution. Let E be the Banach space $C[0,1]$ and let $K \subseteq E$ be the closure of the set of all differentiable functions f having $|f(x)| \leqslant 1$ and $|f'(x)| \leqslant 1$ for all $x \in [0,1]$. Clearly K is closed and convex; it is also equicontinuous (by the mean value theorem) and therefore compact.

let T be the nonlinear map $E \to E$ defined by

$$Tf(x) = \int_0^1 \sin\left(x + f(t)^2\right)\, dt.$$

I claim that T maps K continuously into K. Clearly $\|Tf\| \leqslant 1$ for any f, and standard results show that $Tf(x)$ is differentiable for any f, with derivative

$$\int_0^1 \cos\left(x + f(t)^2\right)\, dt$$

bounded by 1 also. We must prove that T is a continuous map. But

$$Tf(x) - Tg(x) \leqslant \int_0^1 2\cos\left(x + \tfrac{1}{2}(f(t)^2 + g(t)^2)\right) \sin\left(\tfrac{1}{2}(f(t)^2 - g(t)^2)\right)\, dt$$

by a standard trigonometric identity, and this is bounded by $\|f - g\|$ by simple estimates. Thus T is continuous. By Schauder's fixed point theorem, T has a fixed point on K. Such a fixed point is a solution to the original problem.