Math 597A Homework 4 — Due November 17th, 2009

Exercise 1. Let E, F be Banach spaces. Consider the canonical map $(x, y) \mapsto x \otimes y$ of $E \times F$ to the (projective) tensor product $E \otimes_\pi F$. What is the derivative of this map at the point (x_0, y_0)?

Exercise 2. Show that in a Hilbert space H the norm function $x \mapsto \|x\|$ is differentiable at every nonzero point, and find its derivative.

By contrast, show that the norm function on the Banach space ℓ^1 is not differentiable anywhere.

Exercise 3. Let Ω be an open subset of a real Banach space E. A function f from Ω to another real Banach space F is called *quasi-differentiable* at $x_0 \in \Omega$ if there is a continuous linear map $T\colon E \to F$ such that, for every continuous curve $\gamma\colon (-1, 1) \to E$, differentiable at 0 with $\gamma(0) = x_0$, the curve $f \circ \gamma\colon (-1, 1) \to F$ is also differentiable at 0, and $(f \circ \gamma)'(0) = T(\gamma'(0))$.

Show that every differentiable map is quasi-differentiable and that if E is finite-dimensional then every quasi-differentiable map is differentiable.

Exercise 4. Suppose that E is the Banach space of continuous real-valued functions on $[-1, 1]$ (with the sup norm), that $f\colon E \to \mathbb{R}$ is the norm function, and that $h \in E$ is the function $h(t) = 1 - |t|$. Show that f is quasi-differentiable at h but not differentiable.

Exercise 5. Let f be a continuously differentiable map from an open subset Ω of a Banach space E to a Banach space F, and suppose that $Df(x)\colon E \to F$ is a surjective linear map for every $x \in \Omega$. Prove that f is open (that is, $f(U)$ is open in F for every open subset U of Ω.)

Exercise 6. Use the Schauder fixed point theorem to prove the existence of a continuous function $f\colon [0, 1] \to \mathbb{R}$ that satisfies the integral equation

$$f(x) = \int_0^1 \sin\left(x + f(t)^2\right)\, dt.$$