Math 597A Homework 3 — Due October 27th, 2009

Exercise 1. Give (with proof) an example of a distribution Λ on $\mathbb{R}$ and a smooth, compactly supported function f such that $f = 0$ on Support(Λ) but $f\Lambda \neq 0$. Is it possible to give a similar example where $f = 0$ on a *neighborhood* of Support(Λ)?

Solution: Let $\Lambda = \delta'$ (the derivative of the Dirac delta function at the origin) and let f be a smooth compactly supported function that has $f(x) = x$ in a neighborhood of the origin. The support of Λ is the set $\{0\}$, and f vanishes on this set. However, one has by definition

$$(f\Lambda)(g) = \Lambda(fg) = d/dx(xg(x))|_{x=0} = g(0)$$

and this need not vanish.

The definition of the support of a distribution says that Support(Λ) is the complement of the largest open set on which Λ vanishes, meaning that if $f \in \mathscr{D}(\mathbb{R})$ vanishes on a neighborhood of Support(Λ) then $\Lambda(f) = 0$. Indeed we have

$$(f\Lambda)(g) = \Lambda(fg) = 0$$

for any g.

Exercise 2. Let Λ_j be a weakly convergent sequence of distributions on $\mathbb{R}^n$ whose supports are contained in some fixed compact subset $K \subseteq \mathbb{R}^n$. Show that the orders of the distributions Λ_j are (uniformly) bounded.

Solution: Since the distributions $\{\Lambda_j\}$ have support in K, they may all be regarded as elements of the dual of the Fréchet space $\mathscr{D}_L(\mathbb{R}^n)$, where L is a compact set whose interior contains K. (Warning: They may *not* be regarded as elements of the dual of $\mathscr{D}_K(\mathbb{R}^n)$. See the previous question.) Being a weakly convergent sequence they are pointwise bounded and hence equicontinuous by the Banach-Steinhaus theorem. This means that for one of the seminorms $p_N(f) = \sup\{|D^\alpha f(x)| : x \in L, |\alpha| \leqslant N\}$ that define the topology of $\mathscr{D}_L(\mathbb{R}^n)$, we have

$$|\Lambda_j(f)| \leqslant C p_N(f)$$

for all j. But this implies that the distributions Λ_j are all of order $\leqslant N$.

Exercise 3. Let the Schwarz-class function $f \in \mathscr{S}(\mathbb{R})$ satisfy the normalizing condition $\int |f(x)|^2 dx = 1$. Prove the Heisenberg uncertainty relation

$$\left(\int x^2 |f(x)|^2 dx\right)\left(\int t^2 |\hat{f}(t)|^2 dt\right) \geqslant \frac{1}{4}$$

which says that f and $\hat{f}$ cannot simultaneously be "too concentrated". When does equality hold?

Hint: Write

$$1 = \int |f(x)|^2 dx = -\int x \frac{d}{dx}\left(|f(x)|^2\right) dx.$$

Expand the right-hand side, and use the Cauchy-Schwarz inequality together with Plancherel's theorem.

Solution: The equation displayed in the hint comes from integration by parts (integrate by parts over the interval $[-a, a]$, then take the limit as $a \to \infty$). Use the identity $|f(x)|^2 = f(x)\bar{f}(x)$ to expand the right side as

$$-\int x(f'(x)\bar{f}(x) + \bar{f}'(x)f(x))dx$$

and use Cauchy-Schwarz to see that this is less than or equal to

$$2\left(\int x^2 |f(x)|^2 dx\right)^{\frac{1}{2}}\left(\int |f'(x)|^2 dx\right)^{\frac{1}{2}}.$$

The Fourier transform of $f'(x)$ is $it\hat{f}(t)$, so Plancherel's theorem shows that the second parenthesis here is equal to

$$\left(\int t^2 |\hat{f}(t)|^2 dt\right)$$

and this gives the result.

Exercise 4. A locally convex TVS is called a *Montel space* if it is barreled and every closed bounded subset is compact. Show that $\mathscr{D}(\mathbb{R}^n)$ (or $\mathscr{D}(\Omega)$ for any open subset Ω of $\mathbb{R}^n$) is a Montel space.

Prove that if E is a Montel space, and B is a strongly bounded subset of E^*, then the restriction of the strong topology to B is the same as the restriction of the weak-$*$ topology. Deduce that a sequence in E^* converges in the weak-$*$ topology if and only if it converges in the strong topology. (In particular, this applies to sequences of distributions.)

Solution: We proved in an earlier exercise (homework 2) that an inductive limit of barreled spaces is barreled. So $\mathscr{D}(\Omega)$ is barreled. Moreover, a bounded subset B of $\mathscr{D}(\Omega)$ is contained in some $\mathscr{D}_K(\Omega)$, where $K \subseteq \Omega$ is compact, and it is bounded there. That means that there is a sequence of constants C_N such that

$$p_N(f) := \sup\{|D^\alpha f(x)| : x \in K, |\alpha| \leqslant N\} \leqslant C_N$$

for all $f \in B$. By the Mean Value Theorem

$$|D^\alpha f(x) - D^\alpha f(y)| \leqslant C_{|\alpha|+1}|x - y|$$

so each set $\{D^\alpha f : f \in B\}$ is equicontinuous in the sup norm. It follows by the Arzela-Ascoli theorem that any sequence $\{f_n\}$ in B has a subsequence converging in the topology of $\mathscr{D}_K$. Thus, if B is closed, it is compact.

Suppose E is a Montel space and B is a bounded subset of E^*. Then it is equicontinuous (by the Banach-Steinhaus theorem, which applies to barreled spaces.) Theorem 3.12 tells us that, when restricted to B, the topology of pointwise convergence agrees with the topology of uniform convergence on compact subsets of E. But the first of these is just the weak-$*$ topology, and the second is the strong topology (since the compact subsets of a Montel space are just the closed bounded subsets).

Let $\{\phi_n\}$ be a weakly-$*$ convergent sequence in E^*, say with $\phi_n \to \phi_\infty$. Consider the set $S = \{\phi_n : n = 1, 2, \ldots, \infty\}$. This set is weakly-$*$ bounded, hence equicontinuous, and by the previous paragraph the weak-$*$ and strong topologies agree on S. Thus $\phi_n \to \phi$ strongly.

Exercise 5. Let H be a Hilbert space. The space of operators on H of the form $T = \sum_{i=1}^n A_i B_i$, where all the A_i and B_i are Hilbert-Schmidt, is called the space of *trace class operators* on H, and denoted $L^1(H)$. Show that $L^1(H)$ is an ideal in $\mathfrak{B}(H)$ and that for any $T \in L^1(H)$ and any orthonormal basis $\{e_j\}$ for H, the sum

$$\mathrm{Tr}(T) := \sum_j \langle Te_j, e_j\rangle$$

converges absolutely and is independent of the choice of orthonormal basis. Show that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ if both A and B are Hilbert-Schmidt, or if A is trace-class and B is bounded.

Give an example of two bounded operators A and B such that $AB - BA$ is of trace class but $\mathrm{Tr}(AB - BA) \neq 0$. Why does this not contradict the previous result?

Solution: Let $L^2(H)$ denote the space of Hilbert-Schmidt operators on H. As explained in class, $L^2(H)$ is a Hilbert space, with inner product

$$\langle S, T\rangle_{HS} = \sum_j \langle Se_j, Te_j\rangle$$

for any orthonormal basis $\{e_j\}$ for H.

Since $L^2(H)$ is an ideal, the same is true for L^1. If $T = \sum_{i=1}^n A_i B_i$ then we have

$$\mathrm{Tr}(T) = \sum_{i=1}^n \sum_j \langle A_i^* e_j, B_i e_j\rangle = \sum_{i=1}^n \langle A_i^*, B_i\rangle_{HS}$$

which establishes the convergence of the sum and its independence of the choice of orthonormal basis. Moreover if $A, B \in L^2(H)$,

$$\mathrm{Tr}(AB) = \langle A^*, B\rangle_{HS} = \langle B^*, A\rangle_{HS} = \mathrm{Tr}(BA);$$

the middle equality arises by polarization from the equation $\|T\|_{HS} = \|T^*\|_{HS}$ for the Hilbert-Schmidt norm. To obtain the same result when A is trace-class and B is bounded, assume without loss of generality that $A = CD$ where C, D are Hilbert-Schmidt, and then use the previous case and the fact that Hilbert-Schmidt operators for an ideal to write

$$\mathrm{Tr}(AB) = \mathrm{Tr}(C \cdot DB) = \mathrm{Tr}(DB \cdot C) = \mathrm{Tr}(D \cdot BC) = \mathrm{Tr}(BCD) = \mathrm{Tr}(BA).$$

For the final example, let A and B be the operators defined on the sequence space ℓ^2 by

$$A(x_0, x_1, x_2, \dots) = (x_1, x_2, x_3, \dots), \quad B(x_0, x_1, x_2, \dots) = (0, x_0, x_1, \dots).$$

Then $AB - BA$ sends $(x_0, x_1, \dots)$ to $(x_0, 0, 0, \dots)$; it is a trace-class operator of trace 1. The previous reasoning does not apply since AB and BA are not *separately* of trace class.

Warning: Notice that in order for an operator T to be of trace class, it is *not sufficient* that the series $\sum \langle Te_j, e_j\rangle$ converge absolutely for some choice of orthonormal basis. Indeed, for the shift operator (and using the standard basis) all the terms of this series are 0.

Exercise 6. Let $T\colon \mathscr{D}(\mathbb{R}^n) \to \mathscr{D}'(\mathbb{R}^n)$ be an operator defined by a distributional kernel k. Show that k is given by a smooth function (on $\mathbb{R}^n \times \mathbb{R}^n$) if and only if T extends to a continuous linear map $\mathscr{E}'(\mathbb{R}^n) \to \mathscr{E}(\mathbb{R}^n)$ (that is, from compactly supported distributions to smooth functions).

Solution: Note that the space $\mathscr{E}'(\mathbb{R}^n)$ is given its strong topology (as the dual of the Fréchet space $\mathscr{E}(\mathbb{R}^n)$).

(a) Suppose that $Tu(y) = \int k(x,y)u(x)dx$, where k is smooth. For any compactly supported distribution Λ we may define a function $T\Lambda(y) = \Lambda(x \mapsto k(x,y))$. This function is smooth: in fact, we have $D^\alpha T\Lambda(y) = \Lambda(x \mapsto D^\alpha_y k(x,y)$. Thus we have extended T to a map $\mathscr{E}' \to \mathscr{E}$. We must show that this map is continuous.

Let Λ_j be a net in $\mathscr{E}'$ tending strongly to zero. This means that for any bounded subset B of the Fréchet space $\mathscr{E}(\mathbb{R}^n)$, we have $\Lambda_j(f) \to 0$ uniformly for $f \in B$. But the functions $x \mapsto D^\alpha_y k(x,y)$ form a bounded subset of $\mathscr{E}(\mathbb{R}^n)$ as y ranges over a compact subset of $\mathbb{R}^n$ (and α remains fixed). We conclude that $D^\alpha T\Lambda_j(y) \to 0$ uniformly for y in compact subsets of $\mathbb{R}^n$; that is, $T\Lambda_j \to 0$ in $\mathscr{E}(\mathbb{R}^n)$. So T is continuous, as asserted.

(b) Suppose conversely that it is given that $T\colon \mathscr{E}'(\mathbb{R}^n) \to \mathscr{E}(\mathbb{R}^n)$ is continuous. Define

$$k(x,y) = (T\delta_x)(y),$$

where δ_x denotes the delta distribution at x. Now I claim that $x \mapsto \delta_x$ is a continuous map from $\mathbb{R}^n$ to $\mathscr{E}'(\mathbb{R}^n)$. By the definition of the strong topology, to prove this it suffices to show that $f(x)$ is continuous in x, uniformly for f in a bounded subset of $\mathscr{E}(\mathbb{R}^n)$, and this is clearly true by the mean value theorem. Thus $k(x,y)$ is continuous in x (and smooth in y). Moreover a similar argument shows that

$$lim_{h\to 0} h^{-1}(\delta_{x+he_j} - \delta_x) = D^j\delta_x$$

(the limit taken in the strong topology); so (using the continuity of T) the kernel k is in fact differentiable in x with $D^j_x k(x,y) = (-1)(TD^j\delta_x)(y)$. Iterating this argument we find that k is C^∞ in both variables.

It remains to prove that the operator T' defined by the kernel k (which is continuous according to part (a)) is the same as the originally given operator T. By construction, T and T' agree on the span of the delta distributions δ_x so it suffices to show that this span is strongly dense. We can prove this very quickly by appealing to the theorem that every Montel space is reflexive (the dual of its strong dual is the original space). Granted this, by the Hahn-Banach theorem (Proposition 4.7), it suffices to show that the only $f \in \mathscr{E}(X)$ that is annihilated by all δ_x is the zero function; but this is obvious since $\langle \delta_x, f\rangle = f(x)$.

However, we have not discussed reflexivity of Montel spaces so let's give a direct argument for the required density result. It suffices to show that every compactly supported *smooth function* u, when considered as an element of $\mathscr{E}'(\mathbb{R}^n)$, belongs to the closed linear span of the delta distributions (this is because we know that the smooth functions are strongly dense in the distributions). If u is such a function, write

$$u_m = m^{-n} \sum_{k \in \mathbb{Z}^n} u(k/m) \delta_{k/m}.$$

Then u_m is a sequence of linear combinations of δ distributions which converges strongly to u. To see this, let $f \in \mathscr{E}(\mathbb{R}^n)$ range over a bounded set B. Then

$$u_m(f) = m^{-n} \sum_{k \in \mathbb{Z}^n} u(k/m) f(k/m) \to \int u(x) f(x) dx$$

uniformly for $f \in B$, as required.