Math 597A Homework 3 — Due October 27th, 2009

Exercise 1. Give (with proof) an example of a distribution Λ on $\mathbb{R}$ and a smooth, compactly supported function f such that $f = 0$ on Support(Λ) but $f\Lambda \neq 0$. Is it possible to give a similar example where $f = 0$ on a *neighborhood* of Support(Λ)?

Exercise 2. Let Λ_j be a weakly convergent sequence of distributions on $\mathbb{R}^n$ whose supports are contained in some fixed compact subset $K \subseteq \mathbb{R}^n$. Show that the orders of the distributions Λ_j are (uniformly) bounded.

Exercise 3. Let the Schwarz-class function $f \in \mathscr{S}(\mathbb{R})$ satisfy the normalizing condition $\int |f(x)|^2 dx = 1$. Prove the Heisenberg uncertainty relation

$$\left(\int x^2 |f(x)|^2 dx \right) \left(\int t^2 |\hat{f}(t)|^2 dt \right) \geqslant \frac{1}{4}$$

which says that f and $\hat{f}$ cannot simultaneously be "too concentrated". When does equality hold?

Hint: Write

$$1 = \int |f(x)|^2 dx = -\int x \frac{d}{dx} \left(|f(x)|^2 \right) dx.$$

Expand the right-hand side, and use the Cauchy-Schwarz inequality together with Plancherel's theorem.

Exercise 4. A locally convex TVS is called a *Montel space* if it is barreled and every closed bounded subset is compact. Show that $\mathscr{D}(\mathbb{R}^n)$ (or $\mathscr{D}(\Omega)$ for any open subset Ω of $\mathbb{R}^n$) is a Montel space.

Prove that if E is a Montel space, and B is a strongly bounded subset of E^*, then the restriction of the strong topology to B is the same as the restriction of the weak-$*$ topology. Deduce that a sequence in E^* converges in the weak-$*$ topology if and only if it converges in the strong topology. (In particular, this applies to sequences of distributions.)

Exercise 5. Let H be a Hilbert space. The space of operators on H of the form $T = \sum_{i=1}^n A_i B_i$, where all the A_i and B_i are Hilbert-Schmidt, is called the space of *trace class operators* on H, and denoted $L^1(H)$. Show that $L^1(H)$ is an ideal in $\mathfrak{B}(H)$ and that for any $T \in L^1(H)$ and any orthonormal basis $\{e_j\}$ for H, the sum

$$\operatorname{Tr}(T) := \sum_j \langle Te_j, e_j \rangle$$

converges absolutely and is independent of the choice of orthonormal basis. Show that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ if both A and B are Hilbert-Schmidt, or if A is trace-class and B is bounded.

Give an example of two bounded operators A and B such that $AB - BA$ is of trace class but $\mathrm{Tr}(AB - BA) \neq 0$. Why does this not contradict the previous result?

Exercise 6. Let $T\colon \mathscr{D}(\mathbb{R}^n) \to \mathscr{D}'(\mathbb{R}^n)$ be an operator defined by a distributional kernel k. Show that k is given by a smooth function (on $\mathbb{R}^n \times \mathbb{R}^n$) if and only if T extends to a continuous linear map $\mathscr{E}'(\mathbb{R}^n) \to \mathscr{E}(\mathbb{R}^n)$ (that is, from compactly supported distributions to smooth functions).