Math 597A Homework 2 — Due October 6th, 2009

Exercise 1. Let $E_1 \to E_2 \to \dots$ be a (strict) inductive sequence of barreled topological vector spaces (E_i is a closed subspace of E_{i+1} for each i). Show that their inductive limit is also barreled.

Solution. Let B be a barrel in the inductive limit E (closed, convex, balanced, and absorbing). Then $B \cap E_i$ is a closed convex balanced subset of E_i for each i. It is also absorbing in E_i (because B is absorbing in E, any $x \in E_i$ has $\lambda x \in B$ for some $\lambda > 0$; but in fact $\lambda x \in B \cap E_i$ because E_i is a subspace.

Since each E_i is barreled it follows that $B \cap E_i$ is a 0-neighborhood in E_i for each i. By definition of the topology on E, this means that B is an 0-neighborhood in E. (To see this, remark that B is a 0-neighborhood if and only if no net $\{x_\alpha\}$ in $E \setminus B$ can converge to 0. But a convergent net in E must in fact ultimately belong to some E_i and must converge there. Since $E_i \cap B$ is an 0-neighborhood in E_i, the limit of such a net cannot be 0.)

Exercise 2. Let E be the space of *compactly supported* continuous functions on $[0, \infty)$, with the supremum norm. Exhibit a barrel in E that does not contain a neighborhood of the origin. Also, exhibit a pointwise bounded sequence of linear functionals on E that is not uniformly bounded.

Solution Let $B = \{f \in E : |f(n)| \leqslant 1/n \ (n = 1, 2, 3, \dots)\}$. Clearly B is closed, convex, and balanced. It is also absorbing: given any $f \in E$ there exists an integer N such that f is supported in $[0, N]$; and then $(N\|f\|)^{-1} f \in B$. But B contains no neighborhood of 0 since it does not contain any of the balls $B(0; \epsilon)$, $\epsilon > 0$.

Related to this, consider the linear functionals $\phi_n(f) = nf(n)$. For any fixed f, the sequence $\{\phi_n(f)\}$ is pointwise bounded (in fact, it is eventually 0); but $\|\phi_n\| = n$ so the sequence is not *uniformly* bounded.

Exercise 3. Let E be a Fréchet space, E^* its dual space. Which of the following are true? Give proofs or counterexamples as appropriate.

(a) If U is a 0-neighborhood in E, then U° is strongly bounded in E^*.

(b) If B is bounded in E, then B° is a strong 0-neighborhood in E^*.

(c) If V is a strong 0-neighborhood in E^*, then ${}^\circ V$ is bounded in E.

(d) If C is bounded in E^*, then ${}^\circ C$ is a 0-neighborhood in E.

Solution All are true. First, (b) is the definition of the strong dual topology. To prove (a), let $A = U^\circ \subseteq E^*$ and let V be a 0-neighborhood in E^*, which we may assume is of the form V° where B is a bounded set in E. There is λ_0 such that $B \subseteq \lambda U$ for all $\lambda \geqslant \lambda_0$. But this implies $A \subseteq \lambda V$. Thus A is bounded.

Conversely, to prove (c), suppose V is a strong 0-neighborhood in E^*. Then (by definition) it contains an 0-neighborhood of the form B° where B is a bounded set, which we may assume is closed, convex, and balanced. (In any locally convex space the closed, convex, balanced hull of a bounded set is bounded.) It follows that ${}^\circ V \subseteq {}^\circ(B^\circ) = B$ by the bipolar theorem, so ${}^\circ V$ is bounded.

To prove (d), if C is bounded in E^*, then it is equicontinuous (by the Banach-Steinhaus theorem). Therefore there is a 0-neighborhood U in E such that if $\phi \in C$, $\phi(U) \subseteq D(0;1)$. It follows that $U \subseteq {}^\circ C$, which is therefore a 0-neighborhood.

Exercise 4. Let E be the Banach space ℓ^1 (for this question it is convenient to think of this space as made up of functions $f\colon \mathbb{N} \to \mathbb{C}$ such that $\sum_{n=1}^\infty |f(n)| < \infty$). It is known that the dual E^* of E can be identified with the space ℓ^∞ of *bounded* functions on g, with the pairing

$$\langle g, f\rangle = \sum_{n=1}^\infty f(n)g(n).$$

Show that the weak topology on E is different from its strong (norm) topology. Show that, nevertheless, every weakly convergent *sequence* in E also converges in norm. (Hints for the last part: Let $\{f_k\}$ be a sequence converging weakly to zero. Observe that it is enough to show that for each $\epsilon > 0$ there is N such that $\sum_{n=N}^\infty |f_k(n)| < \epsilon$ for all k. To prove this assertion, argue by contradiction: if it is not so one can obtain by induction a sequence of disjoint intervals $[M_\ell, N_\ell]$ and functions f_{k_ℓ} so that

$$\sum_{n=M_\ell}^{N_\ell} |f_{K_m}| \begin{cases} \geqslant \epsilon/2 & (\ell = m) \\ \leqslant 2^{-(\ell+m)} & (\ell \neq m) \end{cases}$$

Use this to contradict weak convergence.)

Solution To see that the topologies are not the same, recall that a basis for the weak 0-neighborhoods comprises the prepolars of finite subsets $\{g_1, \ldots, g_n\}$ of E^*. Any such prepolar however contains the infinite dimensional subspace $\bigcap \mathrm{Ker}\langle g_i, \cdot\rangle$. Since no such subspace is contained in a strong 0-neighborhood, the weak and strong topologies cannot be the same.

Now suppose that f_k is a sequence converging weakly to 0. In particular, each $f_k(n) \to 0$ (n fixed, $k \to \infty$). I claim that it is enough to show that for each $\epsilon > 0$ there is N such that $\sum_{n=N}^{\infty} |f_k(n)| < \epsilon$ for all k. For, if this is proved, then pointwise convergence shows that $\sum_{n=1}^{\infty} |f_k(n)| < 2\epsilon$ for all sufficiently large K, which tells us that $f_k \to 0$ strongly.

Suppose then that the hypothesis is false. Then there is $\epsilon > 0$ with the following property: for any N one can find infinitely many k such that $\sum_{n=N}^{\infty} |f_k(n)| > \epsilon$. We now carry out an inductive construction as suggested in the hint: we choose an increasing sequence of intervals $[M_\ell, N_\ell]$ and subscripts K_ℓ. Supposing that these have been chosen for $\ell < L$ we proceed as follows. First, choose M_L such that $\sum_{n=M_L}^{\infty} |f_{K_\ell}(n)| \leqslant 2^{-\ell+L}$ for $\ell = 1, \ldots, L-1$; this can be done because each of the finitely many functions $f_{K\ell}$ belongs to ℓ^1. Next, choose K_L so that

- $\sum_{n=M_L}^{\infty} |f_{K_L}(n)| \geqslant \epsilon$, and
- $\sum_{n=M_\ell}^{N_\ell} |f_{K_L}(n)| \leqslant 2^{-\ell+L}$ for $\ell = 1, \ldots, L-1$.

This can be done because there are infinitely many functions satisfying the first bullet point, forming a subsequence of $\{f_k\}$ tending weakly to 0; weak convergence implies that the sums appearing on the left hand side in the second bullet tend to 0, so we can choose a member of the sequence making them as small as required. Finally, choose N_L so that $\sum_{n=M_L}^{N+L} |f_{K_L}(n)| \geqslant \epsilon/2$. This completes the inductive construction of disjoint intervals $[M_\ell, N_\ell]$ and functions f_{k_ℓ} so that

$$\sum_{n=M_\ell}^{N_\ell} |f_{K_m}| \begin{cases} \geqslant \epsilon/2 & (\ell = m) \\ \leqslant 2^{-(\ell+m)} & (\ell \neq m) \end{cases}$$

Define $g \in \ell^\infty$ by setting $g(n) = e^{-i\theta_n}$ if $n \in [M_\ell, N_\ell]$ and $f_{K_\ell}(n) = r_n e^{i\theta_n}$, and $g(n) = 0$ otherwise. Then

$$|\langle g, f_{K_\ell}\rangle| \geqslant (\epsilon/2) - \sum_{m \neq \ell} 2^{-(m+\ell)} \geqslant (\epsilon/2) - 2 \cdot 2^{-\ell}.$$

Since the right hand side does not tend to 0, this contradicts weak convergence.

Exercise 5. Let H be the Hilbert space $L^2[-\pi,\pi]$ and let $A \subseteq H$ be the set of all functions $e^{imt} + me^{int}$, where $n > m \geqslant 0$ are integers. Let B be the *weak sequential closure* of A, that is the set of all limits of weakly convergent sequences of members of A.

Find B. Also, show that there is a weakly convergent sequence of members of B whose limit is not a member of B (that is, B is not weakly sequentially closed.)

What is the weak closure of A? (its closure in the weak topology)

Solution. Let $e_k(t) = e^{ikt}$, so that the given functions are $f_{m,n} = e_m + me_n$, $n > m \geqslant 0$. We know that $e_n \to 0$ weakly as $n \to \infty$. It follows that the functions $e_m = \lim_{n\to\infty} f_{m,n}$ belong to the weak sequential closure of A. I claim that they are the only points in this closure, apart from those belonging to A itself.

To see this recall that a weakly convergent sequence is weakly bounded, hence norm bounded. Since $\|f_{m,n}\| = (1+m^2)^{1/2}$ it follows that any weakly convergent sequence in A can involve only finitely many values of m. It must therefore have a subsequence involving only a single value of m, and this subsequence (and therefore the whole original sequence) necessarily must converge to e_m if it involves infinitely many values of n; if not, it converges to a point of A.

Now the weak sequential closure B contains the sequence $\{e_m\}$ which converges weakly to 0, which is *not* a member of B; thus B is not weakly sequentially closed.

I claim that the weak closure of A is $C = B \cup \{0\}$. We have already shown that each point of C is a weak limit point of A, so it suffices to show that C is weakly closed. We will show that the complement of C is weakly open.

If a point $a = \sum a_j e_j$ of H does *not* belong to C there are three possibilities:

- Exactly one a_j is nonzero, and that $a_j \neq 1$ (so a is not an e_m)
- Exactly two a_j are nonzero, and a is not an $f_{m,n}$
- Three or more a_j are nonzero.

In the third case let j_1, j_2, j_3 be the values of j of which $a_j \neq 0$. The map $\psi\colon H \to \mathbb{C}^3$ defined by $x \mapsto (\langle e_{j_1}, x\rangle, \langle e_{j_3}, x\rangle, \langle e_{j_3}, x\rangle)$ is weakly continuous. The inverse image of a neighborhood of $\psi(a) \in \mathbb{C}^3$ defines a weak neighborhood of a not meeting C. The proofs in the other two cases are analogous.

Exercise 6. Let E and F be Banach spaces and endow the space $L(E, F)$ of continuous linear maps from E to F with its usual (norm) topology. Show that the surjective maps form an open subset of $L(E, F)$. Is the same thing true of the injective maps?

Solution Let $T\colon E \to F$ be a surjection of Banach spaces. By the open mapping theorem, T is a homomorphism, and it follows that there is a constant C such that for any $y \in F$, there exists $x \in E$ with $Tx = y$ and $\|x\| \leqslant C\|y\|$.

Suppose now that $S\colon E \to F$ has $\|S - T\| < C^{-1}$ and put $r = C^{-1}\|S - T\| < 1$. Let $y \in F$ be given. Put $y = y_0$ and inductively choose sequences $x_0, x_1, \ldots$ in E and $y_0, y_1, \ldots$ in F with

$$Tx_n = y_n, \quad \|x_n\| \leqslant C\|y_n\|$$

and

$$Sx_n = y_n - y_{n+1}, \quad \|y_{n+1}\| = \|(S - T)x_n\| \leqslant r\|y_n\|.$$

Inductively we find that $\|y_n\| \leqslant r^n\|y\|$ and $\|x_n\| \leqslant Cr^n\|y\|$. Thus the series $x = \sum_{n=0}^{\infty} x_n$ converges in the Banach space E, and

$$Sx = \sum_{n=0}^{\infty} Sx_n = \sum_{n=0}^{\infty}(y_n - y_{n+1}) = y.$$

Hence S is surjective. We have proved that every map sufficiently close (in norm) to the surjection T is also surjective; so the set of surjections is open.

The same does *not* apply to the set of injections. Let E be the Hilbert space ℓ^2 and let $T\colon E \to E$ be the linear map defined by

$$T(a_1, a_2, a_3, \ldots) = (a_1, \tfrac{1}{2}a_2, \tfrac{1}{3}a_3, \ldots).$$

Then T is injective, but for any $\epsilon > 0$ one can find a non-injective map (given by truncating T sufficiently far along the sequence) that is within ϵ of T in norm.