Math 597A Homework 2 — Due October 6th, 2009

Exercise 1. Let $E_1 \to E_2 \to \dots$ be a (strict) inductive sequence of barreled topological vector spaces (E_i is a closed subspace of E_{i+1} for each i). Show that their inductive limit is also barreled.

Exercise 2. Let E be the space of *compactly supported* continuous functions on $[0, \infty)$, with the supremum norm. Exhibit a barrel in E that does not contain a neighborhood of the origin. Also, exhibit a pointwise bounded sequence of linear functionals on E that is not uniformly bounded.

Exercise 3. Let E be a Fréchet space, E^* its dual space. Which of the following are true? Give proofs or counterexamples as appropriate.

(a) If U is a 0-neighborhood in E, then U° is strongly bounded in E^*.

(b) If B is bounded in E, then B° is a strong 0-neighborhood in E^*.

(c) If V is a strong 0-neighborhood in E^*, then ${}^\circ V$ is bounded in E.

(d) If C is bounded in E^*, then ${}^\circ C$ is a 0-neighborhood in E.

Exercise 4. Let E be the Banach space ℓ^1 (for this question it is convenient to think of this space as made up of functions $f\colon \mathbb{N} \to \mathbb{C}$ such that $\sum_{n=1}^{\infty} |f(n)| < \infty$. It is known that the dual E^* of E can be identified with the space ℓ^∞ of *bounded* functions on g, with the pairing

$$\langle g, f \rangle = \sum_{n=1}^{\infty} f(n) g(n).$$

Show that the weak topology on E is different from its strong (norm) topology. Show that, nevertheless, every weakly convergent *sequence* in E also converges in norm. (Hints for the last part: Let $\{f_k\}$ be a sequence converging weakly to zero. Observe that it is enough to show that for each $\epsilon > 0$ there is N such that $\sum_{n=N}^{\infty} |f_k(n)| < \epsilon$ for all k. To prove this assertion, argue by contradiction: if it is not so one can obtain by induction a sequence of disjoint intervals $[M_\ell, N_\ell]$ and functions f_{k_ℓ} so that

$$\sum_{n=M_\ell}^{N_\ell} |f_{K_m}| \begin{cases} \geqslant \epsilon/2 & (\ell = m) \\ \leqslant 2^{-(\ell+m)} & (\ell \neq m) \end{cases}$$

Use this to contradict weak convergence.)

Exercise 5. Let H be the Hilbert space $L^2[-\pi, \pi]$ and let $A \subseteq H$ be the set of all functions $e^{imt} + me^{int}$, where $n > m \geqslant 0$ are integers. Let B be the *weak sequential closure* of A, that is the set of all limits of weakly convergent sequences of members of A.

Find B. Also, show that there is a weakly convergent sequence of members of B whose limit is not a member of B (that is, B is not weakly sequentially closed.)

What is the weak closure of A? (its closure in the weak topology)

Exercise 6. Let E and F be Banach spaces and endow the space $L(E, F)$ of continuous linear maps from E to F with its usual (norm) topology. Show that the surjective maps form an open subset of $L(E, F)$. Is the same thing true of the injective maps?