Math 597A Homework 1 — Due September 15th, 2009
Solutions

Exercise 1. Let E be a Fréchet space (a LCTVS whose topology is defined by countably many seminorms, and which is therefore metrizable). If $x_n \in E$ is a sequence tending to zero, show that there is a sequence λ_n of positive real numbers, tending to infinity, such that that $\lambda_n x_n \to 0$.

Solution. E is metrizable with a translation-invariant metric d (in fact, this is all you need here.) Let $d_n = d(x_n, 0)$ and let λ_n be a sequence of integers tending to infinity such that $\lambda_n d_n \to 0$. (For instance, one could take λ_n to be the integer part of $(d_n)^{-1/2}$, or n if d_n happens to be 0.) Then $d(\lambda_n x_n, 0) \leqslant d(0, x_n) + d(x_n, 2x_n) + \cdots + d((\lambda_n - 1)x_n, \lambda_n x_n) = \lambda_n d_n$ by the triangle inequality and translation-invariance, and this tends to 0. **Warning:** The distance $d(0, x)$ is NOT a norm, so it is incorrect to write $d(0, \lambda x) = \lambda d(0, x)$. the argument above using the triangle inequality is designed to circumvent this point.

Exercise 2. Let F be a subspace of a Hausdorff topological vector space E. One says that F is *complemented* in E if there is a subspace G of E such that $E = F \oplus G$ and the projection maps $E \to F$, $E \to G$ are continuous.

Show that if F is closed and of finite codimension (that is, E/F is finite-dimensional) then it is complemented.

Show that if E is locally convex and F is of finite dimension, then it is complemented.

It is known (Lindenstrauss-Tzafriri 1971) that if E is a Banach space in which *every* closed subspace is complemented, then in fact E is (isomorphic to) a Hilbert space. This is a hard theorem.

Solution. (a) If E/F is finite-dimensional, choose a basis $\{y_1, \ldots, y_n\}$ for E/F and let $\{x_1, \ldots, x_n\}$ be elements of E such that $\pi(x_j) = y_j$ where $\pi \colon E \to E/F$ is the canonical projection. Then $\{x_1, \ldots, x_n\}$ span a closed subspace G which is an *algebraic* complement to F. The map which sends y_j to x_j extends to a linear homeomorphism h between the finite-dimensional spaces E/F and G. The projection of E onto G along F is now just $h \circ \pi$; a composite of continuous maps, so continuous. (It is enough to show that just one of the two projection maps is continuous, since they add up to the identity.)

(b) If F is finite-dimensional, and E is locally convex, choose a basis $\{e_1, \ldots, e_n\}$ for F. The functional ψ_j that sends $\sum \lambda_i e_i \in F$ to the coefficient

λ_j of e_j is continuous. By the Hahn-Banach theorem extend it to a continuous linear functional ϕ_j on E. Let $G = \bigcap_{i=1}^{n} \mathrm{Ker}(\phi_i)$. Then G is an algebraic complement to F and the projection onto F along G is

$$x \mapsto \sum_{i=1}^{n} \phi_i(x) e_i$$

and is continuous.

Warning: Several students attempted to apply the implication "kernel closed implies continuous" to the *projection* maps (in (a) or (b)). This implication is true only for linear *functionals* (maps to the ground field). For infinite-dimensional codomain it is easy to give examples of maps with closed kernel which are not continuous.

Exercise 3. Let E be a Hausdorff TVS. A subset A of E is called *bounded* if it can be absorbed by every neighborhood of the origin: that is, for every 0-neighborhood U there exists $\epsilon > 0$ such that $\lambda A \subseteq U$ for all $|\lambda| < \epsilon$.

Show that this definition agrees with the usual one in a Banach space. Show that the closure of a bounded set is bounded and that the sum or union of two bounded sets is bounded. Show that if E is locally convex, with topology defined by seminorms $\{p_\alpha\}$, then a set A is bounded if and only if $p_\alpha(A)$ is bounded for each α.

Solution. Let E be a Banach space. If A is bounded (in the above sense) then it can be absorbed by the unit ball, so $\{\|a\| : a \in A\}$ is a bounded set of real numbers. Conversely if this holds then A can be absorbed by every ball, hence by every 0-neighborhood, so is bounded.

Let A be a bounded set and let U be a balanced 0-neighborhood. There exists another balanced 0-neighborhood V such that $\overline{V} \subseteq U$ (for instance, if $V + V \subseteq U$, then V has this property). Since A is bounded, $A \subseteq rV$ for some r, and then $\overline{A} \subseteq rU$. So $\overline{A}$ is bounded too. Similarly, if A and B are bounded and U is a balanced 0-neighborhood, there are r, s such that $A \subseteq rU$, $B \subseteq sU$. Then $\lambda(A \cup B) \subseteq U$ for all $\lambda < \max\{r, s\}^{-1}$. So $A \cup B$ is bounded too.

The most interesting case is $A + B$. Let U be a 0-neighborhood. Let V be a balanced 0-neighborhood such that $V + V \subseteq U$. There exist r, s with $A \subseteq rV$, $B \subseteq sV$. Then for $\lambda < \max\{r, s\}^{-1}$, we have $\lambda(A+B) \subseteq V+V \subseteq U$. Hence $A + B$ is bounded. **Warning:** Many students gave incorrect proofs which depended on the false inclusion $rU + sU \subseteq (r + s)U$. This holds if U

is convex but not in general. Since we did not say that E was a LCTVS we may not assume that U is convex.

Suppose the topology of E is defined by seminorms p_α. If A is bounded then for each α it is absorbed by a multiple (say r_α) of the open set $B_\alpha = \{x : p_\alpha(x) < 1\}$. But this simply says that $p_\alpha(x) < r_\alpha$ on A. Conversely, if this condition is satisfied, then A is absorbed by every finite intersection of multiples of B_α's. Such sets form a basis for the 0-neighborhoods, so A is bounded.

Exercise 4. Let E be a Hausdorff LCTVS. Show that the topology of E can be defined by a single norm if and only if there exists a bounded neighborhood of the origin.

Use this fact to show that the topology of the space $C^\infty[0,1]$ cannot be defined by a single norm.

Solution. If the topology of E can be defined by a single norm, then the unit ball is a bounded 0-neighborhood.

Conversely, suppose that U is a bounded, convex, balanced 0-neighborhood and let p be the associated Minkowski functional. It is a seminorm; we prove that, in fact, it is a norm. Let $x \neq 0$; then there is a balanced 0-neighborhood V that does not contain x. There is an r such that $U \subseteq rV$. Now $sx \notin V$ for any $0 \leqslant s \leqslant r$, and so $p(x) \geqslant 1/r$. In particular, $p(x) \neq 0$, and p is a norm. By boundedness, any 0-neighborhood contains a positive multiple of U; so the norm p defines the topology of E.

To apply this to $E = C^\infty[0,1]$, let p_n, $n = 0, 1, \ldots$, be the seminorms $p_n(f) = \sup\{|f^{(n)}(x)|\}$ that define the topology of E. If there were a bounded neighborhood of the origin, there would be one of the form $\{f : p_j(f) \leqslant c_j,\ j = 0, \ldots, N\}$ for some N. Thus, a boundedness on the first N derivatives of a function f would have to imply a bounded on the $(N+1)$st derivative. But examples of the form $f(x) = \epsilon \sin(Kx)$, ϵ small, K large, show that this is not the case.

Exercise 5. Must the set of all extreme points of a compact convex set be compact?

Solution. No. Let C be a strictly convex curve (e.g. a circle) in the xy-plane, passing through the origin. ("Strictly convex" means that each point of the curve is an extreme point of its closed convex hull.) Let p and q be two points on the z-axis (not the origin), one above and one below the

xy-plane. Let K be the closed convex hull of p, q, and C. Then every point of C except the origin is an extreme point of K, but the origin is not.

Exercise 6. Let c_0 be the Banach space of all complex sequences $(a_1, a_2, \ldots)$ that tend to zero (with the supremum norm). Show that the closed unit ball of c_0 is a convex set that has *no* extreme points.

Suppose for a moment that $E = c_0$ were the dual space of another Banach space F. Show that the closed unit ball of E would be a *compact* convex set when equipped with the $\sigma(E, F)$-topology. (Use Tychonoff's theorem applied to a product of copies of the closed unit disc in $\mathbb{C}$, parameterized by the unit ball of F.)

Deduce that, in fact, c_0 is not the dual of any Banach space.

Solution. A closed unit ball is always a convex set. If B is the closed unit ball in c_0, let $\mathbf{a} = (a_1, a_2, \ldots)$ be a point of B. For some N, $|a_N| < \frac{1}{2}$. Let $\mathbf{e}_N$ be the vector with 1 in the Nth slot and 0 elsewhere. Then the two vectors $\mathbf{a} \pm \frac{1}{2}\mathbf{e}_n$ are both in the unit ball B, and of course $\mathbf{a}$ is a proper convex combination of them (their average). So B has no extreme points.

The second part is a special case of Alaoglu's theorem, proved later in the course.

If E were the dual space of F, the unit ball of E would be a compact convex set without extreme points. This would contradict the Krein-Milman theorem.