Math 597A Homework 1 — Due September 15th, 2009

Exercise 1. Let E be a Fréchet space (a LCTVS whose topology is defined by countably many seminorms, and which is therefore metrizable). If $x_n \in E$ is a sequence tending to zero, show that there is a sequence λ_n of positive real numbers, tending to infinity, such that that $\lambda_n x_n \to 0$.

Exercise 2. Let F be a subspace of a Hausdorff topological vector space E. One says that F is *complemented* in E if there is a subspace G of E such that $E = f \oplus G$ and the projection maps $E \to F$, $E \to G$ are continuous.

Show that if F is closed and of finite codimension (that is, E/F is finite-dimensional) then it is complemented.

Show that if E is locally convex and F is of finite dimension, then it is complemented.

It is known (Lindenstrauss-Tzafriri 1971) that if E is a Banach space in which *every* closed subspace is complemented, then in fact E is (isomorphic to) a Hilbert space. This is a hard theorem.

Exercise 3. Let E be a Hausdorff TVS. A subset A of E is called *bounded* if it can be absorbed by every neighborhood of the origin: that is, for every 0-neighborhood U there exists $\epsilon > 0$ such that $\lambda A \subseteq U$ for all $|\lambda| < \epsilon$.

Show that this definition agrees with the usual one in a Banach space. Show that the closure of a bounded set is bounded and that the sum or union of two bounded sets is bounded. Show that if E is locally convex, with topology defined by seminorms $\{p_\alpha\}$, then a set A is bounded if and only if $p_\alpha(A)$ is bounded for each α.

Exercise 4. Let E be a LCTVS. Show that the topology of E can be defined by a single norm if and only if there exists a bounded neighborhood of the origin.

Use this fact to show that the topology of the space $C^\infty[0,1]$ cannot be defined by a single norm.

Exercise 5. Must the set of all extreme points of a compact convex set be compact?

Exercise 6. Let c_0 be the Banach space of all complex sequences $(a_1, a_2, \ldots)$ that tend to zero (with the supremum norm). Show that the closed unit ball of c_0 is a convex set that has *no* extreme points.

Suppose for a moment that $E = c_0$ were the dual space of another Banach space F. Show that the closed unit ball of E would be a *compact* convex set when equipped with the $\sigma(E, F)$-topology. (Use Tychonoff's theorem applied to a product of copies of the closed unit disc in $\mathbb{C}$, parameterized by the unit ball of F.)

Deduce that, in fact, c_0 is not the dual of any Banach space.