Notes on Operator Algebras

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Abstract

These are the lecture notes for the Penn State course Math 520 held in Fall 2000. They will be revised and extended as the course progresses.

1 C^*-Algebra Basics

The key property that relates the norm and the involution on $\mathfrak{B}(H)$ is the C^*-*identity*:

$$\|T^*T\| = \|T\|^2.$$

The proof follows from Cauchy-Schwarz: if $\|v\| = 1$, then

$$\|T\|\|T^*\| \geqslant \|T^*T\| \geqslant |\langle T^*Tv, v\rangle| = \|Tv\|^2$$

and so by taking the supremum over all v we find

$$\|T\|\|T^*\| \geqslant \|T^*T\| \geqslant \|T\|^2$$

whence we obtain both $\|T\| = \|T^*\|$ and the C^*-identity.

(1.1) DEFINITION: *A* C^*-algebra A *is a complex involutive Banach algebra satisfying the* C^**-identity:* $\|a^*a\| = \|a\|^2$ *for all* $a \in A$*.*

One easily deduces that the involution is isometric. We will show eventually that every C^*-algebra is faithfully represented as an algebra of operators on Hilbert space, but it is useful to begin with the abstract definition. Compare 'permutation group theory' versus 'abstract group theory'.

(1.2) REMARK: about units. We shan't necessarily assume that a C^*-algebra A has a unit. Indeed, non-unital C^*-algebras are extremely important — the first fundamental example is the algebra $\mathfrak{K}(H)$ of *compact* operators on a Hilbert space H. However it is the case that every non-unital C^*-algebra A can be embedded in a unital C^*-algebra $\widetilde{A}$ as a closed maximal ideal. To figure out the construction, it helps to consider the case that A is a subalgebra of $\mathfrak{B}(H)$. Then $\widetilde{A}$ should be the algebra generated by A and 1 inside $\mathfrak{B}(H)$. We thus arrive at the following idea: $\widetilde{A}$ is the algebra of pairs (a, λ), with $a \in A$ and $\lambda \in \mathbb{C}$. Multiplication is defined in the obvious way if we think of (a, λ) as $a + \lambda 1$, namely

$$(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda\mu).$$

One shows this is a Banach algebra, and indeed a C^*-algebra with the involution $(a, \lambda) \mapsto (a^*, \bar{\lambda})$, by representing it as an algebra of left multiplication operators on A, and using the operator norm. The algebra $\widetilde{A}$ is called the *unitalization* of A.

An important example of a C^*-algebra is $C(X)$, the algebra of continuous complex-valued functions on a compact Hausdorff space X. If X is just *locally* compact then $C_0(X)$ denotes the algebra of continuous complex-valued functions which vanish at infinity. It is a non-unital C^*-algebra.

An important example of an involutive Banach algebra which is *not* a C^*-algebra is the *disk algebra*: this is the algebra of continuous complex-valued functions on the closed disk in $\mathbb{C}$ which are holomorphic in the interior of the disk, with the involution

$$f^*(z) = \overline{f(\bar{z})}.$$

1.1 Banach Algebra Basics

We review some basic information about Banach algebras. Let A be a unital Banach algebra.

(1.3) DEFINITION: *The* spectrum $\mathrm{sp}(a)$ *of* $a \in A$ *is the set of complex numbers* λ *such that* $a - \lambda 1$ *does* not *have an inverse in* A*. The complement of the spectrum is called the* resolvent set*.*

We'll use the notation $\mathrm{sp}_A(a)$ if it is necessary to specify the algebra A — although, as we shall see later, this will seldom be necessary for C^*-algebras. If A is non-unital, we define the spectrum by passing to the unitalization. For the record, 'invertible' means 'having a two-sided inverse in the algebra A'. Note that this is a purely algebraic notion — the norm of A is not involved.

(1.4) LEMMA: *The resolvent set of* $a \in A$ *is open (and so the spectrum is closed).*

PROOF: It suffices to show that every element sufficiently close to $1 \in A$ is invertible, which follows from the absolutely convergent power series expansion

$$(1-x)^{-1} = 1 + x + x^2 + \cdots ,$$

which is valid for $\|x\| < 1$ in any Banach algebra. ■

(1.5) COROLLARY: *Maximal ideals in a unital Banach algebra* A *are closed. Every algebra homomorphism* $\alpha\colon A \to \mathbb{C}$ *is continuous, of norm* $\leqslant 1$*.*

PROOF: Let $\mathfrak{m}$ be a maximal ideal in A. Then $\mathfrak{m}$ does not meet the open set of invertible elements in A. The closure $\overline{\mathfrak{m}}$ then does not meet the set of invertibles either, so it is a proper ideal, and by maximality $\overline{\mathfrak{m}} = \mathfrak{m}$.

Since the kernel of α is a maximal ideal, it is closed. Thus α is continuous, by the usual continuity criterion for linear functionals. The norm estimate is obtained by refining the argument: if $\|\alpha\| > 1$ then there is $a \in A$ with $\|a\| < 1$ and $\alpha(a) = 1$. But then $1 - a$ is invertible, so $1 - \alpha(a)$ is invertible too, and this is a contradiction. ■

(1.6) PROPOSITION: *Let A be a unital Banach algebra, and let $a \in A$. Then the spectrum* $\mathrm{sp}(a)$ *is a nonempty compact subset of* $\mathbb{C}$. *Moreover, the* spectral radius

$$\mathrm{spr}(a) = \inf\{r \in \mathbb{R}^+ : \mathrm{sp}(a) \subseteq D(0;r)\}$$

is given by the formula

$$\mathrm{spr}(a) = \lim_{n\to\infty} \|a^n\|^{1/n}.$$

PROOF: The power series expansion

$$(\lambda - a)^{-1} = \lambda^{-1} + \lambda^{-2}a + \lambda^{-3}a^2 + \cdots \tag{1.7}$$

converges for $|\lambda| > \|a\|$ and shows that the spectrum of a is contained within $D(0;\|a\|)$ (note in particular that $\mathrm{spr}(a) \leqslant \|a\|$). We already showed that the spectrum is closed, so it is compact. It is nonempty since otherwise the function

$$\lambda \mapsto (\lambda - a)^{-1}$$

would be a bounded, entire function (with values in A), contradicting Liouville's Theorem from complex analysis.[1] Finally let us prove the spectral radius formula. Since by hypothesis the resolvent function $\lambda \mapsto (\lambda - a)^{-1}$ is holomorphic for $|\lambda\| > \mathrm{spr}(a)$, elementary complex analysis (Cauchy's n'th root test) yields the estimate

$$\limsup \|a^n\|^{1/n} \leqslant \mathrm{spr}(a)$$

for the coefficients of the Laurent expansion 1.7 above. On the other hand, it is easy to see that for any polynomial p, the spectrum of $p(a)$ is $p(\mathrm{sp}(a))$. Therefore

$$\mathrm{spr}(a)^n = \mathrm{spr}(a^n) \leqslant \|a^n\|$$

and so

$$\mathrm{spr}(a) \leqslant \inf \|a^n\|^{1/n}.$$

Putting together our two estimates for the spectral radius of a, we complete the proof. ■

(1.8) LEMMA: *(Gelfand-Mazur theorem) The only Banach algebra which is also a field is* $\mathbb{C}$.

PROOF: Let A be such an algebra, $a \in A$. Then a has nonempty spectrum, so there is some $\lambda \in \mathbb{C}$ such that $a - \lambda 1$ is not invertible. But in a field the only non-invertible element is zero; so $a = \lambda 1$ is scalar. ■

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[1] This often seems a surprising argument. In finite dimensions the existence of eigenvalues follows from the Fundamental Theorem of Algebra. But one way to prove the Fundamental Theorem is by way of Liouville's Theorem — if p is a polynomial without root then $1/p$ is a bounded entire function — so the Banach algebra argument is not really so different.

(1.9) PROPOSITION: *Let A be a unital C^*-algebra. If $a \in A$ is normal (that is, commutes with a^*) then $\|a\| = \operatorname{spr}(a)$. For every $a \in A$, $\|a\| = \operatorname{spr}(a^*a)^{1/2}$. Thus the norm on A is completely determined by the algebraic structure!*

PROOF: If a is normal then

$$\|a\|^2 = \|a^*a\| = \|a^*aa^*a\|^{1/2} = \|a^2a^{*2}\|^{1/2} = \|a^2\|$$

using the C^*-identity several times. Thus the spectral radius formula gives $\|a\| = \operatorname{spr}(a)$. Whatever a is, the element $b = a^*a$ is normal (indeed selfadjoint), so the second part follows from the first applied to b, together with the C^*-identity again. ■

Gelfand used the results above to give a systematic analysis of the structure of *commutative* unital Banach algebras. If A is such an algebra then its *Gelfand dual* $\widehat{A}$ is the space of maximal ideals of A; equivalently, by 1.8 and 1.5 above, it is the space of algebra-homomorphisms $A \to \mathbb{C}$. Every such homomorphism is of norm $\leqslant 1$, as we saw, so $\widehat{A}$ may be regarded as a subset of the unit ball of the Banach space dual A^* of A (namely, it is the subset comprising all those linear functionals α which are also *multiplicative* in the sense that $\alpha(xy) = \alpha(x)\alpha(y)$). This is a weak-star closed subset of the unit ball of A^*, and so by the Banach-Alaoglu Theorem it is a compact Hausdorff space in the weak-star topology. If $a \in A$ then we may define a continuous function $\hat{a}$ on $\widehat{A}$ by the usual double dualization:

$$\hat{a}(\alpha) = \alpha(a).$$

In this way we obtain an algebra-homomorphism $\mathcal{G}\colon a \mapsto \hat{a}$, called the *Gelfand transform*, from A to $C(\widehat{A})$, the algebra of continuous functions on the Gelfand dual.

(1.10) THEOREM: *Let A be a commutative unital Banach algebra. An element $a \in A$ is invertible if and only if its Gelfand transform $\hat{a} = \mathcal{G}a$ is invertible. Consequently, the Gelfand transform preserves spectrum: the spectrum of $\mathcal{G}a$ is the same as the spectrum of a.*

PROOF: If a is invertible then $\mathcal{G}a$ is invertible, since $\mathcal{G}$ is a homomorphism. On the other hand, if a is *not* invertible then it is contained in some maximal ideal $\mathfrak{m}$, which corresponds to a point of the Gelfand dual on which $\mathcal{G}a$ vanishes. So $\mathcal{G}a$ isn't invertible either. ■

The algebra $C(\widehat{A})$ has an involution, given by pointwise complex conjugation. One might thus ask: if A is a Banach $*$-algebra, must $\mathcal{G}$ be a $*$-homomorphism, that is, preserve the involutions? To see that the answer is no, look at the disk algebra again. The Gelfand dual of the disk algebra is the closed disk (exercise!) and the

Gelfand transform is the identity map, but the involution on the disk algebra is of course not pointwise complex conjugation.

It turns out that this is related to the existence in the disk algebra of "selfadjoint" elements whose spectrum is not real (one says that the disk algebra is not *symmetric*). For instance the function z is "selfadjoint" in the disk algebra, but its spectrum is the whole disk. One has a simple

(1.11) LEMMA: *Let A be a commutative unital Banach $*$-algebra. Then the Gelfand transform is a $*$-homomorphism if and only if the spectrum of every "selfadjoint" element of A is real.*

PROOF: In view of the Theorem above, the hypothesis implies that a "selfadjoint" element of A has real-valued Gelfand transform. Now split a general $a \in A$ into real and imaginary parts:

$$a = \frac{a + a^*}{2} + \frac{a - a^*}{2}$$

and apply the Gelfand transform separately to each. ■

(1.12) PROPOSITION: *A selfadjoint element of a C^*-algebra has real spectrum. Consequently, the Gelfand transform for a commutative C^*-algebra is a $*$-homomorphism.*

PROOF: Let $a \in A$ with $a = a^*$ where A is a unital C^*-algebra. Let $\lambda \in \mathbb{R}$. Then from the C^*-identity

$$\|a \pm \lambda i\|^2 = \|a^2 + \lambda^2 1\| \leqslant \|a\|^2 + \lambda^2.$$

Consequently the spectrum of a lies within the lozenge-shaped region shown in the diagram **??**. As $\lambda \to \infty$ the intersection of all these lozenge-shaped regions is the interval $[-\|a\|, \|a\|]$ of the real axis. ■

(1.13) COROLLARY: *The Gelfand transform for a commutative C^*-algebra is an* isometric *$*$-homomorphism (in particular, it is injective).*

PROOF: This follows from the facts that $\mathcal{G}$ preserves the spectrum and the involution, and that the norm on a C^*-algebra is determined by the spectral radius. ■

What about the *range* of the Gelfand transform?

(1.14) LEMMA: *The Gelfand transform for a commutative C^*-algebra is surjective.*

PROOF: The range of $\mathcal{G}$ is a $*$-subalgebra of $C(\widehat{A})$ which separates points of $\widehat{A}$ (for tautological reasons). By the Stone-Weierstrass theorem, then, the range of $\mathcal{G}$ is dense. But the range is also *complete*, since it is isometrically isomorphic (via $\mathcal{G}$) to a Banach algebra; hence it is closed, and thus is the whole of $C(\widehat{A})$. ■

Putting it all together.

(1.15) THEOREM: *If A is a commutative unital C^*-algebra, then A is isometrically isomorphic to $C(X)$, for some (uniquely determined up to homeomorphism) compact Hausdorff space X. Any unital $*$-homomorphism of commutative unital C^*-algebras is induced (contravariantly) by a continuous map of the corresponding compact Hausdorff spaces. Consequently, the Gelfand transform gives rise to an equivalence of categories between the category of commutative unital C^*-algebras and the opposite of the category of compact Hausdorff spaces.*

This is the *Gelfand-Naimark Theorem*. There is a non-unital version as well, which is proved by unitalization: every commutative C^*-algebra is of the form $C_0(X)$ for some locally compact Hausdorff space X.

Let A be a unital C^*-algebra, not necessarily commutative, and let $a \in A$ be a normal element. Then the unital C^*-subalgebra $C^*(a) \subseteq A$ generated by a is commutative, so according to Gelfand-Naimark it is of the form $C(X)$ for some compact Hausdorff space X. We ask: What is X? Notice that every homomorphism $\alpha\colon C^*(a) \to \mathbb{C}$ is determined by the complex number $\alpha(a)$. Thus $\hat{a}$ defines a continuous injection $\widehat{C^*(a)} \to \mathbb{C}$.

(1.16) PROPOSITION: *Let a be a normal element of a unital C^*-algebra A. Then the Gelfand transform identifies $C^*(a)$ with $C(\mathrm{sp}(a))$, the continuous functions on the spectrum (in A) of a. Under this identification, the operator a corresponds to the identity function $z \mapsto z$.*

Notice that it is a consequence of this proposition that, if $a \in A \subseteq B$, where A and B are C^*-algebras, then $\mathrm{sp}_A(a) = \mathrm{sp}_B(a)$. It is for this reason that we do not usually need to specify the C^*-algebra when discussing the spectrum of an element.

PROOF: What is the image of the injection $\hat{a}\colon \widehat{C^*(a)} \to \mathbb{C}$ defined above? It is simply the spectrum of the function $\hat{a}$ considered as an element of the commutative Banach algebra $C(\widehat{C^*(a)})$. By Gelfand's theorem above, this is the same thing as the spectrum of a, considered as an element of the commutative Banach algebra $C^*(a)$. Thus $\hat{\alpha}$ is a homeomorphism[2] from $\widehat{C^*(a)}$ to $\mathrm{sp}_{C^*(a)}(a)$.

It remains to prove that $\mathrm{sp}_{C^*(a)}(a) = \mathrm{sp}_A(a)$. Clearly, if $x \in C^*(a)$ is invertible in $C^*(a)$, then it is invertible in A. On the other hand suppose that $x \in C^*(a)$ *fails* to be invertible in $C^*(a)$. Then $\hat{x}$ is a continuous function on a compact Hausdorff space which takes the value zero somewhere, say at $\alpha \in \widehat{C^*(a)}$. Using bump functions supported in neighborhoods of α one can produce, for each $\varepsilon > 0$, an element $y_\varepsilon \in C^*(a)$ such that $\|y_\varepsilon\| = 1$ and $\|xy_\varepsilon\| < \varepsilon$. But if x

[2]A continuous bijective map of compact Hausdorff spaces is a homeomorphism.

were invertible in A, this would be impossible for sufficiently small ε (less than $\|x^{-1}\|^{-1}$). ■

A slight reformulation of this yields the so-called *functional calculus* for C^*-algebras. Namely, given a a normal element of a C^*-algebra A, and given a continuous function f on $\mathrm{sp}(a)$, there is a unique element of A (in fact of $C^*(a)$) which corresponds to f under the Gelfand transform for $C^*(a)$. What is usually called the functional calculus is simply the following:

DECISION: We denote the element described above by $f(a)$.

This procedure has all the properties which the notation would lead you to expect (so long as you confine yourself to thinking about only a single normal element). For instance, $(f+g)(a) = f(a)+g(a)$, $(f \cdot g)(a) = f(a)g(a)$, $(f \circ g)(a) = f(g(a))$ and so on. The proofs all follow from the fact that the Gelfand transform is an isomorphism, except the last which uses polynomial approximation.

(1.17) REMARK: Finally we make some comments about how all this goes in the non-unital case. Let A be a non-unital C^*-algebra and let $a \in A$ be normal. We define the *spectrum* of a in this case to be its spectrum in $\widetilde{A}$. It always contains zero, and it is easy to see that

$$C_0^*(a) \cong C_0(\mathrm{sp}(a) \setminus \{0\}),$$

where $C_0^*(a)$ denotes the non-unital C^*-subalgebra of A generated by a. So, in this case we have a functional calculus for continuous functions on the spectrum that vanish at zero.

1.2 Positive Elements and Order

Let T be a selfadjoint operator on a Hilbert space H. One says that T is *positive*[3] if the quadratic form defined by T is positive semidefinite, that is, if

$$\langle Tv, v \rangle \geqslant 0 \quad \text{for all } v \in H.$$

The positive operators form a *cone* — the sum of two positive operators is positive, as is any positive real multiple of a positive operator. It is easy to prove that the following are equivalent for a selfadjoint $T \in \mathfrak{B}(H)$:

(a) The spectrum of T is contained in the positive reals.

[3] This means 'non-negative' in this context, not 'strictly positive'.

(b) T is the square of a selfadjoint operator.

(c) $T = S^*S$ for some operator S.

(d) T is positive.

Indeed, (a) implies (b) by using the functional calculus to construct a square root of T, (b) obviously implies (c) which implies (d) since $\langle S^*Sv, v\rangle = \|Sv\|^2$, and finally if (d) holds let us define

$$T_+ = f(T), T_- = g(T) \tag{1.18}$$

where $f(x) = \max(x, 0)$ and $g(x) = -\min(x, 0)$. We have $T = T^+ - T^-$, and for any $w \in H$,

$$\langle TT_-w, T_-w\rangle = -\langle T_-^2 w, T_-w\rangle = -\|T_-^{3/2}w\|^2 \leqslant 0.$$

So positivity implies that $T_- = 0$, which yields (a).

A key challenge in setting up the theory of abstract C^*-algebras is to prove these properties without using quadratic forms on Hilbert space as a crutch. This was done by Kelley and Vaught some years after C^*-algebras were invented! (This time lapse is responsible for a now-obsolete terminological difference in the older literature: there was a distinction between 'B^*-algebras' and 'C^*-algebras', and Kelley and Vaught's theorem showed that this distinction was otiose.)

(1.19) PROPOSITION: *Let A be a unital C^*-algebra. The following conditions on a selfadjoint $a \in A$ are equivalent:*

(a) The spectrum of a is contained in the positive reals.

(b) a is the square of a selfadjoint element of A.

(c) $\|t1 - a\| \leqslant t$ for all $t \geqslant \|a\|$.

(d) $\|t1 - a\| \leqslant t$ for some $t \geqslant \|a\|$.

PROOF: Conditions (a) and (b) are equivalent by a functional calculus argument. Suppose that (a) holds. Then $\mathrm{sp}(a)$ is contained in the interval $[0, \|a\|]$ of the real axis and so the supremum of the function $t1 - \lambda$, for $\lambda \in \mathrm{sp}(a)$, is at most t; this proves (c) which implies (d). Finally, assume (d). Then $|t - \lambda| < t$ for all $\lambda \in \mathrm{sp}(a)$; so λ is positive on $\mathrm{sp}(a)$, yielding (a). ■

We will call $a \in A$ *positive* if it is selfadjoint and satisfies one of these four conditions. Part (d) of this characterization implies that the positive elements form a cone:

(1.20) PROPOSITION: *The sum of two positive elements of A is positive.*

PROOF: Let a', a'' be positive elements of A. Let t', t'' be as in (d). Let $a = a' + a''$ and $t = t' + t''$. Then by the triangle inequality $t \geqslant \|a\|$ and $\|t1 - a\| \leqslant \|t'1 - a'\| + \|t''1 - a''\| \leqslant t' + t'' = t$. So a is positive. ■

We can therefore define a (partial) order on A_{sa} by setting $a \leqslant b$ if $a - b$ is positive. This order relation is very useful, especially in connection with von Neumann algebras, but it has to be handled with care — not every 'obvious' property of the order is actually true. For instance, it is not true in general that if $0 \leqslant a \leqslant b$, then $a^2 \leqslant b^2$. In Proposition 1.22 we will give some true properties of the order relation. These depend on the following tricky result.

(1.21) THEOREM: (KELLEY–VAUGHT) *An element a of a unital C^*-algebra A is positive if and only if $a = b^*b$ for some $b \in A$.*

PROOF: Any positive element can be written in this form with $b = a^{1/2}$. To prove the converse we will argue first that b^*b cannot be a *negative* operator (except zero). Indeed, put $b = x + iy$ with x and y selfadjoint; then

$$b^*b + bb^* = 2(x^2 + y^2) \geqslant 0.$$

Thus if b^*b is negative, $bb^* = 2(x^2 + y^2) - b^*b$ is positive. But the spectra of b^*b and bb^* are the same (apart possibly from zero), by a well-known algebraic result; we deduce that $\operatorname{sp}(b^*b) = \{0\}$, and hence $b^*b = 0$.

Now for the general case: suppose that $a = b^*b$; certainly a is selfadjoint, so we may write $a = a_+ - a_-$ where the positive elements a_+ and a_- are defined as in Equation 1.18. We want to show that $a_- = 0$. Let $c = ba_-^{1/2}$. Then we have

$$c^*c = a_-^{1/2} a a_-^{1/2} = -a_-^2$$

so c^*c is negative. Hence it is zero, by the special case above; so $a_- = 0$ and a is positive. ■

(1.22) PROPOSITION: *Let x and y be positive elements of a unital C^*-algebra A, with $0 \leqslant x \leqslant y$. Then*

(a) We have $x \leqslant \|x\|1$ (and similarly for y); moreover $\|x\| \leqslant \|y\|$.

(b) For any $a \in A$, $a^*xa \leqslant a^*ya$.

(c) If x and y are invertible, we have $0 \leqslant y^{-1} \leqslant x^{-1}$.

(d) For all $\alpha \in [0, 1]$ we have $x^\alpha \leqslant y^\alpha$.

PROOF: The first part of (a) is obvious from the functional calculus, and it implies the second part by arguing that

$$x \leqslant y \leqslant \|y\|1$$

so $x \leqslant \|y\|1$ whence $\|x\| \leqslant \|y\|$. Item (b) becomes obvious if we use Theorem 1.21; if $x \leqslant y$ then $y - x = c^*c$, so $a^*ya - a^*xa = (ca)^*ca$ is positive. For (c), if $x \leqslant y$ put $z = x^{1/2}y^{-1/2}$; then $z^*z = y^{-1/2}xy^{-1/2} \leqslant 1$ (by (b)). But $z^*z \leqslant 1$ if and only if $zz^* \leqslant 1$, which is to say that $x^{1/2}y^{-1}x^{1/2} \leqslant 1$; and applying (b) again we get $y^{-1} \leqslant x^{-1}$. Finally (d) follows from (c) once we observe that the function $\lambda \mapsto \lambda^\alpha$, defined on $\mathbb{R}^+$, is a norm limit of convex combinations of the functions

$$g_\xi(\lambda) = \frac{1}{\xi}\left(1 - \frac{1}{1+\xi\lambda}\right);$$

this follows from the integral formula

$$\lambda^\alpha = c_\alpha \int g_\xi(\lambda)\xi^\alpha \, d\xi$$

which is valid for $0 \leqslant \alpha \leqslant 1$. ■

1.3 Moore–Smith Convergence

The strong topology and other related matters for C^* and von Neumann algebras are frequently discussed in terms of *nets*. Here is a brief reminder about their theory. The proofs will be left to the reader.

(1.23) DEFINITION: *A* directed set Λ *is a partially ordered set in which any two elements have an upper bound. A* net *in a topological space X is a function from a directed set to X. Usually we denote a net like this: $\{x_\lambda\}_{\lambda\in\Lambda}$.*

For example, a sequence is just a net based on the directed set $\mathbb{N}$. The following example of a directed set will be important to us:

(1.24) LEMMA: *Let A be a C^*-algebra and let $D \subseteq A$ be the set of all $u \in A_{\mathsf{sa}}$ with $0 \leqslant u < 1$. Then D is directed by the order relation.*

PROOF: We must show that any two elements of D have an upper bound. Let

$$f(t) = \frac{t}{1-t}, \quad g(t) = \frac{t}{1+t} = 1 - \frac{1}{1+t}.$$

The function f is a homeomorphism of $[0,1)$ on $[0,\infty)$, and g is its inverse. Given $x', x'' \in D$ let $x = g(f(x') + f(x''))$, defined by the functional calculus. Then x is in D also. Clearly $f(x') \leqslant f(x') + f(x'')$, and since the function g is order-preserving by Proposition 1.22(c) we get $x' \leqslant x$. Similarly $x'' \leqslant x$, so x will do as the required upper bound. ■

Note that since the functions f and g send 0 to 0, this argument is valid even in a *non-unital* C^*-algebra.

(1.25) DEFINITION: *Let* $U \subseteq X$. *A net* $\{x_\lambda\}_{\lambda\in\Lambda}$ *in* X *is* eventually in U *if there is some* $\lambda_0 \in \Lambda$ *such that* $x_\lambda \in U$ *for all* $\lambda \geqslant \lambda_0$. *If there is some point* $x \in X$ *such that* $\{x_\lambda\}$ *is eventually in every neighborhood of* x, *we say that* $\{x_\lambda\}$ converges *to* X.

As an exercise in manipulating this definition, the reader may prove that a space X is Hausdorff if and only if every net converges to at most one point. Similarly a function $f\colon X \to Y$ is continuous if and only if, whenever $\{x_\lambda\}$ is a net in X converging to x, then net $\{f(x_\lambda)\}$ converges in Y to $f(x)$. The basic idea here is to use the system of open neighborhoods of a point, directed by (reverse) inclusion, as the parameter space of a net.

Warning: a convergent net in a metric space need not be bounded.

A *subnet* of a net is defined as follows. Let $\{x_\lambda\}_{\lambda\in\Lambda}$ be a net based on the directed set Λ. Let Ξ be another directed set. A function $\lambda\colon \Xi \to \Lambda$ is called a *final* function if, for every $\lambda_0 \in \Lambda$, there is $\xi_0 \in \Xi$ such that $\lambda(\xi) \geqslant \lambda_0$ whenever $\xi \geqslant \xi_0$. For such a function the net $\{x_{\lambda(\xi)}\}_{\xi\in\Xi}$ is a net based on the directed set Ξ; it is called a *subnet* of the originally given one. Notice that this notion of subnet is more relaxed than the usual one of subsequence; we allow some repetition and backtracking.

(1.26) PROPOSITION: *A topological space* X *is compact if and only if every net in* X *has a convergent subnet.*

For completeness we mention the notion of *universal net*. A net is *universal* if, for every subset A of X, the net is either eventually in A or eventually in the complement of A. Using the axiom of choice it can be shown that every net has a universal subnet. This makes possible a rather short proof of Tychonoff's theorem.

(1.27) THEOREM: *A product of compact topological spaces is compact.*

PROOF: Let $X = \prod X_\alpha$ be such a product, π_α the coordinate projections. Pick a universal net $\{x_\lambda\}$ in X; then a one-line proof shows that each the nets $\{\pi_\alpha(x_\lambda)\}$ is universal in X_α, and hence is convergent (a universal net with a convergent subnet must itself be convergent), say to x_α. But then by definition of the Tychonoff topology the net $\{x_\lambda\}$ converges to the point x whose coordinates are x_α. Thus every universal net in X converges, and since every net has a universal subnet, this proves that X is compact. ■

1.4 Approximate Units, Ideals, and Quotients

(1.28) DEFINITION: *Let* A *be a* C^**-algebra, probably without unit. An* approximate unit *for* A *is a net* $\{u_\lambda\}$ *in* A *which is increasing, has* $0 \leqslant u_\lambda \leqslant 1$ *for all* λ, *and such that moreover for each* $a \in A$, $\lim u_\lambda a = \lim a u_\lambda = a$.

For example, let $A = C_0(\mathbb{R})$. Then the sequence of functions

$$u_n(x) = e^{-x^2/n} \in A$$

tends to 1 uniformly on compact sets, and so constitutes an approximate unit. A very similar argument produces an approximate unit for any *commutative* C^*-algebra $C_0(X)$ (if A is separable, so that X is second countable, our approximate unit will be a sequence; otherwise we must use a net).

(1.29) THEOREM: *Every* C^**-algebra has an approximate unit.*

PROOF: For our approximate unit we just take the set D of *all* elements u with $0 \leqslant u < 1$), ordered by the usual order relation. It is directed (Lemma 1.24).

To prove the approximate unit property we first note that for any *positive* $a \in A$ there is a *sequence* $\{u_n\}$ in D with $\|(1-u_n)x\|$ and $\|x(1-u_n)\|$ tending to zero. Indeed, $C_0^*(a)$ is a commutative, separable C^*-algebra and we may take $\{u_n\}$ to be an approximate unit for it, as in the example above. If $u, u' \in D$ with $u \leqslant u'$, then[4]

$$\begin{aligned}\|(1-u')x\|^2 = \|x(1-u')^2x\| \leqslant \|x(1-u')x\| \\ \leqslant \|x(1-u)x\| \leqslant \|x\|\|(1-u)x\|.\end{aligned}$$

Since D is directed this allows us to infer that the existence of the sequence $\{u_n\}$ in D for which $\lim_n \|(1-u_n)x\| = 0$ implies that the limit $\lim_{u\in D} \|(1-u)x\|$ exists and is zero also. This shows that D acts as an approximate identity at least on positive elements, and since x^*x is positive for any x, another simple argument shows that D acts as an approximate identity on all elements. ■

By an *ideal* in a C^*-algebra we mean a norm-closed, two-sided ideal. We could also require $*$-closure but this fact is automatic:

(1.30) PROPOSITION: *Any ideal* J *in a* C^**-algebra* A *is closed under the adjoint operation (that is* $J = J^*$*).*

PROOF: Let $x \in J$ and let $\{u_\lambda\}$ be an approximate unit for the C^*-algebra $J \cap J^*$. Using the C^*-identity

$$\|x^*(1-u_\lambda)\|^2 = \|xx^*(1-u_\lambda) - u_\lambda xx^*(1-u_\lambda)\|$$

which tends to zero. It follows that $x^*u_\lambda \to x^*$, whence $x^* \in J$. ■

[4]It is convenient here and elsewhere to write the approximate unit condition in the form $\lim \|x(1-u)\| = 0$. Formally speaking this is an identity in the *unitalization* of A. However, the elements $x(1-u) = x - xu$ in fact belong to A itself.

It is an important fact that the quotient of a C^*-algebra by an ideal (which is certainly a Banach algebra in the induced norm) is in fact a C^*-algebra. We need approximate units to check this.

(1.31) THEOREM: *Let A be a C^*-algebra and J an ideal in A. Then A/J is a C^*-algebra with the induced norm and involution.*

PROOF: A/J is a Banach algebra with involution (note that we need $J = J^*$ here). Recall that the quotient norm is defined as follows:

$$\|[a]\| = \inf_{x \in J} \|a + x\|.$$

Here $[a]$ denotes the coset (in A/J) of the element $a \in A$. I claim that

$$\|[a]\| = \lim_u \|a(1-u)\|$$

where u runs over an approximate identity for J. Granted this it is plane sailing to the C^*-identity, since we get

$$\begin{aligned} \|[a^*][a]\| = \lim_u \|a^*a(1-u)\| \geqslant \lim_u \|(1-u)a^*a(1-u)\| \\ = \lim_u \|a(1-u)\|^2 = \|[u]\|^2. \end{aligned}$$

To establish the claim notice that certainly $\|a(1-u)\| \geqslant \|[a]\|$ for every a. On the other hand, given $\varepsilon > 0$ there exists $x \in J$ with $\|a - x\| \leqslant \|[a]\| + \varepsilon$. It is frequently true that $\|x(1-u)\| \leqslant \varepsilon$ also and then

$$\|a(1-u)\| \leqslant \|[a]\| + 2\varepsilon$$

by the triangle inequality. This establishes the claim. ■

We can use this result to set up the fundamental 'isomorphism theorems' in the category of C^*-algebras.

(1.32) LEMMA: *Any injective homomorphism of C^*-algebras is an isometry.*

PROOF: Let $\alpha\colon A \to B$ be such a homomorphism and let $a \in A$. Assume without loss of generality that A, B are unital. It suffices to show that $\|a^*a\| = \|\alpha(a^*a)\|$, by the C^*-identity. But now we may restrict attention to the C^*-subalgebras generated by a^*a and its image, and thus we may assume without loss of generality that A and B are commutative. In this case $A = C(X)$, $B = C(Y)$, and α is induced by a surjective map $Y \to X$; this makes the result obvious. ■

(1.33) COROLLARY: *Let $\alpha\colon A \to B$ be any homomorphism of C^*-algebras. Then α is continuous with norm $\leqslant 1$, and it can be factored as*

$$A \qquad A/\ker(\alpha) \qquad B$$

where the first map is a quotient map and the second is an isometric injection whose range is a C^*-subalgebra of B.

PROOF: Continuity of α follows from the spectral radius formula, and the rest of the proof is standard from algebra. ■

We can also prove the 'diamond isomorphism theorem'.

(1.34) PROPOSITION: *Let A be a C^*-algebra. Let J be an ideal in A and let B be a C^*-subalgebra of A. Then $B + J$ is a C^*-subalgebra of A and there is an isomorphism $B/(B \cap J) = (B + J)/J$.*

PROOF: The $*$-homomorphism $B \to A \to A/J$ has range a C^*-subalgebra of A/J, isomorphic to $B/B \cap J$ by the previous theorem. Thus the inverse image in A of this C^*-subalgebra (namely $B + J$) is closed, and so is a C^*-algebra also. The result now follows by the usual algebra argument. ■

When B is another ideal J there is a useful addendum to this result: for two ideals I and J in A, their intersection is equal to their product. To see this, observe

$$(I \cap J)^2 \subseteq IJ \subseteq I \cap J.$$

But for any C^*-algebra B, we have $B = B^2$, by an easy functional calculus argument.

1.5 Topologies on $\mathfrak{B}(H)$

Let H be a Hilbert space. The C^*-algebra $\mathfrak{B}(H)$ carries a number of important topologies in addition to the familiar *norm topology*. These are *weak topologies*, in other words they are defined to have the minimal number of open sets required to make certain maps continuous.

The *weak operator topology* is the weakest topology making the maps $\mathfrak{B}(H) \to \mathbb{C}$ given by $T \mapsto \langle Tu, v\rangle$ continuous, for all fixed $u, v \in H$. Similarly, the *strong operator topology* is the weakest topology making the maps $\mathfrak{B}(H) \to H$ given by $T \mapsto Tu$ continuous. It is clear that the weak operator topology is weaker (fewer open sets) than the strong operator topology, which in turn is weaker than the norm topology. These relations are strict.

(1.35) REMARK: A set $W \subseteq \mathfrak{B}(H)$ is a strong neighborhood of $T_0 \in \mathfrak{B}(H)$ if there exist $u_1, \ldots, u_n \in H$ such that

$$\sum_{i=1}^{n} \|(T - T_0)u_i\|^2 < 1 \implies T \in W.$$

Similarly, W is a weak neighborhood of T_0 if there exist $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ such that

$$\sum_{i=j=1}^{n} |\langle (T - T_0)u_i, v_i \rangle| < 1 \implies T \in W.$$

Here are some easy properties of these topologies. Addition is jointly continuous in both topologies; multiplication is not jointly continuous in either topology. However, multiplication is jointly strongly continuous when restricted to the unit ball. Multiplication by a *fixed* operator (on the left or the right) is both weakly and strongly continuous. The unit ball of $\mathfrak{B}(H)$ is weakly compact, but not strongly compact. A weakly (or strongly) convergent *sequence* is bounded (use the uniform boundedness principle).

It is easy to see that the adjoint operation $T \mapsto T^*$ is weakly continuous. However it is *not* strongly continuous: if V denotes the unilateral shift operator on ℓ^2, defined by $V(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, \ldots)$, then the sequence $(V^*)^n$ tends strongly to 0 while the sequence V^n does not. It is sometimes worth noting that the adjoint operation *is* strongly continuous on the set $\mathfrak{N}$ of normal operators. The identity

$$\|Tu\|^2 = \langle T^*Tu, u \rangle = \langle TT^*u, u \rangle = \|T^*u\|^2$$

which is valid for $T \in \mathfrak{N}$, shows that if a net T_λ of normal operators converges strongly to zero, then the net T_λ^* converges strongly to zero also. Unfortunately the general case does not follow from this, since $\mathfrak{N}$ is not closed under addition. Instead, use the identity (valid for $S, T \in \mathfrak{N}$)

$$\|(S - T)^*u\|^2 = \|Su\|^2 - \|Tu\|^2 + 2\Re\langle (T - S)T^*u, u \rangle$$

to obtain the inequality

$$\|(S - T)^*u\|^2 \leqslant \|(S - T)u\|(\|Su\| + \|Tu\|) + 2\|(T - S)T^*u\|\|u\|.$$

From this it follows that if S_λ is a net of normal operators converging strongly to a normal operator T, then S_λ^* converges strongly to T^*. **Warning:** It is important that T be normal too.

(1.36) PROPOSITION: *The only strongly continuous linear functionals on* $\mathfrak{B}(H)$ *are of the form*

$$T \mapsto \sum_{i=1}^{n} \langle Tu_i, v_i \rangle$$

for certain vectors $u_1, \ldots, u_n$ *and* $v_1, \ldots, v_n$ *in* H. *(Notice that these functionals are also weakly continuous.)*

PROOF: Let $\varphi\colon \mathfrak{B}(H) \to \mathbb{C}$ be a strongly continuous linear functional. Then by definition of the strong operator topology there exist vectors $u_1, \ldots, u_n$ such that

$$|\varphi(T)| \leqslant \left(\sum_{i=1}^{n} \|Tu_i\|^2 \right)^{\frac{1}{2}}.$$

Let L be the linear subspace of H^n comprised of those vectors of the form $(Tu_1, \ldots, Tu_n)$ for $T \in \mathfrak{B}(H)$. We may define a linear functional of norm $\leqslant 1$ on L by

$$(Tu_1, \ldots, Tu_n) \mapsto \varphi(T).$$

Extend this functional to the whole of H^n by the Hahn-Banach theorem, and then represent it by an inner product with a vector $(v_1, \ldots, v_n)$ in H^n. Restricting back down to L we find that

$$\varphi(T) = \sum_{i=1}^{n} \langle Tu_i, v_i \rangle$$

as required. ∎

Since the weak and strong duals of $\mathfrak{B}(H)$ are the same we find

(1.37) COROLLARY: *The weak and strong operator topologies have the same closed convex sets.*

There is an intimate relation between positivity and the strong (and weak) operator topologies:

(1.38) LEMMA: *If a bounded net of positive operators converges weakly to zero, then it converges strongly to zero.*

PROOF: Let $\{T_\lambda\}$ be such a net in $\mathfrak{B}(H)$. Let $v \in H$. By the Cauchy-Schwarz inequality

$$\|T_\lambda v\|^4 = \langle T_\lambda^{1/2} v, T_\lambda^{3/2} v \rangle^2 \leqslant \langle T_\lambda v, v \rangle \langle T_\lambda^3 v, v \rangle \leqslant \|T_\lambda\|^3 \|v\|^2 \langle T_\lambda v, v \rangle$$

and the right-hand side tends to zero by weak convergence. ∎

(1.39) PROPOSITION: *Let $\{T_\lambda\}$ be an increasing net of selfadjoint operators in $\mathfrak{B}(H)$, and suppose that $\{T_\lambda\}$ is bounded above, by S say. Then $\{T_\lambda\}$ is strongly convergent to an operator T with $T \leqslant S$.*

PROOF: For each fixed $v \in H$ the limit of the increasing net $\langle T_\lambda v, v\rangle$ of real numbers exists and is bounded by $\langle Sv, v\rangle$. The function $v \mapsto \lim_\lambda \langle T_\lambda v, v\rangle$ is a quadratic form, and using the polarization identity one may define a linear operator T such that

$$\langle Tv, v\rangle = \lim_\lambda \langle T_\lambda v, v\rangle.$$

By construction, the net T_λ converges to T in the weak operator topology. So $T - T_\lambda$ is a bounded net of positive operators converging weakly to zero; by Lemma 1.38 it converges strongly to zero. ■

1.6 Continuity of the Functional Calculus

In this section we'll look at the following question: let f be a function on $\mathbb{R}$, and consider the map

$$T \mapsto f(T)$$

from the selfadjoint part of $\mathfrak{B}(H)$ to $\mathfrak{B}(H)$. When is this a continuous map, relative to the various topologies that we have been discussing on $\mathfrak{B}(H)$?

(1.40) PROPOSITION: *Let $f\colon \mathbb{R} \to \mathbb{C}$ be any continuous function. Then $T \mapsto f(T)$ is continuous relative to the norm topologies on $\mathfrak{B}(H)_{\mathrm{sa}}$ and $\mathfrak{B}(H)$.*

PROOF: Since neighborhoods in the norm topology are bounded, and the spectral radius is bounded by the norm, only the behavior of f on a compact subinterval of $\mathbb{R}$ is significant for continuity. Thus we may and shall assume that $f \in C_0(\mathbb{R})$.

If $f(\lambda) = (\lambda + z)^{-1}$, for $z \in \mathbb{C} \setminus \mathbb{R}$, then $T \mapsto f(T)$ is continuous; for we have

$$f(S) - f(T) = f(S)(T - S)f(T) \tag{1.41}$$

and $\|f(S)\|$ and $\|f(T)\|$ are bounded by $|\Im z|^{-1}$.

Let A denote the collection of all functions $f \in C_0(\mathbb{R})$ for which $T \mapsto f(T)$ is norm continuous. It is easy to see that A is a C^*-subalgebra. Since it contains the functions $\lambda \mapsto (z + \lambda)^{-1}$ it separates points of $\mathbb{R}$. Thus $A = C_0(\mathbb{R})$ by the Stone-Weierstrass theorem. ■

Now we will look at continuity in the *strong* topology. Note that strong neighborhoods need not be bounded, so that the behavior of the function f at infinity will become important. We say that a function $f\colon \mathbb{R} \to \mathbb{C}$ is *of linear growth* if $|f(t)| \leqslant A|t| + B$ for some constants A and B.

(1.42) PROPOSITION: *Let $f: \mathbb{R} \to \mathbb{C}$ be a continuous function of linear growth. Then $T \mapsto f(T)$ is continuous relative to the strong topologies on $\mathfrak{B}(H)_{\mathrm{sa}}$ and $\mathfrak{B}(H)$.*

PROOF: Let V denote the vector space of all continuous functions $f: \mathbb{R} \to \mathbb{C}$ for which $T \mapsto f(T)$ is strongly continuous, and let V^b denote the bounded elements of V. Note that if f belongs to V (or V^b), then so does $\bar{f}$; this follows from the strong continuity of the adjoint on the set $\mathfrak{N}$ of normal operators. We claim first that $V^b V \subseteq V$; this follows from the identity

$$fg(S) - fg(T) = f(S)(g(S) - g(T)) + (f(S) - f(T))g(T)$$

together with the following properties of the strong topology: if $T_\lambda \to T$ strongly, then $(T_\lambda - T)R \to 0$ strongly for any fixed operator R, while $R_\lambda(T_\lambda - T) \to 0$ strongly for any bounded net R_λ. In particular, V^b is closed under multiplication, so it is an algebra. One sees easily that V^b is a C^*-algebra. Arguing with Equation 1.41 exactly as above, and using the properties of multiplication in the strong topology that we just mentioned, we see that $V^b \supseteq C_0(\mathbb{R})$.

Now let f be a function of linear growth. Then the function $t \mapsto (1+t^2)^{-1}f(t)$ belongs to $C_0(\mathbb{R})$, hence to V^b. The identity function of course belongs to V, hence so does the function $g: t \mapsto t(1 + t^2)^{-1}f(t)$, as it is the product of a member of V and a member of V^b. The function g is bounded, so belongs to V^b, hence the function $t \mapsto tg(t) = t^2(1 + t^2)^{-1}f(t)$ belongs to V. Finally, then, the function

$$t \mapsto (1 + t^2)^{-1}f(t) + t^2(1 + t^2)^{-1}f(t) = f(t)$$

belongs to V, as required. ■

1.7 Von Neumann Algebras

Let H be a Hilbert space.

(1.43) DEFINITION: *A unital $*$-subalgebra of $\mathfrak{B}(H)$ which is closed in the strong operator topology*[5] *is called a* von Neumann algebra *of operators on H.*

If M is any selfadjoint[6] set of operators on H its *commutant* M' is the set

$$M' = \{T \in \mathfrak{B}(H) : ST = TS \;\forall\, S \in M\}.$$

It is a von Neumann algebra. For tautological reasons, M'' (the *double commutant*) contains M. The *double commutant theorem* is

[5]By Corollary 1.37 it makes no difference to replace 'strong' by 'weak' here.

[6]This means that $T \in M$ implies $T^* \in M$.

(1.44) THEOREM: *A unital C^*-subalgebra of $\mathfrak{B}(H)$ is a von Neumann algebra if and only if it is equal to its own double commutant. In general, the double commutant of $S \subseteq \mathfrak{B}(H)$ is the smallest von Neumann algebra of operators on H that contains S.*

PROOF: The thing we have to prove is this: if A is a unital C^*-subalgebra of $\mathfrak{B}(H)$ and $T \in A''$, then T is the strong limit of a net of operators in A. In other words, we have to show that for any finite list $v_1, \ldots, v_n$ of vectors in H one can find an operator $R \in A$ such that

$$\sum_{i=1}^{n} \|(T - R)v_i\|^2 < 1.$$

Let us first address the case $n = 1$, and $v_1 = v$. Let P be the orthogonal projection operator onto the A-invariant closed subspace generated by v (that is $\overline{Av}$). Then P belongs to the commutant A' of A, because if $R \in A$ then R preserves the range of P and R^* preserves the kernel of P. Since $T \in A''$, T commutes with P, and in particular $PTv = TPv = Tv$, so Tv belongs to the range of P. That is, Tv is a limit of vectors of the form Rv, $R \in A$, which is what was required.

Now consider the general case. Let $K = H \oplus \cdots \oplus H$ (n times) and let $v \in K$ be the vector $(v_1, \ldots, v_n)$. Let $\rho\colon \mathfrak{B}(H) \to \mathfrak{B}(K)$ be the 'inflation' $*$-homomorphism which is defined by

$$T \mapsto \begin{pmatrix} T & & \\ & \ddots & \\ & & T \end{pmatrix}$$

(in a more sophisticated language this is $T \mapsto T \otimes I_n$). A routine calculation shows that $\rho(A)'$ consists of $n \times n$ matrices all of whose elements belong to A', and therefore that $\rho(A)'' = \rho(A'')$. Now apply the previous case to the C^*-algebra $\rho(A)$, the operator $\rho(T)$, and the vector v. We find that there is $\rho(R) \in \rho(A)$ such that $\|(\rho(R) - \rho(T))v\| < 1$, and when written out componentwise this is exactly what we want. ■

It follows from the double commutant theorem that a von Neumann algebra is weakly closed. This can also be proved directly; in fact, each strongly closed convex set in $\mathfrak{B}(H)$ is weakly closed.

(1.45) REMARK: Our hypothesis that A must be unital is stronger than necessary. Suppose simply that A is *non-degenerate*, meaning that AH is dense in H, or equivalently that $Ru = 0$ for all $R \in A$ implies $u = 0$. Then the conclusion of the double commutant theorem holds. To see this, we need only check that the vector

v still belongs to $\overline{Av}$ — the rest of the proof will go through as before. Indeed, one can write $v = Pv + (1 - P)v$, where P is the projection onto $\overline{Av}$ as above, and since $P \in A'$ we have, for every $R \in A$,

$$R(1 - P)v = (1 - P)Rv = Rv - Rv = 0;$$

so by non-degeneracy $(1 - P)v = 0$ and $v = Pv \in \overline{Av}$.

The *Kaplansky density theorem* strengthens the double commutant theorem by showing that every element of A'' can be strongly approximated by a *bounded* net from A (note that strongly convergent nets need not be bounded). The main step in the proof has already been taken: it is the strong continuity of the functional calculus discussed in the preceding section.

(1.46) THEOREM: *Let A be a unital C^*-subalgebra of $\mathfrak{B}(H)$ and let $M = A''$ be the von Neumann algebra that it generates. The the unit ball of A is strongly dense in the unit ball of M, the unit ball of A_{sa} is strongly dense in the unit ball of M_{sa}, and the unit chunk of A_+ is strongly dense in the unit ball of M_+. Finally, the unitary group of A is strongly dense in the unitary group of M.*

PROOF: Let's start with the case of M_{sa}. Let $T \in M_{\mathsf{sa}}$ with $\|T\| \leqslant 1$. Using the Double Commutant Theorem, we see that T is the strong limit of a net T_λ of selfadjoint elements of A.[7] The norms of the operators T_λ need not be bounded; but if we replace T_λ by $f(T_\lambda)$, where $f(x) = \min(1, \max(-1, x))$, then $f(T_\lambda) \to f(T) = T$ by Proposition 1.39. The same argument works for M_+, and for the unitary group if we use the fact (a consequence of the spectral theorem) that in any von Neumann algebra M, any unitary U can be written $e^{2\pi i T}$ with T selfadjoint and of norm $\leqslant 1$. The last remaining conclusion is that the whole unit ball of A is dense in the unit ball of M; this can be proved by a 2×2 matrix trick, or as a consequence of the Russo–Dye Theorem below. ■

One pleasant feature of von Neumann algebras is the existence in them of a polar decomposition theory, analogous to the familiar representation $z = re^{i\theta}$ of a complex number as a product of a positive real number and a complex number of modulus one.

(1.47) DEFINITION: *Let H be a Hilbert space. A* partial isometry *on H is an operator $V \in \mathfrak{B}(H)$ which satisfies one of the following four (equivalent) conditions:*

[7] There is a technical point here: The double commutant theorem might produce a net T_λ which is *not* selfadjoint. We can replace T_λ by $\frac{1}{2}(T_\lambda + T_\lambda^*)$ but now we get a net which only converges *weakly* to T. However, by taking convex combinations, we can replace weak convergence by strong convergence (Corollary 1.37), and of course a convex combination of selfadjoint operators is selfadjoint.

(a) V^*V *is a projection (the 'initial projection'),*

(b) VV^* *is a projection (the 'final projection'),*

(c) $V = VV^*V$*,*

(d) $V^* = V^*VV^*$.

It is easy to see that these conditions are equivalent, since (c) and (d) are adjoint are one another, (c) implies (a) (multiply on the left by V^*), and if (a) is true then $I - V^*V$ is a projection with range $\mathrm{Ker}(V)$, so that $V(I - V^*V) = 0$ which is (c). One should think of V as giving an isomorphism from the range of the initial projection to the range of the final projection.

(1.48) DEFINITION: *Let* $T \in \mathfrak{B}(H)$*. A* polar decomposition *for* T *is a factorization* $T = VP$*, where* P *is a positive operator and* V *is a partial isometry with initial space* $\overline{\mathrm{ran}}(P)$ *and final space* $\overline{\mathrm{ran}}(T)$*.*

Here we are using the notation $\overline{\mathrm{ran}}(T)$ for the closure of the range of the operator T. Note that V and P need not commute, so that there are really two notions of polar decomposition (left and right handed); we have made the conventional choice.

(1.49) PROPOSITION: *Every operator* $T \in \mathfrak{B}(H)$ *has a unique polar decomposition* $T = VP$*. The operator* $P = |T| = (T^*T)^{\frac{1}{2}}$ *belongs to* $C^*(T)$*. The partial isometry* V *belongs to* $W^*(T)$*, the von Neumann algebra generated by* T*. If* T *happens to be invertible, then* V *is unitary; it belongs to* $C^*(T)$ *in this case.*

PROOF: Let $T = VP$ be a polar decomposition. Then $T^*T = P^*V^*VP = P^2$, so P is the unique positive square root of T^*T, which we denote by $|T|$. Fixing this P let us now try to construct a polar decomposition. For $v \in H$,

$$\|Pv\|^2 = \langle P^*Pv, v\rangle = \langle T^*Tv, v\rangle = \|Tv\|^2$$

so an isometry $\overline{\mathrm{ran}}(P) \to \overline{\mathrm{ran}}(T)$ is defined by $Pv \mapsto Tv$. Extending by zero on the orthogonal complement we get a partial isometry V with the property that $T = VP$; and it is uniquely determined since the definition of polar decomposition requires that $V(Pv) = Tv$ and that $V = 0$ on the orthogonal complement of $\overline{\mathrm{ran}}(P)$.

Let us check that $V \in W^*(T)$; using the bicommutant theorem, it is enough to show that V commutes with every operator S that commutes with T and T^*. We have $H = \mathrm{Ker}(P) \oplus \overline{\mathrm{ran}}(P)$ so it suffices to check that $VSv = SVv$ separately for $v \in \mathrm{Ker}(P) = \mathrm{Ker}(T)$ and for $v = Py$. In the first case, $STv = TSv = 0$ so $Sv \in \mathrm{Ker}(T)$, and then $Vv = 0$, $VSv = 0$. In the second case

$$VSv = VSPy = VPSy = TSy = STy = SVPy = SVv$$

using the fact that P belongs to the C^*-algebra generated by T.

Finally, if T is invertible then so is T^*T, and we have $V = T(T^*T)^{-\frac{1}{2}}$; this shows that $V \in C^*(T)$ in this case, and unitarity is a simple check. ∎

2 Representation Theory

We will discuss representations of C^* and von Neumann algebras on Hilbert space, and prove various forms of the spectral theorem.

2.1 Representations and States

(2.1) DEFINITION: *Let A be a C^*-algebra. A* representation *of A is a $*$-homomorphism $\rho\colon A \to \mathfrak{B}(H)$ for some Hilbert space H. The representation is* faithful *if ρ is injective, and* non-degenerate *if $\rho(A)H = H$. The representation is* irreducible *if there are no proper closed subspaces of H that are invariant under $\rho(A)$.*

(2.2) REMARK: Suppose that $\rho\colon A \to \mathfrak{B}(H)$ is a representation and that K is a closed subspace of H. Then the orthogonal projection P onto K belongs to the commutant $\rho[A]'$ if and only if K is invariant under the action of A. It is a consequence of the spectral theorem that every von Neumann algebra larger than $\mathbb{C}1$ contains a proper projection, and so we see that ρ is irreducible if and only if the commutant $\rho[A]'$ is equal to $\mathbb{C}1$, or equivalently if and only if the bicommutant $\rho[A]''$ is equal to $\mathfrak{B}(H)$.

From the bicommutant theorem, $\rho[A]''$ is the strong closure of $\rho[A]$. It follows then that, if ρ is irreducible, then given any two vectors $u, v \in H$ with $u \neq 0$, and given $\varepsilon > 0$, there is $a \in A$ such that $\|\rho(a)u - v\| < \varepsilon$. By an iterative argument, Kadison improved this result: he showed that in fact there is $a \in A$ with $\rho(a)u = v$. This is called *Kadison's transitivity theorem*, and it if course implies that an irreducible representation of a C^*-algebra has *no* invariant subspaces, closed or not. We will (perhaps) prove this theorem later on.

Representations are studied by way of *states*.

(2.3) DEFINITION: *A linear functional $\varphi\colon A \to \mathbb{C}$ on a C^*-algebra A is called* positive *if $\varphi(a^*a) \geqslant 0$ for all $a \in A$. A* state *is a positive linear functional of norm one.*

Any positive linear functional is bounded; for, if φ were positive and unbounded, we could find elements $a_k \in A_+$ of norm one with $\varphi(a_k) \geqslant 4^k$, and then $a = \sum 2^{-k}a_k$ is an element of A_+ satisfying $\varphi(a) \geqslant 2^{-k}\varphi(a_k) \geqslant 2^k$ for all k, contradiction. Thus there is little loss of generality in restricting attention to states. A state on $C(X)$ is a Radon probability measure on X, by the Riesz representation theorem from measure theory. If $\rho\colon A \to \mathfrak{B}(H)$ is a representation and $v \in H$ is a unit vector, a state $\sigma_v\colon A \to \mathbb{C}$ (called a *vector state*) is defined by $\sigma_v(a) = \langle \rho(a)v, v\rangle$. This is the link between states and representations. Note that the state defined by the Radon probability measure μ is a vector state of $C(X)$;

take $H = L^2(X,\mu)$, ρ the representation by multiplication operators, and v the constant function 1.

If σ is a state (or just a positive linear functional) on A then we may define a (semidefinite) 'inner product' on A by

$$\langle a, b\rangle_\sigma = \sigma(b^*a).$$

The Cauchy–Schwarz inequality continues to hold (with the usual proof): we have

$$|\langle a, b\rangle_\sigma|^2 \leqslant \langle a, a\rangle_\sigma \,\langle b, b\rangle_\sigma.$$

Note that $\langle a, a\rangle_\sigma$ is different from the C^*-norm $\|a\|^2$.

(2.4) PROPOSITION: *A bounded linear functional $\varphi\colon A \to \mathbb{C}$ is positive if and only if $\lim \varphi(u_\lambda) = \|\varphi\|$ for some approximate unit u_λ for A; and in particular, if and only if $\varphi(1) = \|\varphi\|$ in case A happens to be unital.*

PROOF: There is no loss of generality in supposing that $\|\varphi\| = 1$.

Suppose that φ is positive. Then $\varphi(u_\lambda)$ is an increasing net of real numbers, bounded above by 1, and therefore convergent to some real $r \leqslant 1$. For each a in the unit ball of A, we have by Cauchy–Schwarz

$$|\varphi(au_\lambda)|^2 \leqslant \varphi(a^*a)\varphi(u_\lambda^2) \leqslant \varphi(u_\lambda) \leqslant r.$$

Thus, since $\{u_\lambda\}$ is an approximate unit, $|\varphi(a)|^2 \leqslant r$, whence $r \geqslant 1$ on taking the supremum over a in the unit ball of A. Since we already know $r \leqslant 1$, this gives $r = 1$ as required.

Conversely suppose that $\lim \varphi(u_\lambda) = 1$. First we will prove that φ takes real values on selfadjoint elements. Take a selfadjoint element $a \in A$, of norm one, and write $\varphi(a) = x + iy \in \mathbb{C}$; we want to prove that $y = 0$; assume for a contradiction that $y > 0$. For each n one can extract a u_n from the approximate unit which has $\varphi(u_n) > 1 - 1/n^3$ and commutes well enough with a that we have the estimate

$$\|nu_n + ia\|^2 \leqslant n^2 + 2$$

(of course if it were the case that $u_n a = au_n$ exactly then the bound would be $n^2 + 1$). Therefore

$$x^2 + \left(n(1 - 1/n^3)^2 + y\right)^2 \leqslant |\varphi(nu_n + ia)|^2 \leqslant n^2 + 2$$

as this yields a contradiction as $n \to \infty$.

We have now shown that φ is a selfadjoint linear functional; to show positivity, let $x \in A$ with $0 \leqslant x \leqslant 1$. Then for each λ, $u_\lambda - x$ is in the unit ball and so $\varphi(u_\lambda - x) \leqslant 1$. Let $\lambda \to \infty$ to obtain $1 - \varphi(x) \leqslant 1$, whence $\varphi(x) \geqslant 0$ as required.
■

(2.5) COROLLARY: *Every state on a non-unital C^*-algebra extends uniquely to a state on its unitalization.*

PROOF: Let $\sigma\colon A \to \mathbb{C}$ be a state, and define $\tilde{\sigma}$ on the unitalization $\widetilde{A}$ in the only possible way,

$$\tilde{\sigma}(a + \mu 1) = \sigma(a) + \mu.$$

We need only to know that $\tilde{\sigma}$ is positive (and hence continuous), and this follows if we write

$$\tilde{\sigma}(a + \mu 1) = \lim \sigma(a + \mu u_\lambda)$$

using the previous proposition. ■

2.2 The Gelfand–Naimark–Segal construction

(2.6) DEFINITION: *Let A be a C^*-algebra. A representation $\rho\colon A \to \mathfrak{B}(H)$ is* cyclic *if there is a vector $v \in H$ (called a* cyclic vector*) such that $\rho[A]v$ is dense in H.*

(2.7) PROPOSITION: *Every non-degenerate representation of a C^*-algebra is a (possibly infinite) direct sum of cyclic representations.*

PROOF: Let $\rho\colon A \to \mathfrak{B}(H)$ be a representation. For each unit $v \in H$, let H_v be the cyclic subspace of H generated by v, that is the closure of $\rho[A]v$. By non-degeneracy, $v \in H_v$ (see Remark 1.45), and it clearly follows that H_v is a cyclic subrepresentation of H with cyclic vector v. Define a partially ordered set $\mathcal{S}$ as follows: a member of $\mathcal{S}$ is a set S of unit vectors of H such that the corresponding cyclic subspaces are pairwise orthogonal, and $\mathcal{S}$ is partially ordered by setwise inclusion. Zorn's Lemma clearly applies and produces a maximal $S \in \mathcal{S}$. Let $K = \bigoplus_{v \in S} H_v \leqslant H$; I claim that $K = H$. If not, there is a vector u orthogonal to K, and then H_u is orthogonal to K since K is $\rho[A]$-invariant. Thus u may be added to S, contradicting maximality. The claim, and thus the Proposition, follows. ■

(2.8) THEOREM: *Let A be a C^*-algebra and let σ be a state on A. Then there exist a representation ρ of A on a Hilbert space H_σ, and a unit cyclic vector $u \in H_\sigma$, such that*

$$\sigma(a) = \langle \rho(a)u, u\rangle_{H_\sigma};$$

that is, σ is the vector state corresponding to u.

PROOF: The Hilbert space H_σ is built out of A with the inner product $\langle a, b\rangle_\sigma = \sigma(b^*a)$ defined above, and the representation is the left regular representation. That is the idea: here are the details.

Let $N \subseteq A$ be defined by $N = \{a \in A : \langle a, a\rangle_\sigma = 0\}$. By virtue of the Cauchy–Schwarz inequality, N is a closed subspace: in fact

$$N = \{a \in A : \langle a, b\rangle = 0 \forall b \in A\}.$$

Since $\langle ca, b\rangle_\sigma = \langle a, c^* b\rangle_\sigma$, N is actually closed under multiplication on the left by members of A; it is a *left ideal*.

The quotient space A/N inherits a well-defined positive definite inner product from the inner product on A (this uses the fact that N is a left ideal). We complete this quotient space to obtain a Hilbert space H_σ. Since N is a left ideal, the left multiplication representation of A on A descends to a representation ρ (clearly a $*$-representation) of A on A/N. This representation is norm-decreasing because

$$\|\rho(a)\|^2 = \sup_{\sigma(b^*b) \leqslant 1} \sigma(b^* a^* a b) \leqslant \|a^* a\| = \|a\|^2,$$

where for the inequality we used the relation

$$b^* a^* a b \leqslant \|a^* a\| b^* b$$

between positive operators. Thus ρ extends by continuity to a representation of A on H_σ.

Now we must produce the cyclic vector u in H_σ. Let u_λ be an approximate unit for A. The inequality

$$\|[u_\lambda] - [u_{\lambda'}]\|^2 = \sigma((u_\lambda - u_{\lambda'})^2) \leqslant \sigma(u_\lambda - u_{\lambda'})$$

shows that the classes $[u_\lambda]$ form a Cauchy net in H_σ, convergent say to a vector u. For each $a \in A$ we have $\rho(a)u = \lim[a u_\lambda] = [x]$, so $\rho(A)u$ is dense in H_σ and u is cyclic. Finally, for any positive element $a^* a$ of A,

$$\langle \rho(a^* a)u, u\rangle_\sigma = \langle [a], [a]\rangle_\sigma = \sigma(a^* a).$$

Hence by linearity $\langle \rho(b)u, u\rangle_\sigma = \sigma(b)$ for all $b \in A$. ■

(2.9) REMARK: There is a uniqueness theorem to complement the existence theorem provided by the GNS construction. Namely, suppose that $\rho\colon A \to \mathfrak{B}(H)$ is a cyclic representation with cyclic vector u. Then a state of A is defined by $\sigma(a) = \langle \rho(a)u, u\rangle$, and from this state one can build according to the GNS construction a cyclic representation $\rho_\sigma\colon A \to \mathfrak{B}(H_\sigma)$ with cyclic vector u_σ. This newly-built representation is unitarily equivalent to the original one: in other words there is a unitary isomorphism $U\colon H \to H_\sigma$, intertwining ρ and ρ_σ, and mapping u to u_σ. It is easy to see this: from the GNS construction the map

$$\rho(a)u \mapsto [a]$$

is a well-defined isometric map from the dense subspace $\rho[A]u$ of H to the dense subspace A/N of H_σ; so it extends by continuity to the required unitary isomorphism.

We are now going to show that every C^*-algebra has plenty of states, enough in fact to produce a faithful Hilbert space representation by the GNS construction.

(2.10) LEMMA: *Let A be a C^*-algebra. For any positive $a \in A$, there is a state σ of A for which $\sigma(a) = \|a\|$.*

PROOF: In view of Corollary 2.5 we may assume that A is unital. The commutative C^*-algebra $C^*(a) = C(\mathrm{sp}(a))$ admits a state φ (in fact, a $*$-homomorphism to $\mathbb{C}$) corresponding to a point of $^{(}a)$ where the spectral radius is attained; for this φ, one has $\varphi(a) = \|a\|$, and of course $\varphi(1) = 1$, $\|\varphi\| = 1$. Using the Hahn–Banach theorem we may extend φ to a linear functional σ on A of norm one. Since $\sigma(1) = 1$, Proposition 2.4 shows that σ is a state. ■

(2.11) REMARK: By the same argument we may show that if $a \in A$ is self-adjoint, there is a state σ such that $|\sigma(a)| = \|a\|$.

This allows us to complete the proof of the Gelfand–Naimark Representation Theorem.

(2.12) THEOREM: *Every C^*-algebra is $*$-isomorphic to a C^*-subalgebra of some $\mathfrak{B}(H)$.*

PROOF: Let A be a C^*-algebra. For each $a \in A$, use the previous lemma to manufacture a state σ_a having $\sigma_a(a^*a) = \|a\|^2$, and then use the GNS construction to build a representation $\rho_a \colon A \to \mathfrak{B}(H_a)$, with cyclic vector u_a, from the state σ_a. We have $\|\rho_a(a)u_a\|^2 = \langle \rho(a^*a)u_a, u_a\rangle = \sigma_a(a^*a) = \|a\|^2$. It follows that $\rho = \bigoplus_{a\in A} \rho_a$ is a faithful representation, as required. ■

When A is separable the enormously large Hilbert space we used above can be replaced by a separable one; this is a standard argument.

(2.13) REMARK: We have applied the GNS construction to C^*-algebras. The construction, however, has a wider scope. Let A be a unital complex $*$-algebra (without topology). A linear functional $\varphi \colon A \to \mathbb{C}$ is said to be *positive* if $\varphi(a^*a) \geqslant 0$ for all $a \in A$. It is *representable* if, in addition, for each $x \in A$ there exists a constant $c_x > 0$ such that

$$\varphi(y^*x^*xy) \leqslant c_x \varphi(y^*y)$$

for all $y \in A$.

By following through the GNS construction exactly as above, one can show that *to any representable linear functional φ there corresponds a Hilbert space representation $\rho \colon A \to \mathfrak{B}(H)$ and a cyclic vector u such that $\varphi(a) = \langle \rho(a)u, u\rangle$.*

Representability is automatic for positive linear functionals on (unital) *Banach* algebras:

(2.14) LEMMA: *Any positive linear functional φ on a unital Banach $*$-algebra*[8] *A is continuous and representable.*

PROOF: Suppose $x = x^* \in A$ with $\|x\| < 1$. Using Newton's binomial series we may construct a selfadjoint

$$y = (1-x)^{\frac{1}{2}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \cdots$$

with $y^2 = 1 - x$. We conclude that $\varphi(1-x) \geqslant 0$ by positivity, so $\varphi(x) \leqslant \varphi(1)$. Now for any $a \in A$ we have

$$|\varphi(a)|^2 \leqslant \varphi(1)\varphi(a^*a) \leqslant \varphi(1)^2\|a^*a\| \leqslant \varphi(1)^2\|a\|^2$$

(the first inequality following from Cauchy-Schwarz and the second from the preceding discussion), so φ is continuous and $\|\varphi\| \leqslant \varphi(1)$. To show representability, we apply the preceding discussion to the positive linear functional $a \mapsto \varphi(y^*ay)$ to get the desired conclusion with $c_x = \|x^*x\|$. ■

It was shown by Varopoulos that this theorem extends to non-unital Banach algebras having bounded approximate units.

Remark on universal algebras defined by generators and relations.

2.3 Abelian von Neumann Algebras and the Spectral Theorem

In this section we discuss the GNS construction in the abelian case, and what can be obtained from it. Let us begin by constructing a standard example of an abelian von Neumann algebra.

Let (X, μ) be a finite measure space. For each essentially bounded function $f \in L^\infty(X, \mu)$, the corresponding multiplication operator M_f on the Hilbert space $H = L^2(X, \mu)$ is bounded, with operator norm equal to the L^∞-norm of f; and in this way one identifies $L^\infty(X, \mu)$ with a $*$-subalgebra of $\mathfrak{B}(H)$.

(2.15) PROPOSITION: *The algebra $L^\infty(X, \mu)$ is a von Neumann algebra of operators on $L^2(X, \mu)$.*

PROOF: We will show that $M = L^\infty(X, \mu)$ is equal to its own commutant. Suppose that $T \in \mathfrak{B}(H)$ commutes with each M_f, and let $g = T \cdot 1$, where

[8] By definition, we require that the involution on a Banach $*$-algebra should be isometric.

1 denotes the constant function 1, considered as an element of H. Then for all $f \in L^\infty(X,\mu) \subseteq L^2(X,\mu)$ we have

$$Tf = TM_f 1 = M_f T1 = M_f g = fg.$$

Since L^∞ is dense in L^2 it follows that $T = M_g$, and standard estimates show that $\|g\|_\infty \leqslant \|T\|$, so that $g \in L^\infty$. Thus $M' \subseteq M$; but $M \subseteq M'$ since M is abelian, and consequently $M = M'$. ■

The proposition still holds for a σ-finite measure space (X,μ), with essentially the same proof.

The content of Spectral Theorem is that this is essentially the *only* example of an abelian von Neumann algebra. However, to explain what is meant here we need to distinguish between two notions of isomorphism for von Neumann algebras (or other algebras of operators on Hilbert space). Let $M \subseteq \mathfrak{B}(H)$ and $N \subseteq \mathfrak{B}(K)$ be von Neumann algebras. We will say that they are *abstractly isomorphic* if there is a $*$-isomorphism (an isomorphism of C^*-algebras) from M to N; and we will say that they are *spatially isomorphic* if there is a unitary $U\colon H \to K$ such that $UMU^* = N$. Clearly spatial isomorphism implies abstract isomorphism but the converse is not true: the 1-dimensional von Neumann algebras generated by the identity operators on two Hilbert spaces of different finite dimension are abstractly isomorphic, but they are not spatially isomorphic.

(2.16) PROPOSITION: *Let X be a compact metrizable space. Every cyclic representation of $C(X)$ is unitarily equivalent to a representation of $C(X)$ by multiplication operators on $L^2(X,\mu)$, where μ is some regular Borel probability measure on X.*

PROOF: Let ρ be a cyclic representation of $A = C(X)$ with cyclic vector u. Then $\sigma(f) = \langle \rho(f)u, u\rangle$ defines a state on A. By the uniqueness of the GNS construction (Remark 2.9), ρ is unitarily equivalent to the GNS representation constructed from the state σ. Now we appeal to the Riesz Representation Theorem from measure theory (see Rudin, *Real and complex analysis*, Theorem 2.14, for example): each state σ of $C(X)$ is the functional of integration with respect to some regular Borel probability measure μ. The GNS construction then produces the space $L^2(X,\mu)$ and the multiplication representation. ■

Let M be an abelian von Neumann algebra in $\mathfrak{B}(H)$. We say that M is *maximal abelian* if it is contained in no larger abelian von Neumann algebra. It is easy to see that this condition is equivalent to $M' = M$. In particular, the proof of Proposition 2.15 shows that $L^\infty(X,\mu)$ is a maximal abelian von Neumann algebra of operators on $L^2(X,\mu)$.

(2.17) THEOREM: *Let M be an abelian von Neumann algebra in $\mathfrak{B}(H)$, where H is separable. Then the following are equivalent:*

(a) M has a cyclic vector;

(b) M is maximal abelian;

(c) M is spatially equivalent to $L^\infty(X,\mu)$ acting on $L^2(X,\mu)$, for some finite measure space X.

PROOF: Suppose that M has a cyclic vector. We begin by remarking that M has a *separable* C^*-subalgebra A such that $A'' = M$. (Proof: The unit ball of $\mathfrak{B}(H)$ is separable and metrizable in the strong topology, hence second countable. Thus the unit ball of M is second countable, hence M itself is separable, in the strong topology. Let A be the C^*-algebra generated by a countable strongly dense subset of M.) The representation of A on $\mathfrak{B}(H)$ is cyclic (since A is strongly dense in M) so by Proposition 2.16 it is unitarily equivalent to the representation of $C(X)$ on $L^2(X,\mu)$ by multiplication operators. Now $L^\infty(X,\mu)$ is a von Neumann algebra which contains $C(X)$ as a strongly dense subset (by the Dominated Convergence Theorem) so the unitary equivalence must identify $L^\infty(X,\mu)$ with M. This proves that (a) implies (c). We have already remarked that (c) implies (b). Finally, suppose that M is *any* abelian von Neumann algebra of operators on H. Decompose H as a countable direct sum of cyclic subspaces for the commutant M', say with unit cyclic vectors u_n; and let $u = \sum 2^{-n}u_n$. The projection $P_n\colon H \to H_n$ belongs to $M'' = M \subseteq M'$, using the bicommutant theorem and the fact that M is abelian. Hence each $u_n = 2^n P_n u$ belongs to $M'x$, so u is a cyclic vector for M'. We have shown that the commutant of any abelian von Neumann algebra on a separable Hilbert space has a cyclic vector; in particular if M is maximal abelian, then $M = M'$ and M has a cyclic vector. ■

We can use this to get the structure of a general abelian von Neumann algebra, up to abstract (not spatial!) isomorphism:

(2.18) THEOREM: *Each commutative von Neumann algebra on a separable Hilbert space is abstractly isomorphic to some $L^\infty(X,\mu)$.*

PROOF: Let M be an abelian von Neumann algebra on $\mathfrak{B}(H)$. Recall from the last part of the preceding proof that the commutant M' has a cyclic vector u; let P denote the orthogonal projection onto $\overline{Mu}$. Then P is a projection in M'. The map $\alpha\colon T \mapsto TP$ is a $*$-homomorphism from M onto the von Neumann algebra MP, and, by construction, u is a cyclic vector for MP (thought of as acting on the range of P). Thus MP is isomorphic to an algebra $L^\infty(X,\mu)$ by the previous theorem. I claim that $\mathrm{Ker}(\alpha) = 0$; this will show that M and MP are isomorphic and so will complete the proof. If $T \in \mathrm{Ker}(\alpha)$ then $Tu = 0$. For any $R \in M$", then, $TRu = RTu = 0$; but since u is cyclic for M', the vectors Ru are dense in H, and it follows that $T = 0$. ■

(2.19) REMARK: (about spectral multiplicity theory)

Let us specialize these constructions to the case of a single operator.

(2.20) COROLLARY: *Let T be a normal operator on a separable Hilbert space H. Then there exist a measure space (Y, μ) and a unitary equivalence $U\colon H \to L^2(Y, \mu)$, such that UTU^* is the multiplication operator by some L^∞ function on Y.*

PROOF: Let X be the spectrum of T, so that the functional calculus gives a representation ρ of $C(X)$ on H. By Proposition 2.7, we can break ρ up into a direct sum of cyclic representations; there are only countably many summands since H is separable. According to Proposition 2.16, each such cyclic representation ρ_n is unitarily equivalent to the representation of $C(X)$ by multiplication operators on $L^2(X, \mu_n)$ for some regular Borel probability measure μ_n on X. Put now $(Y, \mu) = \bigsqcup_n (X, \mu_n)$ to get the result. ■

As a corollary we obtain the *Borel functional calculus*. Given any bounded Borel function f on the spectrum of T, we may define $f(T)$ as follows: if T is unitarily equivalent to multiplication by some L^∞ function g on $L^2(Y, \mu)$ (note that the closure of the essential range of g is the spectrum of T), then $f(T)$ is the bounded operator that is unitarily equivalent to multiplication by $f \circ g$. Observe that if f_n is a uniformly bounded sequence of Borel functions that tends pointwise to f, then $f_n(T)$ tends strongly to $f(T)$, since

$$\int_Y |(f_n - f) \circ g(y)|^2 |u(y)|^2 \, d\mu(y) \to 0$$

as $n \to \infty$, for all $u \in L^2(Y, \mu)$, by Lebesgue's dominated convergence theorem.

(2.21) REMARK: An alternative, classic formulation of these results makes use of the notion of *spectral measure* (also called *resolution of the identity*). Let H be a Hilbert space and let $(X, \mathfrak{M})$ be a measurable space, that is a space equipped with a σ-algebra of subsets. A *spectral measure* associated to the above data is a map E from $\mathfrak{M}$ to the collection $P(H)$ of selfadjoint projections in H, having the following properties:

(a) $E(\emptyset) = 0$, $E(X) = 1$;

(b) $E(S \cap T) = E(S)E(T)$;

(c) if $S \cap T = \emptyset$, then $E(S \cup T) = E(S) + E(T)$;

(d) E is countably additive relative to the strong operator topology; that is, if S_n is a sequence of mutually disjoint subsets of X with union S, then $E(S) = \sum E(S_n)$ (with strong operator convergence).

Given such a spectral measure it is straightforward to define an 'integral'

$$\int_X f(x)dE(x) \in \mathfrak{B}(H)$$

for every function f on X which is bounded and $\mathfrak{M}$-measurable; one simply mimics the usual processes of measure theory. For instance if f is the characteristic function of some $S \in \mathfrak{M}$, then we define $\int_X f(x)dE(x) = E(S)$. By linearity we extend this definition to simple functions (finite linear combinations of characteristic functions), and then by a limit argument we extend further to all measurable bounded functions.

We may then formulate the spectral theorem as follows: for every normal operator T on a separable Hilbert space, there is a resolution of the identity E on the σ-algebra of Borel subsets of $\mathrm{sp}(T)$, such that

$$T = \int_{\mathrm{sp}(T)} \lambda dE(\lambda).$$

For the proof, one need only verify that the definition $E(S) = \chi_S(T)$ (using the Borel functional calculus described above) describes a resolution of the identity. To show that T has an integral decomposition as above one approximates the integral by Riemann sums and uses the fact that, when restricted to the range of the projection $E((k/n, (k+1)/n])$, the operator T lies within $1/n$ in norm of the operator of multiplication by k/n.

2.4 Pure States and Irreducible Representations

In this section we will begin a more detailed study of the representation theory of a C^*-algebra A. By application to group C^*-algebras, this theory will include the theory of unitary representations of groups. We'll discuss this connection.

Recall that a representation $\rho\colon A \to \mathfrak{B}(H)$ is *irreducible* if there are no closed A-invariant subspaces of H. In Remark 2.2 we drew attention to Kadison's transitivity theorem, which states that our topological notion of irreducibility is equivalent to the algebraic notion according to which there are no invariant subspaces at all, closed or not. We will now prove this theorem.

(2.22) KADISON TRANSITIVITY THEOREM *Let ρ be a (topologically) irreducible representation of a C^*-algebra A on a Hilbert space H. Then for any two vectors*

$u, v \in H$, with u nonzero, there exists $a \in A$ with $\rho(a)u = v$. In particular, there are no A-invariant subspaces of H, closed or not.

PROOF: We begin by establishing a quantitative, approximate version of the theorem. Suppose that u and v are unit vectors in H. We assert that there exists $a \in A$ with $\|a\| < 2$ and $\|\rho(a)u - v\| < \frac{1}{2}$. Indeed, irreducibility implies that $\rho[A]'' = \mathfrak{B}(H)$ (see Remark 2.2), so by the Kaplansky Density Theorem the unit ball of $\rho[A]$ is strongly dense in the unit ball of $\mathfrak{B}(H)$. In particular there is an element $\rho(a)$ of this unit ball such that $\|\rho(a)u - v\| < \frac{1}{2}$; and we may take $\|a\| < 2$ since ρ is the composite of a quotient map and an isometric injection (Corollary 1.33).

Now we apply this argument, and an obvious rescaling, iteratively as follows: given u and $v = v_0$, produce a sequence $\{a_j\}$ in A and a sequence $\{v_j\}$ of vectors of H with

$$v_{j+1} = v_j - \rho(a_j)u, \quad \|v_j\| < 2^{-j}, \quad \|a_j\| < 4 \cdot 2^{-j}.$$

The series $a = \sum a_j$ then converges in A to an element of norm $\leqslant 4$ having $\rho(a)u = v$. (With more care we can reduce the norm estimate from 4 to $1 + \varepsilon$. We can also move n linearly independent u's simultaneously onto arbitrary v's, rather than just one.) ■

Let A be a C^*-algebra (unital for simplicity). Recall that a linear functional $\sigma\colon A \to \mathbb{C}$ is a state if it has norm one and moreover $\sigma(1) = 1$. Thus the collection of states on A is a weak-star compact[9], convex subset of A^*. It is called the *state space* of A, and denoted by $S(A)$.

The general yoga of functional analysis encourages us to consider the extreme points of $S(A)$, called the *pure states*. By the Krein–Milman Theorem, $S(A)$ is the closed convex hull of the pure states.

(2.23) LEMMA: *Let σ be a state of a C^*-algebra A, and let $\rho\colon A \to \mathfrak{B}(H)$ be the associated GNS representation, with unit cyclic vector u. For any positive functional $\varphi \leqslant \sigma$ on A there is an operator $T \in \rho[A]'$ such that*

$$\varphi(a) = \langle \rho(a)Tu, u \rangle$$

and $0 \leqslant T \leqslant 1$.

This is a kind of 'Radon-Nikodym Theorem' for states.

PROOF: Recall that H is obtained by completing the inner product space A/N, where N is the set $\{a \in A : \sigma(a^*a) = 0\}$. Define a sesquilinear form on A/N by

[9]Warning: In the non-unital case the state space is not closed in the unit ball of A^*, and hence is not compact; because the condition $\lim \sigma(u_\lambda) = 1$ is not preserved by weak-star convergence. This causes some technical gyrations, see Pedersen's book for details.

$(a, b) \mapsto \varphi(b^*a)$. This form is well-defined, positive semidefinite, and bounded by 1, so there is an operator T on H such that

$$\varphi(b^*a) = \langle T[a], [b]\rangle = \langle T\rho(a)u, \rho(b)u\rangle.$$

The computation

$$\langle \rho(a)T[b], [c]\rangle = \langle T[b], [a^*c]\rangle = \varphi(c^*ab) = \langle T\rho(a)[b], [c]\rangle$$

shows that T commutes with $\rho[A]$. ■

(2.24) PROPOSITION: *A state is pure if and only if the associated GNS representation is irreducible.*

PROOF: Suppose first that the representation $\rho\colon A \to \mathfrak{B}(H)$ is reducible, say $H = H_1 \oplus H_2$ where H_1 and H_2 are closed, A-invariant subspaces. Write a cyclic vector $v = v_1 + v_2$ where $v_i \in H_i$, $i = 1, 2$. We cannot have $v_1 = 0$, otherwise $\rho[A]v \subseteq H_2$; and for similar reasons we cannot have $v_2 = 0$. Now we have

$$\sigma(a) = \langle \rho(a)v, v\rangle = \langle \rho(a)v_1, v_1\rangle + \langle \rho(a)v_2, v_2\rangle$$

which is a nontrivial convex combination of the states corresponding to the unit vectors $v_1/\|v_1\|$ and $v_2/\|v_2\|$. Thus σ is impure.

Conversely, suppose that $\sigma = t\sigma_1 + (1-t)\sigma_2$ where σ_1, σ_2 are states. By the Lemma above there is an operator T in the commutant $\rho[A]'$ such that $t\sigma_1(a) = \langle \rho(a)Tu, u\rangle$. If ρ is irreducible then $\rho[A]$ has trivial commutant and so $T = cI$ for some constant c and thus $\sigma_1 = c\sigma$. Normalization shows $\sigma_1 = \sigma$. Similarly $\sigma_2 = \sigma$ and thus σ is pure. ■

(2.25) REMARK: It is important to know that there are enough *irreducible* unitary representations of a C^*-algebra A to separate points of A, that is to say C^*-algebras are *semi-simple*. To show this we need to refine Lemma 2.10 to show that, for any positive $a \in A$, there exists a *pure* state σ with $\sigma(a) = \|a\|$. Here is how to do this: Lemma 2.10 shows that states having this property exist. The collection Σ of all states having this property is a closed convex subset of the state space, so (by the Krein–Milman Theorem) it has extreme points — indeed it is the closed convex hull of these extreme points. Let σ be an extreme point of Σ. If $\sigma = t\sigma_1 + (1-t)\sigma_2$, $0 < t < 1$, then the inequalities

$$\sigma(a) = \|a\|, \quad \sigma_1(a) \leqslant \|a\|, \quad \sigma_2(a) \leqslant \|a\|$$

imply that $\sigma_1, \sigma_2 \in \Sigma$. Hence $\sigma_1 = \sigma_2 = \sigma$ since σ is extreme in Σ. Thus σ is extreme in S and we are done.

(2.26) DEFINITION: *Let A be a C^*-algebra. The* spectrum *of A, denoted $\widehat{A}$, is the collection of unitary (= spatial) equivalence classes of irreducible representations of A. We topologize it as a quotient of the pure state space: that is, we give $\widehat{A}$ the topology which makes the surjective map $P(A) \to \widehat{A}$, which associates to each pure state its GNS representation, open and continuous. This is called the* Fell topology.

(2.27) REMARK: Let $A = C(X)$ be commutative. The states of A are measures on X; the pure states are the Dirac δ-measures at the points of X. On the other hand, it is shown in the exercises that each irreducible representation of A is 1-dimensional, that is, is a homomorphism $A \to \mathbb{C}$. These homomorphisms are again classified by the points of X. In this case therefore we have $P(A) = \widehat{A} = X$. Needless to say the general case is not so simple.

Let $\rho\colon A \to \mathfrak{B}(H)$ be a representation. Then the kernel of ρ is an ideal of A.

(2.28) DEFINITION: *An ideal in a C^*-algebra A is called* primitive *if it arises as the kernel of an irreducible representation. The space of primitive ideals is denoted* $\mathrm{Prim}(A)$.

(2.29) LEMMA: *Every primitive ideal of a C^*-algebra is prime.*

PROOF: We begin by proving a general fact. Suppose that $\rho\colon A \to \mathfrak{B}(H)$ is a representation and J is an ideal in A. Then *the orthogonal projection onto $\rho[J]H$[10] lies in the center of the bicommutant $\rho[A]''$*. To see this, note that the subspace $\rho[J]H$ is both $\rho[A]$-invariant (because J is an ideal) and $\rho[J]'$-invariant. Therefore P lies in $\rho[A]' \cap \rho[J]''$. But this is contained in $\rho[A]' \cap \rho[A]''$, which is simply the center of the bicommutant.

If we apply this when ρ is irreducible, the bicommutant of $\rho[A]$ is $\mathfrak{B}(H)$ which has trivial center; so $\rho[J]H$ is either zero or H, and $J \subseteq \mathrm{Ker}(\rho)$ precisely when $\rho[J]H = 0$. It is now easily seen that if J_1 and J_2 are ideals such that $J_1 J_2 \subseteq \mathrm{Ker}(\rho)$, then at least one of them is itself contained in $\mathrm{Ker}(\rho)$. That is, $\mathrm{Ker}(\rho)$ is prime. ■

The *converse* of this result is also true, at least for separable C^*-algebras. However, to prove this will require something of a detour.

Let R be any ring. The collection of prime ideals of R always has a topology, called the *Zariski topology*, according to which the closed sets of prime ideals are just those sets of the form

$$\mathrm{hull}(X) = \{\mathfrak{p} : X \subseteq \mathfrak{p}\}$$

[10] This subspace is closed by the Cohen Factorization Theorem.

as X ranges over the set of subsets of R. (To prove that this defines a topology you need to use the defining property of a prime ideal, namely that if an intersection $\mathfrak{j}_1 \cap \mathfrak{j}_2 \subseteq \mathfrak{p}$, then either $\mathfrak{j}_1$ or $\mathfrak{j}_2$ already lies in $\mathfrak{p}$. The closure of any set S of prime ideals is the set $\mathrm{hull}\left(\bigcap_{\mathfrak{p}\in S}\mathfrak{p}\right)$. The intersection appearing here is sometimes called the 'kernel' of S, a terminology that I find confusing.) Since primitive ideals in a C^*-algebra A are prime, the Zariski topology gives rise to a topology on $\mathrm{Prim}(A)$, which in this context is called the *Jacobson topology* or the *hull-kernel topology*.

(2.30) REMARK: The Zariski topology is rather bad from the point of view of the separation axioms. A single point of $\mathrm{Prim}(A)$ is closed iff it represents a maximal ideal. Consequently, unless all prime ideals are maximal (a rare occurrence) the topology is not even T_1, let alone Hausdorff. However, it is T_0 — given any two points, at least one of them is not in the closure of the other one.

(2.31) THEOREM: *Let A be a separable C^*-algebra. The canonical map $\widehat{A} \to \mathrm{Prim}(A)$ is a topological pullback: that is, the open sets in $\widehat{A}$ are precisely the inverse images of the open sets in* $\mathrm{Prim}(A)$.

We will give the proof in a moment.

(2.32) COROLLARY: *The canonical map $P(A) \to \mathrm{Prim}(A)$ is a quotient map.*

Here is an important deduction from this:

(2.33) PROPOSITION: (DIXMIER) *In a* separable *C^*-algebra A every closed prime ideal is primitive.*

PROOF: (Recall our standing assumption that A is unital.) The space $P(A)$ is a closed convex subset of the unit ball of A^*, and hence is compact and metrizable; in particular it is second countable and it is a Baire space. This latter assertion means that the Baire category theorem holds: each countable intersection of dense open sets is dense. Both properties pass to quotients, and so we conclude that $\mathrm{Prim}(A)$ is also a second countable Baire space.

Now suppose that 0 is a prime ideal of A (the general case can be reduced to this one by passing to quotients), in other words that any two nontrivial ideals of A have nontrivial intersection. We want to prove that 0 is a primitive ideal, that is, A has a faithful irreducible representation. Since nontrivial ideals intersect non-trivially, it follows that we can't write $\mathrm{Prim}(A)$ as the union of two proper closed subsets; which is to say that any two nonempty open subsets of $\mathrm{Prim}(A)$ intersect; so every nonempty open subset is dense. Applying the Baire category theorem to a countable base for the topology we find that there is a dense point in $\mathrm{Prim}(A)$. This point corresponds to a primitive ideal which is contained in *every* primitive ideal, and since the irreducible representations separate points of A, the only possibility for such an ideal is (0) which is therefore primitive, as required. ■

Now we will work towards a proof of Theorem 2.31.

(2.34) LEMMA: *Let A be a C^*-algebra, let $\mathcal{E}$ be a collection of states of A, and suppose that for every $a \geqslant 0$ in A we have*

$$\sup\{\sigma(a) : \sigma \in \mathcal{E}\} = \|a\|.$$

Then the weak-$*$ closure of $\mathcal{E}$ contains every pure state.

PROOF: Once again, we stick with unital A. We'll show in fact that the weak-$*$ closed convex hull C of $\mathcal{E}$ is the entire state space S of A. This will suffice, since the extreme points of the closed convex hull of $\mathcal{E}$ must belong to the closure of $\mathcal{E}$.

Think of A_{sa} as a real Banach space, and think of C and S as compact convex subsets of the LCTVS E which is the dual of A_{sa} equipped with the weak-$*$ topology. The dual of E is then A_{sa} once again. Suppose for a contradiction that $C \neq S$, so that there exists $\varphi \in S \setminus C$. By the Hahn-Banach theorem applied to the space E, there exist $a \in A_{\mathsf{sa}}$ and $c \in \mathbb{R}$ such that $\varphi(a) > c$ whereas $\sigma(a) < c$ for all $\sigma \in C$. We may assume a is positive (add a large multiple of 1). Now we get $\varphi(a) > \sup\{\sigma(a) : \sigma \in \mathcal{E}\} = \|a\|$ which is a contradiction. ■

PROOF OF THEOREM 2.31: Let $\Pi \subseteq \widehat{A}$ be a set of irreducible representations of A, and let ρ be another such representation. Then ρ belongs to the closure of Π for the quotient topology from $P(A)$ if and only if

(a) some state associated to ρ is a weak-$*$ limit of states associated to representations $\pi \in \Pi$.

On the other hand, ρ belongs to the closure of Π for the pull-back of the Jacobson topology from $\mathrm{Prim}(A)$ if and only if

(b) the kernel $\mathrm{Ker}(\rho)$ contains $\bigcap\{\mathrm{Ker}(\pi) : \pi \in \Pi\}$.

We need therefore to prove that conditions (a) and (b) are equivalent.

Suppose (b). By passing to the quotient, we may assume without loss of generality that $\bigcap\{\mathrm{Ker}(\pi) : \pi \in \Pi\} = 0$. The direct sum of all the representations π is then faithful, and so for every $a \in A$

$$\sup\{\sigma(a) : \sigma \in \mathcal{E}\} = \|a\|,$$

where $\mathcal{E}$ denotes the collection of vector states associated to the representations belonging to Π. By the lemma, every pure state of A (and every state associated to ρ in particular) belongs to the weak-$*$ closure of $\mathcal{E}$. This proves (a).

Conversely suppose (a). Let J denote the ideal $\bigcap\{\mathrm{Ker}(\pi) : \pi \in \Pi\}$. Suppose that the state $\sigma(a) = \langle \rho(a)v, v\rangle$ is a weak-$*$ limit of states associated to representations $\pi \in \Pi$. All such states vanish on every $a \in J$, so $\sigma(a) = 0$ for all $a \in J$. But then for all $y, z \in A$,

$$\langle \rho(a)\rho(y)v, \rho(z)v\rangle = \sigma(z^* ay) = 0$$

since J is an ideal. Because v is a cyclic vector for ρ this implies that $\rho(a) = 0$, so $\mathrm{Ker}(\rho) \supseteq J$, which is (b). ■

(2.35) REMARK: Using Voiculescu's theorem one can show that two points of $\widehat{A}$ map to the same point of $\mathrm{Prim}(A)$ if and only if the corresponding representations are approximately unitarily equivalent.

2.5 Representations of compact operators

To begin our study of representation theory, we need a few lemmas about representations of algebras of compact operators.

(2.36) LEMMA: *Let A be a C^*-algebra, let J be an ideal in A, and let $\rho_J\colon J \to \mathfrak{B}(H)$ be a non-degenerate representation. There is a unique extension of ρ_J to a representation $\rho_A\colon A \to \mathfrak{B}(H)$. Moreover, ρ_A is irreducible if and only if ρ_J is irreducible.*

PROOF: Since we went to the trouble of proving it in an exercise, let's use the Cohen Factorization Theorem here (we could avoid this by a marginal complication of the argument). By Cohen, every $v \in H$ is of the form $v = \rho_J(j)w$ for some $j \in J$, $w \in H$. Define then

$$\rho_A(a)v = \rho_J(aj)w.$$

To see that this is well-defined, take an approximate unit u_λ for J and write

$$\rho_J(aj)w = \lim \rho_J(au_\lambda j)w = \lim \rho_J(au_\lambda)v,$$

which also shows that $\rho_A(a)$ is the strong limit of the operators $\rho_J(au_\lambda)$ and therefore has norm bounded by $\|a\|$. It is routine to verify that ρ_A is a $*$-representation of A.

It is obvious that if ρ_A is reducible, so is ρ_J. On the other hand, if ρ_J has a proper closed invariant subspace H' say, the construction by way of an approximate unit shows that ρ_A will map H' to H'. So ρ_A is reducible too. ■

(2.37) LEMMA: *Let H be a Hilbert space and let $A \subseteq \mathfrak{K}(H)$ be a C^*-algebra of compact operators on H that is irreducibly represented on H. Then $A = \mathfrak{K}(H)$.*

PROOF: Since A consists of compact operators, it contains finite-rank projections by the functional calculus. Let P be such a projection, having minimal rank. I claim that P is of rank one. To see this, note first that $PTP = \lambda_T P$ for all $T \in A$ (one may assume T to be selfadjoint, and if the assertion were false then a nontrivial spectral projection of PTP would have smaller rank than P, contradiction). Now if there are orthogonal vectors u, v in the range of P, then for all $T \in A$,

$$\langle Tu, v\rangle = \langle PTPu, v\rangle = \lambda_T \langle u, v\rangle = 0.$$

Thus Au is a nontrivial invariant subspace, contradicting irreducibility. This establishes the claim that P has rank one.

Any rank-one projection P has the form

$$P(w) = \langle w, u\rangle u$$

for some unit vector u. Let v', v'' be two more unit vectors. According to Kadison's transitivity theorem there are $S', S'' \in A$ such that $S'u = v'$ and $S''v'' = u$. Thus A contains the operator $S'PS''$, which is given by

$$w \mapsto \langle w, v''\rangle v'.$$

The operators of this sort span all the operators of finite rank, hence they generate $\mathfrak{K}(H)$ as a C^*-algebra. ■

(2.38) COROLLARY: *The algebra $\mathfrak{K}(H)$ is simple (it has no non-trivial ideals).*

PROOF: Let $J \subseteq \mathfrak{K}(H)$ be a non-zero ideal. The subspace $J \cdot H$ of H is invariant under $\mathfrak{K}(H)$, so it is all of H. Thus J is non-degenerately represented on H. By Lemma 2.36, this representation of J is irreducible. By Lemma 2.37, $J = \mathfrak{K}(H)$. ■

(2.39) COROLLARY: *If A is an irreducible C^*-subalgebra of $\mathfrak{B}(H)$ which contains a single non-zero compact operator, then A contains all of $\mathfrak{K}(H)$.*

PROOF: Consider the ideal $J = A \cap \mathfrak{K}(H)$ in A. The space $J \cdot H$ is A-invariant, and A is irreducible, so $JH = H$ and J is non-degenerate. Now we argue as before: by Lemma 2.36 J is irreducible, and then by Lemma 2.37, $J = \mathfrak{K}(H)$. ■

(2.40) REMARK: Of course it follows that $\mathrm{Prim}(\mathfrak{K}(H))$ consists of a single point (corresponding to the zero ideal). Let us also go on to prove that $\mathfrak{K}(H)$ has only one irreducible representation, so that $\widehat{\mathfrak{K}(H)}$ also consists just of a single point.

Let $\rho\colon \mathfrak{K}(H) \to \mathfrak{B}(H')$ be an irreducible representation. Let P be a rank-one projection in $\mathfrak{K}(H)$. Since $\mathfrak{K}(H)$ is simple, ρ is faithful; so $\rho(P)$ is a nonzero projection on H'. Now let $v' \in H'$ be a unit vector in the range of the projection $\rho(P)$, and let σ be the associated state, that is

$$\sigma(T) = \langle \rho(T)v', v' \rangle.$$

Since ρ is irreducible, v' is a cyclic vector. By the GNS uniqueness theorem, ρ is unitarily equivalent to the GNS representation associated to the state σ. But

$$\sigma(T) = \langle \rho(PTP)v', v' \rangle = \lambda_T = \mathrm{Trace}(PTP)$$

and the GNS representation associated to this state is the standard one.

2.6 The Toeplitz algebra and its representations

Let $H = \ell^2$ be the usual sequence space. The *unilateral shift operator* V is the isometry $H \to H$ given by

$$V(x_0, x_1, x_2, \ldots) = (0, x_0, x_1, \ldots).$$

Notice that $V^*V = 1$, whereas $VV^* = 1 - P$, where P is the rank one projection onto the span of the basis vector $e_0 = (1, 0, 0, \ldots)$. Thus V is a Fredholm operator, of index -1.

(2.41) DEFINITION: *The* Toeplitz C^*-algebra $\mathfrak{T}$ *is the C^*-subalgebra of* $\mathfrak{B}(H)$ *generated by the unilateral shift operator.*

(2.42) LEMMA: *The algebra $\mathfrak{T}$ is irreducible and contains the compact operators.*

PROOF: Suppose that $K \subseteq H$ is a closed invariant subspace for $\mathfrak{T}$. If $K \neq 0$, then K contains a nonzero vector, and by applying a suitable power of V^* if necessary we may assume that K contains a vector $v = (x_0, x_1, x_2, \ldots)$ with $x_0 \neq 0$. Since $P \in \mathfrak{T}$ we find that $Pv = v_0 e_0 \in K$ and so $e_0 \in K$. Now applying powers of V we find that each $e_n = V^n e_0 \in K$, so $K = H$. Thus $\mathfrak{T}$ is irreducible. Since it contains the compact operator P, the second statement follows from Corollary 2.39. ■

The unilateral shift operator is unitary modulo the compacts, and so its *essential spectrum*, that is the spectrum of its image in the Calkin algebra $\mathfrak{Q}(H) = \mathfrak{B}(H)/\mathfrak{K}(H)$, is a subset of the unit circle S^1.

(2.43) LEMMA: *The essential spectrum of V is the whole unit circle.*

PROOF: This is a consequence of Fredholm index theory. Suppose the contrary: then there is a continuous path of complex numbers λ, running from $\lambda = 0$ to $\lambda = 2$, that avoids the essential spectrum of V. But then $\lambda \mapsto (V - \lambda)$ is a continuous path of Fredholm operators, and $\operatorname{Index} V = -1$ as we observed above, whereas $\operatorname{Index}(V - 2) = 0$ since $V - 2$ is invertible. This is a contradiction. ■

We see therefore that the C^*-algebra $\mathfrak{T}/\mathfrak{K}$ is commutative, generated by a single unitary element whose spectrum is the whole unit circle. Consequently there is a short exact sequence of C^*-algebras

$$0 \to \mathfrak{K} \to \mathfrak{T} \to C(S^1) \to 0$$

called the *Toeplitz extension*.

IT is helpful to make the link with complex function theory.

(2.44) DEFINITION: *Let S^1 denote the unit circle in $\mathbb{C}$. The* Hardy space $H^2(S^1)$ *is the closed subspace of $L^2(S^1)$ spanned by the functions z^n, for $n \geqslant 0$. A* Toeplitz operator *on $H^2(S^1)$ is a bounded operator T_g of the form*

$$T_g(f) = P(gf) \qquad (f \in H^2(S^1)),$$

where $g \in L^\infty(S^1)$ and P is the orthogonal projection from $L^2(S^1)$ onto $H^2(S^1)$. The function g is called the symbol *of T_g.*

(2.45) LEMMA: *The C^*-algebra generated by the Toeplitz operators is isomorphic to $\mathfrak{T}$, and the map $g \mapsto T_g$ is a linear splitting for the quotient map $\mathfrak{T} \to C(S^1)$ that appears in the Toeplitz extension.*

PROOF: We have $V = T_z$ (in the obvious basis) so certainly the C^*-algebra generated by the Toeplitz operators contains $\mathfrak{T}$. In the other direction, suppose that $p(z) = a_{-n}\bar{z}^n + \cdots + a_0 + \cdots + a_n z^n$ is a trigonometric polynomial. Then $T_p = a_{-n}(V^*)^n + \cdots + a_0 1 + \cdots + a_n V^n$ belongs to $\mathfrak{T}$. Since $\|T_g\| \leqslant \|g\|$, we find using the Stone-Weierstrass theorem that T_g belongs to $\mathfrak{T}$ for each continuous function g, and therefore that the C^*-algebra generated by the T_g is contained in $\mathfrak{T}$. ■

Toeplitz operators serve as a prototype for the connection between C^*-algebras, index theory, and K-homology. It is not the purpose of this course to focus on these connections, but we should at least mention the fundamental index theorem for Toeplitz operators. Recall that the *winding number* $\operatorname{Winding}(g) \in \mathbb{Z}$ of a continuous function $g\colon S^1 \to \mathbb{C} \setminus \{0\}$ is its class in the fundamental group $\pi_1(\mathbb{C} \setminus \{0\})$, which we identify with $\mathbb{Z}$ in such a way that the winding number of the function $g(z) = z$ is $+1$.

(2.46) TOEPLITZ INDEX THEOREM *If T_g is a Toeplitz operator on $H^2(S^1)$ whose symbol is a continuous and nowhere vanishing function on S^1 then T_g is a Fredholm operator and*

$$\mathrm{Index}(T_g) = -\,\mathrm{Winding}(g).$$

PROOF: Denote by M_g the operator of pointwise multiplication by g on $L^2(S^1)$. The set of all continuous g for which the commutator $PM_g - M_gP$ is compact is a C^*-subalgebra of $C(S^1)$. A direct calculation shows that $g(z) = z$ is in this C^*-subalgebra, for in this case the commutator is a rank-one operator. Since the function $g(z) = z$ generates $C(S^1)$ as a C^*-algebra (by the Stone–Weierstrass Theorem) it follows that $PM_g - M_gP$ is compact for every continuous g. Identifying T_g with PM_g, we see that

$$\begin{aligned} T_{g_1}T_{g_2} &= PM_{g_1}PM_{g_2} = PPM_{g_1}M_{g_2} + \text{compact operator} \\ &= PM_{g_1g_2} + \text{compact operator} \\ &= T_{g_1g_2} + \text{compact operator}, \end{aligned}$$

for all continuous g_1 and g_2. In particular, if g is continuous and nowhere zero then $T_{g^{-1}}$ is inverse to T_g, modulo compact operators, and so by Atkinson's Theorem, T_g is Fredholm. Now the continuity of the Fredholm index shows that $\mathrm{Index}(T_g)$ only depends on the homotopy class of the continuous function $g\colon S^1 \to \mathbb{C} \setminus \{0\}$. So it suffices to check the theorem on a representative of each homotopy class; say the functions $g(z) = z^n$, for $n \in \mathbb{Z}$. This is an easy computation, working in the orthogonal basis $\{\, z^n \mid n \geqslant 0 \,\}$ for $H^2(S^1)$. ■

(2.47) REMARK: on C^*-algebras defined by generators and relations. Wold–Coburn theorem: $\mathfrak{T}$ is the universal C^*-algebra generated by an isometry.

Now let us classify the irreducible representations of the Toeplitz algebra $\mathfrak{T}$. Let $\rho\colon \mathfrak{T} \to \mathfrak{B}(H)$ be such a representation. We consider two cases:

(a) ρ annihilates the ideal $\mathfrak{K} \subseteq \mathfrak{T}$;

(b) ρ does not annihilate the ideal $\mathfrak{K} \subseteq \mathfrak{T}$.

In case (b), $\rho(\mathfrak{K}) \cdot H$ is a $\mathfrak{T}$-invariant subspace, hence is equal to H; so the restriction of ρ to $\mathfrak{K}(H)$ is a non-degenerate representation. Hence it is irreducible (Lemma 2.36) and thus it is unitarily equivalent to the identity representation. It follows (Lemma 2.36 again) that the representation ρ of T is unitarily equivalent to the identity representation of $\mathfrak{T}$ on $H^2(S^1)$.

In case (a), ρ passes to an irreducible representation of $\mathfrak{T}/\mathfrak{K} = C(S^1)$. Such a representation is 1-dimensional, determined by a point of S^1; conversely all such 1-dimensional representations of $\mathfrak{T}$ are (of course!) irreducible.

Thus $\widehat{\mathfrak{T}} = \mathrm{Prim}(\mathfrak{T})$ consists of the union of S^1 and a disjoint point. What is the topology? It is easy to see that the circle S^1 gets its usual topology. On the other hand, the disjoint point corresponds to the zero ideal, which is of course contained in every other ideal; so that the only neighborhood of that point is the whole space. Thus the 'extra point' cannot be separated from any other representation. We summarize:

(2.48) PROPOSITION: *The space $\widehat{\mathfrak{T}}$ of irreducible representations of the Toeplitz algebra $\mathfrak{T}$ is of the form $S^1 \sqcup \{p_0\}$, where S^1 has its usual topology and the only open set containing p_0 is the whole space.*

3 Unitary Group Representations

3.1 Basics on Group C^*-Algebras

Let G be a group. Usually, later on in the course, we will assume that G is a *countable, discrete* group; but for the moment we will allow continuous groups as well, so let us assume that G is a *topological group*, with a *Hausdorff and second countable* topology. A *unitary representation* of G is a (group) homomorphism $\pi\colon G \to U(H)$, where $U(H)$ is the unitary group of a Hilbert space H, and π is required to be continuous relative to the *strong* topology on $U(H)$. That is, if $g_\lambda \to g$, then $\pi(g_\lambda)v \to \pi(g)v$, for each fixed vector $v \in H$.

(3.1) REMARK: It would *not* be a good idea to require that π be *norm* continuous. To see why, recall that a locally compact topological group G carries an (essentially unique) *Haar measure* μ: a regular Borel measure which is invariant under left translation. One can therefore form the Hilbert space $H = L^2(G, \mu)$ and the unitary representation of G on H by left translation

$$\pi(g)f(h) = f(g^{-1}h)$$

which is called the *regular representation*. It is a simple consequence of the Dominated Convergence Theorem that this representation is strongly continuous; but it is not norm continuous for instance when $G = \mathbb{R}$.

The Haar measure μ on G, which is invariant under *left* translation, is usually *not* invariant under *right* translation. However, for each fixed $g \in G$, the map $E \mapsto \mu(Eg)$ is *another* left-translation-invariant measure on G, and hence it differs from μ only by a scalar multiple which we denote $\Delta(g)$: thus

$$\mu(Eg) = \Delta(g)\mu(E).$$

It is easily checked that $\Delta\colon G \to \mathbb{R}^+$ is a topological group homomorphism. Δ is called the *modular function* of G; and groups such as discrete groups or abelian groups, for which $\Delta \equiv 1$, are called *unimodular*. An example of a non-unimodular group is the "$(ax + b)$ group" — the Lie group of matrices

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$$

with $a > 0$. (insert discussion of the modular function and Lie groups here.)

We'll use the notation dg instead of $d\mu(g)$ for integration with respect to the Haar measure. Notice the identities (for fixed h)

$$d(hg) = dg, \quad d(gh) = \Delta(h)dg, \quad d(g^{-1}) = \Delta(g^{-1})dg.$$

To prove the last identity note that both sides define *right* Haar measures, so they agree up to a scalar multiple; consider integrating the characteristic function of a small symmetric neighborhood of the identity (where $\Delta \approx 1$) to show that the scalar is 1.

(3.2) DEFINITION: *We make $L^1(G,\mu)$ into a Banach $*$-algebra by defining*

$$f_1 * f_2(g) = \int_G f_1(h) f_2(h^{-1}g)dh, \quad f^*(g) = \Delta(g)^{-1}\bar{f}(g^{-1}).$$

Exercise: Check that the involution is isometric and that $(f_1 * f_2)^* = f_2^* * f_1^*$.

(3.3) THEOREM: *There is a one-to-one correspondence between strongly continuous unitary representations of G and non-degenerate $*$-representations of the Banach algebra $L^1(G)$.*

PROOF: To keep matters simple let us assume first of all that G is discrete. (In this case the Haar measure on G is just counting measure, and we write $\ell^1(G)$ for $L^1(G)$.) Let $\pi\colon G \to U(H)$ be a unitary representation. We define a representation $\rho\colon L^1(G) \to U(H)$ by

$$\rho(\sum a_g[g]) = \sum a_g \pi(g);$$

the series on the right converges in norm. This representation has $\rho([e]) = 1$, so it is non-degenerate. Conversely, given a representation ρ as above, the map $\pi\colon G \to U(H)$ defined by

$$\pi(g) = \rho([g])$$

is a unitary representation of G.

In the general case we must replace summation by integration, handle the complications caused by the modular function, and use an approximate unit to deal with the fact that the 'Dirac masses' at points of G are no longer elements of $L^1(G)$. This is routine analysis, the details are on page 183 of Davidson's book. ■

The L^1-norm on $L^1(G)$ is *not* a C^*-norm in general. To manufacture a C^*-algebra related to the representation theory of G, we should form a *completion* of $L^1(G)$ in an appropriate C^*-norm. There are several different ways to form such a completion (as will become apparent, this kind of issue turns up many times in C^*-algebra theory.)

(3.4) DEFINITION: *The* maximal *C^*-algebra of G is the enveloping C^*-algebra of the Banach $*$-algebra $L^1(G)$.*

Here is the explanation of the term 'enveloping C^*-algebra'. As we saw in Remark 2.13, the GNS construction can be applied in any Banach $*$-algebra (having a norm-bounded approximate unit) and it shows that each state of such an algebra (continuous positive linear functional of norm one) gives rise to a cyclic Hilbert space representation. In particular there exists a *universal* representation $\rho_A \colon A \to \mathfrak{B}(H_A)$ for each Banach $*$-algebra A, obtained as the direct sum of the GNS representations coming from all the states; and the C^*-subalgebra of $\mathfrak{B}(H_A)$ generated by $\rho_A[A]$ is called the *enveloping* C^*-algebra of A. It has the universal property that any $*$-homomorphism from A to a C^*-algebra B factors uniquely through the enveloping C^*-algebra. In particular, then, there is a 1:1 correspondence between unitary representations of G and non-degenerate representations of $C^*(G)$. We usually denote $\widehat{C^*(G)}$ by $\widehat{G}$, regarding it as the space of irreducible unitary representations of G — the 'unitary dual'.

There is a canonical $*$-homomorphism from any Banach $*$-algebra A to its C^*-envelope. In general this $*$-homomorphism need not be injective (think about the disk algebra again!); it will be injective precisely when A has a faithful Hilbert space representation. To see that this is so in the case of $A = L^1(G)$ we consider once again the left regular representation λ of $L^1(G)$ on $L^2(G)$:

$$(\lambda(f_1)f_2)(g) = \int_G f_1(h)f_2(h^{-1}g)dh.$$

It is an easy measure-theoretic fact that this representation is faithful. Thus the $*$-homomorphism $L^1(G) \to C^*(G)$ is injective.

The special rôle of the regular representation in the above argument prompts us to define another 'group C^*-algebra'. The *reduced* C^*-algebra $C^*_r(G)$ of G is the C^*-subalgebra of $\mathfrak{B}(L^2(G))$ generated by the image of the left regular representation. Thus we have a surjective $*$-homomorphism

$$C^*(G) \to C^*_r(G)$$

coming from λ via the universal property. Dually, the spectrum of $C^*_r(G)$ is identified with a closed subspace of the spectrum $\widehat{G}$ of $C^*(G)$; this is the *reduced dual* $\widehat{G}_r$ of G.

Of course we want to know what all these weird spaces are and whether there is any difference between them. Let us first consider the case of *abelian* groups.

(3.5) LEMMA: *Let G be a locally compact abelian group. Then its unitary dual $\widehat{G}$ is equal to its* Pontrjagin dual $\mathrm{Hom}(G, \mathbb{T})$*, the space of* characters *of G, that is continuous homomorphisms of G into the circle group. The topology on $\widehat{G}$ is the restriction of the weak-$*$ topology on $L^\infty(G)$; that is, a net α_λ of characters* *converges to the character α if and only if*

$$\int_G \alpha_\lambda(g) f(g) dg \to \int_G \alpha(g) f(g) dg$$

for all $f \in L^1(G)$. If G is discrete, therefore, the topology on $\widehat{G}$ is the topology of pointwise convergence.

(if G isn't discrete $\widehat{G}$ isn't compact — here my insistence on considering only representations of unital C^*-algebras is coming back to haunt me.)

PROOF: Since G is commutative, so is $C^*(G)$. Thus by the theory of commutative C^*-algebras, $C^*(G) = C_0(\widehat{G})$, and $\widehat{G}$ is the space of $*$-homomorphisms from $C^*(G) \to \mathbb{C}$, that is one-dimensional representations of $C^*(G)$. By the discussion above, these are the same thing as one-dimensional unitary representations of G, that is characters. The topologies match up since (by construction) states on $C^*(G)$ are all obtained by extending states on $L^1(G)$, which themselves are given by (positive) L^∞ functions on G. ■

(3.6) LEMMA: *Let G be a locally compact abelian group. Then $C^*(G) = C^*_r(G) = C_0(\widehat{G})$.*

PROOF: All we need to do is to show that $C^*(G) = C^*_r(G)$, and for this it will be enough to show that every character α of G extends to a continuous linear map $C^*_r(G) \to \mathbb{C}$.

Let us first establish the following: pointwise multiplication by a character of G preserves the norm on the regular representation. In other words, if $f \in L^1(G)$ and α is a character, then

$$\|\lambda(\alpha f)\| = \|\lambda(f)\|.$$

To see this write

$$\lambda(\alpha f) = U_\alpha \lambda(f) U^*_\alpha$$

where $U_\alpha \colon L^2(G) \to L^2(G)$ is the unitary operator of pointwise multiplication by α.

It follows that if α is any character of G, and β is a character which extends continuously to $C^*_r(G)$ (that is, $\beta \in \widehat{G}_r$), then the pointwise product $\alpha\beta$ belongs to $\widehat{G}_r$ also. Since $\widehat{G}$ is a group under pointwise product, and $\widehat{G}_r$ is nonempty (after all, $C^*_r(G)$ must have *some* characters), we deduce that $\widehat{G} = \widehat{G}_r$. ■

(3.7) LEMMA: *Let G be a discrete (or a locally compact) group. The commutant of the left regular representation of G on $L^2(G)$ is the von Neumann algebra generated by the right regular representation (and vice versa of course).*

PROOF: Obviously the left and right regular representations commute. The commutant $\mathcal{R} = \lambda[G]'$ of the left regular representation consists of those operators on $L^2(G)$ that are invariant under the left-translation action of G on $L^2(G)$, and the commutant $\mathcal{L} = \rho[G]'$ of the right regular representation consists of those operators that are invariant under the right-translation action. It will be enough to show that $\mathcal{L}$ and $\mathcal{R}$ commute, since then $\rho[G] \subseteq \mathcal{R} \subseteq (\mathcal{L})' = \rho[\mathcal{G}]''$.

So let $S \in \mathcal{L}$ and $T \in \mathcal{R}$. For simplicity, assume that G is discrete. Then (by translation invariance) S is completely determined by $S[e] = \sum s_x[x]$, and similarly T is completely determined by $T[e] = \sum t_y[y]$.

For group elements g and h we have

$$\langle TS[g], [h] \rangle = \langle S[g], T^*[h] \rangle = \left\langle \sum s_x[gx], \sum \bar{t}_{y^{-1}}[yh] \right\rangle = \sum s_x t_{hx^{-1}g^{-1}}.$$

By a similar calculation

$$\langle ST[g], [h] \rangle = \sum t_y s_{g^{-1}y^{-1}h}.$$

The substitution $x = g^{-1}y^{-1}h$ shows that these sums agree, and thus $TS = ST$. ■

3.2 Commutative Harmonic Analysis

We will use our results to discuss the bare bones of the Pontrjagin duality theory for abelian groups.

For starters let G be a discrete abelian group. As we saw above, $\widehat{G}$ is a compact abelian also, and $C^*(G) = C^*_r(G) = C(\widehat{G})$. The implied isomorphism $C^*_r(G) \to C(\widehat{G})$ is called the *Fourier transform*, and denote $\mathcal{F}$. For an element $f \in L^1(G)$ it can be defined by the formula

$$\mathcal{F}f(\alpha) = \int_G f(g)\alpha(g)dg$$

where α is a character of G. (Of course this integral is really a sum in the discrete case, but we use notation which is appropriate to the continuous case as well.) Notice that it is immediate that the Fourier transform converts convolution into multiplication.

(3.8) PLANCHEREL THEOREM *The Fourier transform extends to a unitary isomorphism of $L^2(G)$ with $L^2(\widehat{G})$ (for an appropriate choice of Haar measure).*

The phrase ‘for an appropriate choice of Haar measure’ conceals all the various powers of 2π which show up in the classical theory of Fourier analysis.

PROOF: The regular representation of $C^*_r(G)$ is a cyclic representation with cyclic vector $[e]$. The associated vector state

$$\tau(f) = \langle f * [e], [e] \rangle$$

corresponds to a measure on $\widehat{G}$ (by the Riesz representation theorem). Translation by a character α on $C(\widehat{G})$ corresponds to pointwise multiplication by α on $C_r^*(G)$; and since $\alpha(e) = 1$ we see that the measure corresponding to τ is translation invariant, so it is (up to a multiple) the Haar measure ν on $\widehat{G}$. By the uniqueness of the GNS construction, the regular representation of $C_r^*(G)$ is spatially equivalent to the multiplication representation of $C(\widehat{G})$ on $L^2(\widehat{G}, \nu)$, with cyclic vector equal to the constant function 1. In particular there is a unitary $U\colon L^2(G) \to L^2(\widehat{G})$ such that

$$U(f * h) = (\mathcal{F}f) \cdot (Uh)$$

for $f \in L^1(G)$, $h \in L^2(G)$. In particular, taking $h = [e]$ (which is the cyclic vector, so that $Uh = 1$), we see that $Uf = \mathcal{F}f$ for $f \in L^1$. Thus the unitary U is an extension of the Fourier transform, as asserted. ■

The Plancherel theorem extends to non-discrete locally compact groups too. Now the representing measure ν on $\widehat{G}$ is not finite, and we cannot appeal directly to the GNS theory. Instead we build the representing measure by hand, in the following way. For each $x \in L^1 \cap L^2(G)$ we define a measure ν_x by the GNS construction, so that

$$\int_{\widehat{G}} \mathcal{F}f d\nu_x = \langle f * x, x \rangle_{L^2(G)}.$$

If both x and y belong to $L^1 \cap L^2(G)$ then a simple argument using commutativity shows that

$$|\mathcal{F}x|^2 \nu_y = |\mathcal{F}y|^2 \nu_x$$

as measures on $\widehat{G}$. For any compact subset K of $\widehat{G}$ one can find an $x \in L^1 \cap L^2(G)$ such that $\mathcal{F}x$ is positive on K. Now we may define an unbounded measure ν on $\widehat{G}$ via the Riesz representation theorem as a positive linear functional on $C_c(\widehat{G})$: if $f \in C_c(\widehat{G})$ is supported within K we put

$$\int f d\nu = \int \frac{f}{|\mathcal{F}x|^2} d\nu_x,$$

for any x with $\mathcal{F}x > 0$ on K; and this is independent of the choice of x. Thus we obtain a measure ν on $\widehat{G}$.

Let u_λ be an approximate unit for $C_r^*(G)$; then the functions $\mathcal{F}u_\lambda$ increase pointwise to the constant function 1. For $g \in L^1 \cap L^2(G)$ we have

$$\|g\|^2 = \lim \langle u_\lambda * g, g \rangle = \lim \int \mathcal{F}u_\lambda d\nu_g = \int d\nu_g = \int |\mathcal{F}g|^2 d\nu.$$

Consequently, the Fourier transform extends to a unitary equivalence between $L^2(G)$ and $L^2(\widehat{G})$. A similar argument with approximate units shows that ν is a Haar measure on $\widehat{G}$.

The next step in the duality theory is to show that $\widehat{\widehat{G}} = G$. We won't do this here.

3.3 Positive Definite Functions and States

From now on we will restrict attention to discrete, but not necessarily abelian, groups. Let G be such a group. A function p on such a group is called *positive definite* if for every positive integer n and every n-tuple $(x_1, \ldots, x_n)$ of group elements, the $n \times n$ matrix whose entries are $p(x_j^{-1}x_i)$ is positive as a matrix; that is, for all $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ we have

$$\sum_{i,j=1}^{n} \alpha_i \bar{\alpha}_j p(x_j^{-1}x_i) \geqslant 0.$$

The collection of positive definite functions on G is denoted by $B_+(G)$. Let us observe that every positive definite function is bounded: since for every such function the matrix

$$\begin{bmatrix} p(e) & p(x) \\ p(x^{-1}) & p(e) \end{bmatrix}$$

is positive, which implies that $|p(x)| \leqslant p(1)$ for all x.

(3.9) LEMMA: *Let φ be a positive linear functional on $C^*(G)$. Then the mapping $p\colon G \to \mathbb{C}$ defined by $p(x) = \varphi([x])$ is positive definite; and every positive definite function on G arises in this way from a unique positive linear functional. The states of $C^*(G)$ correspond to the positive definite functions with $p(e) = 1$.*

PROOF: If φ is positive linear and we define p as suggested above, then

$$\sum_{i,j=1}^{n} \alpha_i \bar{\alpha}_j p(x_j^{-1}x_i) = \varphi(f^* * f) \geqslant 0$$

where $f = \sum_{i=1}^{n} \alpha_i [x_i]$. Thus p is positive definite. Conversely if p is positive definite and we define φ to be the linear extension of p to $\ell^1[G]$, then the same computation shows that φ is a positive linear functional on the Banach algebra $\ell^1[G]$. Every such functional is automatically continuous for the C^*-norm and so extends to a positive linear functional on $C^*[G]$. ■

We let $B_+(G)$ denote the space of positive definite functions on G, and we let $B(G)$ denote the vector space of complex-valued functions on G spanned by $B_+(G)$, that is, the set $\{p_1 - p_2 + i(p_3 - p_4)\}$ where $p_1, \ldots, p_4$ are positive definite functions. It is not hard to show (Exercise 6.22) that every self-adjoint linear functional on a C^*-algebra is the difference of two positive linear functionals and thus we find that $B(G)$ is isomorphic, as a vector space, to the dual of $C^*(G)$. We make $B(G)$ a Banach space by giving it the norm it acquires from $C^*(G)^*$ via this isomorphism. (Notice that the weak-$*$ topology on $C^*(G)^*$ corresponds to the topology of pointwise convergence on $B(G)$.)

(3.10) PROPOSITION: *The pointwise product of two positive definite functions is again positive definite. Consequently, $B(G)$ is a Banach algebra under pointwise multiplication.*

PROOF: To prove that the product of positive definite functions is again positive definite, it is necessary to prove that the pointwise product (Schur product) of positive matrices is positive. This is a standard result of matrix theory. We reproduce the proof. Let $A = (a_{ij})$ and $B = (b_{ij})$ be positive matrices. Their Schur product $C = (a_{ij}b_{ij})$. Now because B is positive, $B = X^*X$ for some matrix X; that is, $b_{ij} = \sum_k x_{ik}\bar{x}_{jk}$. It follows that for any vector $\boldsymbol{\xi}$,

$$\boldsymbol{\xi}^*C\boldsymbol{\xi} = \sum_k \sum_{i,j} a_{ij}(\bar{x}_{jk}\bar{\xi}_j)(x_{ik}\xi_i)$$

and for each k the inner sum is positive. Hence C is a positive matrix, as required.

To finish the proof it is necessary to show that $\|pq\| \leqslant \|p\|\|q\|$ for the norm of $B(G)$. This is obvious if p, q are positive definite because positive definite functions attain their norm at the identity element of G. The general case follows from this same fact if we decompose p and q as differences of positive definite functions. ■

Can every positive definite function on G be approximated (in some sense) by *compactly supported* positive definite functions? This turns out to be a delicate question, and its answer leads us to consider the relationship between the maximal C^*-algebra $C^*(G)$ and the reduced C^*-algebra $C^*_r(G)$ which is associated to the regular representation. To see why, suppose that $p(g) = \langle \lambda(g)v, v\rangle$ is the positive definite function associated to a vector state of the regular representation, $v \in \ell^2(G)$. Then we have

$$\|p\|_{B(G)} \leqslant \|v\|^2;$$

and therefore, since vectors v of compact (= finite) support are dense in $\ell^2(G)$, we find that every positive definite function associated to a vector state of the regular representation of G belongs to the closure (in $B(G)$) of the space of functions of finite support. Conversely we also have:

(3.11) LEMMA: *Any positive definite function of finite support is associated to a vector state of the regular representation.*

PROOF: Let p be such a function. The operator T of right convolution by p is bounded on $\ell^2(G)$, by finiteness of the support; and the fact that p is a positive definite function shows that T is a positive operator. Moreover, T commutes with the left regular representation $\lambda[G]$. Now put $v = T^{\frac{1}{2}}[e]$ (notice that v need not have finite support). Then for $g \in G$,

$$\langle \lambda(g)v, v\rangle = \langle \lambda(g)T^{\frac{1}{2}}[e], T^{\frac{1}{2}}[e]\rangle = \langle [g], T[e]\rangle = \langle [g], p\rangle = p(g)$$

as required. ■

Denote by $A_+(G)$ the space of positive definite functions on G of the form

$$p(g) = \sum_i \langle \lambda(g)v_i, v_i\rangle$$

where v_i is a sequence in $\ell^2(G)$ having $\sum \|v_i\|^2 < \infty$ (notice that this implies that the above sequence converges uniformly to a positive definite function.) More canonically we may write

$$p(g) = \mathrm{Trace}(\lambda(g)T)$$

where T is a positive trace-class operator on $\ell^2(G)$. We have $\|p\|_{B(G)} = \sum \|v_i\|^2 = \|T\|_1$, the trace norm of T, and it follows from the fact that the space of trace-class operators is a Banach space under the trace norm that $A_+(G)$ is a closed cone in $B_+(G)$. In fact the arguments above prove

(3.12) PROPOSITION: *$A_+(G)$ is the closure of $\mathbb{C}[G] \cap B_+(G)$ (the compactly supported functions on G) in $B_+(G)$.*

Indeed, Lemma 3.11 shows that every compactly supported element of $B_+(G)$ belongs to $A_+(G)$, and the remarks preceding the lemma show that every element of $A_+(G)$ can be approximated by compactly supported elements of $B_+(G)$.

We can also introduce the notation $A(G)$ for the subalgebra of $B(G)$ generated by $A_+(G)$. It is the closure of $\mathbb{C}[G]$ in $B(G)$. Elements of $A_(G)$ are functions of the form

$$p(g) = \sum \langle \lambda(g)v_i, w_i\rangle = \mathrm{Trace}(\lambda(g)T)$$

where $\sum \|v_i\|^2 < \infty$ and $\sum \|w_i\|^2 < \infty$ (the trace-class operator T need no longer be positive). In other words, the corresponding elements of $C^*(G)^*$ are precisely those that arise, via the regular representation, from the ultraweakly continuous linear functionals on $\mathfrak{B}(\ell^2(G))$.

(3.13) REMARK: The algebra $A(G)$ is called the *Fourier algebra* of G; $B(G)$ is the *Fourier–Stieltjes* algebra. To explain this terminology, consider the case $G = \mathbb{Z}$. The positive definite functions on $\mathbb{Z}$ are the Fourier transforms of positive measures on the circle, and thus the algebra $B(G)$ is just the algebra of Fourier transforms of measures on the circle, classically known as the Stieltjes algebra. On the other hand the algebra $A(G)$ is generated by sequences of the form

$$n \mapsto \int_0^{2\pi} e^{int} u(t)\bar{v}(t)\, dt$$

where $u, v \in L^2(\mathbb{T})$. This is the sequence of Fourier coefficients of the L^1 function $u\bar{v}$, and the collection of products of this sort has dense span in L^1, so we see that $A(\mathbb{Z})$ is just the algebra of Fourier transforms of L^1 functions on the circle.

(3.14) LEMMA: *Let G be a discrete group. The weak-$*$ topology on a bounded subset of $C^*(G)^*$ is equivalent to the topology of pointwise convergence on the corresponding bounded subset of $B(G)$.*

PROOF: It is apparent that weak-$*$ convergence in the dual of $C^*(G)$ implies pointwise convergence in $B(G)$. Conversely suppose that p_λ is a bounded net in $B(G)$ converging pointwise to p. Then the corresponding functionals

$$\varphi_\lambda\bigl(\sum a_g[g]\bigr) = \sum a_g p_\lambda(g)$$

converge to

$$\varphi\bigl(\sum a_g[g]\bigr) = \sum a_g p(g)$$

for all $a = \sum a_g[g]$ belonging to $\ell^1(G)$. Since the $\{\varphi_\lambda\}$ are uniformly bounded and $\ell^1(G)$ is norm dense in $C^*(G)$, it follows that $\varphi_\lambda(a) \to \varphi(a)$ for all $a \in C^*(G)$. ■

(3.15) THEOREM: *Let G be a countable discrete group. Then the following are equivalent:*

(a) $C^*_r(G) = C^*(G)$,

(b) The Banach algebra $A(G)$ possesses a bounded approximate unit,

(c) Every function in $B(G)$ can be pointwise approximated by a norm bounded sequence of compactly supported functions.

PROOF: Suppose (c). Then (by lemma 3.14) each state σ of $C^*(G)$ is a weak-$*$ limit of vector states $x \mapsto \langle \lambda(x)v, v \rangle$, where v is a unit vector in the regular representation. It follows that for each such state σ,

$$\|\sigma(a)\| \leqslant \|\lambda(a)\|$$

for all $a \in C^*(G)$; and consequently $\|a\|_{C^*(G)} \leqslant \|\lambda(a)\|$, whence $C^*(G) = C_r^*(G)$, which is (a).

Now suppose (a), that $C^*(G) = C_r^*(G)$. Let $a \in C^*(G)$ have $\varphi(a) = 0$ for each ultraweakly continuous linear functional φ on $\mathfrak{B}(\ell^2(G))$. Then $\lambda(a) = 0$ and thus $a = 0$ since λ is faithful on $C^*(G)$. It follows that the space of ultraweakly continuous functionals on the regular representation is weak-$*$ dense in $C^*(G)^*$. Using lemma 3.14, this implies that $A_+(G)$ is dense in $B_+(G)$ in the topology of pointwise convergence. Now the constant function 1 belongs to $B_+(G)$ (it corresponds to the trivial representation of G) and thus there is a net in $A_+(G)$ converging pointwise to 1. Since $A(G)$ is generated by compactly supported functions, such a net is necessarily an approximate unit for $A(G)$, and it is bounded because the norm in $A(G)$ is given by evaluation at e. This proves (b).

Finally, suppose (b). Let f_λ be an approximate unit for $A(G)$. We may assume without loss of generality that f_λ consists of compactly supported functions. Let $f \in B(G)$; then ff_λ is a net in $A(G)$ and it converges pointwise to f. ■

To connect this discussion with the geometry of G we introduce the notion of *amenability*.

3.4 Amenability

Let G be a discrete group. An *invariant mean* on G is a state $\mathfrak{m}$ on the commutative C^*-algebra $\ell^\infty(G)$ which is invariant under left translation by G: that is $\mathfrak{m}(L_g f) = \mathfrak{m}(f)$, where $L_g f(x) = f(g^{-1}x)$.

(3.16) DEFINITION: *A group G that possesses an invariant mean is called* amenable.

EXAMPLE: The group $\mathbb{Z}$ is amenable. Indeed, consider the states of $\ell^\infty(\mathbb{Z})$ defined by

$$\sigma_n(f) = \frac{1}{2n+1} \sum_{j=-n}^{n} f(j),$$

and let σ be a weak-$*$ limit point of these states (which exists because of the compactness of the state space). Since for each $k \in \mathbb{Z}$, and each f,

$$|\sigma_n(L_k f - f)| \leqslant \frac{2k}{2n+1}\|f\|$$

tends to zero as $n \to \infty$, we deduce that σ is an invariant mean.

EXAMPLE: Generalizing the construction above, suppose that G is a group and that $\{G_n\}$ is a sequence of finite subsets having the property that for each fixed $h \in G$,

$$\frac{\# G_n \triangle hG_n}{\# G_n} \to 0$$

as $n \to \infty$. (Such a sequence is called a Følner sequence.) Then any weak-$*$ limit point of the sequence of states

$$f \mapsto \frac{1}{\# G_n} \sum_{g \in G_n} f(g)$$

is an invariant mean on G. It can be proved that every amenable group admits a Følner sequence, but we will not do this here.

EXAMPLE: The free group on two or more generators is *not* amenable. To see this, denote the generators of the free group G by x and y, and define a bounded function f_x on the group G as follows: $f_x(g)$ is equal to 1 if the reduced word for g begins with the letter x, and otherwise it is zero. Notice then that $f_x(xg) - f_x(g) \geqslant 0$, and that $f_x(xg) - f_x(g) = 1$ if the reduced word for g begins with y or y^{-1}. Define f_y similarly; then we have

$$[f_x(xg) - f_x(g)] + [f_y(yg) - f_y(g)] \geqslant 1$$

for all $g \in G$. But this contradicts the hypothesis of amenability: an invariant mean $\mathfrak{m}$ would assign zero to both terms on the left-hand side, yet would assign 1 to the term on the right.

There are numerous equivalent characterizations of amenability. For our purposes the most relevant one is the following.

(3.17) PROPOSITION: *A countable discrete group G is amenable if and only if there is a sequence $\{v_n\}$ of unit vectors in $\ell^2(G)$ such that the associated positive definite functions*

$$g \mapsto \langle \lambda(g) v_n, v_n \rangle$$

tend pointwise to 1.

PROOF: Suppose that we can find such a sequence $\{v_n\}$. Since v_n and $\lambda(g)v_n$ are unit vectors we may write

$$\|v_n - \lambda(g)v_n\|^2 = 2 - 2\Re\langle \lambda(g)v_n, v_n\rangle \to 0$$

so that, for each fixed g, $\lambda(g)v_n - v_n \to 0$ as $n \to \infty$. Now define states σ_n on $\ell^\infty(G)$ by

$$\sigma_n(f)\langle M_f v_n, v_n\rangle$$

where M_f is the multiplication operator on ℓ^2 associated to f, and let σ be a weak-$*$ limit point of the states $\{\sigma_n\}$. Since

$$|\sigma_n(L_g f - f)| \leqslant \|f\|\|v_n - \lambda(g)v_n\|$$

we see that $\sigma(L_g f - f) = 0$, that is, σ is an invariant mean.

Conversely, suppose that G is amenable and let $\mathfrak{m}$ be an invariant mean. Then $\mathfrak{m}$ is an element of the unit ball of $\ell^\infty(G)^* = \ell^1(G)^{**}$. A theorem of functional analysis (Goldstine's Theorem — a consequence of the Hahn-Banach theorem) states that for any Banach space E, the unit ball of E is weak-$*$ dense in the unit ball of E^{**}. Thus we can find a net $\{\varphi_\lambda\}$ in the unit ball of $\ell^1(G)$ that converges weak-$*$ to the invariant mean $\mathfrak{m}$. We have

$$\sum |\varphi_\lambda(g)| = 1, \quad \sum \varphi_\lambda(g) \leqslant 1;$$

and these facts together show that we may assume without loss of generality that each φ_λ is a positive function of norm one. For each $g \in G$ we have $L_g\varphi_\lambda - \varphi_\lambda \to 0$ weakly in $\ell^1(G)$ as $\lambda \to \infty$. Now we use the fact (a consequence of the Hahn-Banach Theorem) that the weak and norm topologies on ℓ^1 have the same closed convex sets to produce a sequence $\{\psi_1, \psi_2, \ldots\}$ of convex combinations of the $\{\varphi_\lambda\}$ such that for each $g \in G$, $L_g\psi_n - \psi_n \to 0$ in the norm topology $\ell^1(G)$. Each ψ_n is a positive function of ℓ^1-norm equal to one. Take v_n to be the pointwise square root of ψ_n. ∎

Now we can finally prove

(3.18) THEOREM: *A discrete group G is amenable if and only if $C^*_r(G) = C^*(G)$.*

PROOF: Suppose that G is amenable. Then by Proposition 3.17 there is a sequence $\{p_n\}$ of positive definite functions on G of norm one , associated to vector states of the regular representation, that tend pointwise to 1. These functions p_n constitute a bounded approximate unit for $A(G)$, so by Theorem 3.15, $C^*(G) = C^*_r(G)$.

Conversely suppose that $C^*(G) = C^*_r(G)$. Then $A(G)$ has a bounded approximate unit, which we may assume without loss of generality is made up of compactly supported positive definite functions of norm one. By Lemma 3.11, such functions are associated to vector states of the regular representation. By Proposition 3.17, G is amenable. ∎

3.5 Example: A crystallographic group

In the next few sections, following Davidson, we shall look in detail at the C^*-algebras of some discrete groups. Our first example is a *crystallographic group* — the group G of isometries of the plane generated by the translation $\tau_{0,1}\colon (x,y)\mapsto (x,y+1)$ together with the glide reflection $\sigma\colon (x,y)\mapsto (x+\frac{1}{2},-y)$. One verifies easily that σ^2 is the translation $\tau_{1,0}$ and that the translations

$$\tau_{m,n}\colon (x,y)\mapsto (x+m,y+n)$$

form a free abelian subgroup of G of index two. Thus G fits into a short exact sequence

$$0\to\mathbb{Z}^2\to G\to\mathbb{Z}/2\to 0.$$

A discussion analogous to the present one can be given for any extension of a free abelian group by a finite group. Notice that G is amenable so $C^*(G)=C^*_r(G)$.

We partition G as the union of two cosets of $A=\mathbb{Z}^2$: $G=A\sqcup A\sigma$. Thus $\ell^2(G)=\ell^2(A)\oplus\ell^2(A)=L^2(\mathbb{T}^2)\oplus L^2(\mathbb{T}^2)$ by Fourier analysis. The elements of A act diagonally with respect to this decomposition so that the subalgebra generated by A in $C^*_r(G)$ is equal to $C^*_r(\mathbb{Z}^n)=C(\mathbb{T}^2)$ acting diagonally: it consists of matrices $\left(\begin{smallmatrix} f & 0\\ 0 & f\end{smallmatrix}\right)$ where $f\in C(\mathbb{T}^2)$.

The relations $\tau_{m,n}\sigma=\sigma\tau_{m,-n}$, $\sigma^2=\tau_{1,0}$ show that the action of σ in this representation is by the matrix $\left(\begin{smallmatrix} 0 & zJ\\ J & 0\end{smallmatrix}\right)$, where J is induced by the automorphism $(z,w)\mapsto(z,\bar w)$ of the torus $\mathbb{T}^2$. It is convenient to adjust this representation slightly by applying J to the second $L^2(\mathbb{T}^2)$ summand, that is by conjugating by the matrix $\left(\begin{smallmatrix} 1 & 0\\ 0 & J\end{smallmatrix}\right)$. Then the actions of the group generators are given in the following table:

$$\sigma\mapsto\begin{pmatrix} 0 & z\\ 1 & 0\end{pmatrix},\quad \tau_{1,0}\mapsto\begin{pmatrix} z & 0\\ 0 & z\end{pmatrix},\quad \tau_{0,1}\mapsto\begin{pmatrix} w & 0\\ 0 & \bar w\end{pmatrix}.$$

It follows that $C^*(G)=C^*_r(G)$ may be identified with the C^*-algebra of two by two matrices of the form

$$\begin{pmatrix} f(z,w) & zg(z,\bar w)\\ g(z,w) & f(z,\bar w)\end{pmatrix}\tag{3.19}$$

where f and g are continuous functions on $\mathbb{T}^2$.

A function of this kind is completely determined by its values on the cylinder (half-torus) consisting of those points (z,w) where $\Im w\geqslant 0$. We may therefore write $w=e^{i\pi t}$ with $0\leqslant t\leqslant 1$ and identify $C^*(G)$ with the algebra of functions $\mathbb{T}\times[0,1]\to M_2(\mathbb{C})$ generated by

$$\sigma\mapsto\begin{pmatrix} 0 & z\\ 1 & 0\end{pmatrix},\quad \tau_{1,0}\mapsto\begin{pmatrix} z & 0\\ 0 & z\end{pmatrix},\quad \tau_{0,1}\mapsto\begin{pmatrix} e^{i\pi t} & 0\\ 0 & e^{-i\pi t}\end{pmatrix}.$$

By the nature of the construction, we obtain a two-dimensional representation $V_{z,t}$ of $C^*(G)$ by fixing $z \in \mathbb{T}$ and $t \in [0,1]$.

(3.20) PROPOSITION: *For $t \in (0,1)$ the representation $V_{z,t}$ is irreducible. For $t = 0$ and $t = 1$, $V_{z,t}$ splits as a direct sum $W_{t,u} \oplus W_{t,-u}$, where $W_{t,u}$ is the one-dimensional representation on which $\tau_{0,1}$ acts as $e^{i\pi t} \in \{\pm 1\}$, and σ acts as $u \in \mathbb{T}$ with $u^2 = z$.*

PROOF: We look at the commutant. If $t \in (0,1)$ then $\tau_{0,1}$ acts as a diagonal matrix with distinct eigenvalues; the only things that commute with such a matrix are other diagonal operators. And one easily checks that the only diagonal operators that commute with the matrix $\left(\begin{smallmatrix} 0 & z \\ 1 & 0 \end{smallmatrix}\right)$ representing σ are multiples of the identity. Thus the representation is irreducible in this case. If $t = 0$ then $\tau_{0,1}$ acts as the identity. The projection $\frac{1}{2}\left(\begin{smallmatrix} 1 & u \\ \bar{u} & 1 \end{smallmatrix}\right)$ commutes with the action of σ and thus with the whole representation; it decomposes it into the two one-dimensional representations mentioned above. Similarly for $t = 1$. ■

(3.21) PROPOSITION: *The only irreducible representations of $C^*(G)$ are those mentioned above.*

PROOF: Left to the reader. Here is the start of the proof. Since $\sigma^2 = \tau_{1,0}$ is in the center of G, it must act as a scalar zI in any irreducible representation, with $z \in \mathbb{T}$ by unitarity. Since σ itself acts unitarily it is not hard to deduce that it must act either as a scalar uI, $u^2 = z$, or as $\left(\begin{smallmatrix} 0 & z \\ 1 & 0 \end{smallmatrix}\right)$ with respect to some orthogonal direct sum decomposition. In the second case, use the relation $\sigma^{-1}\tau_{0,1}\sigma\tau_{0,1} = e$ to deduce that $\tau_{0,1}$ acts as a diagonal matrix $\left(\begin{smallmatrix} w & 0 \\ 0 & \bar{w} \end{smallmatrix}\right)$. Continue from here… ■

The *character* of a finite-dimensional representation π of a group G is the class function $g \mapsto \operatorname{tr}\pi(g)$ on G; it is invariant under isomorphism of representations. The character $\chi_{z,t}$ of $V_{z,t}$ sends $\tau_{1,0}$ to $2z$ and $\tau_{0,1}$ to $2\cos\pi t$. Thus the representations $V_{z,t}$ are mutually non-isomorphic.

(3.22) COROLLARY: *The spectrum of the algebra $A = C^*_r(G)$ is equal to its primitive ideal space. It is a non-Hausdorff space consists of an open cylinder $T \times (0,1)$ together with two copies of $\mathbb{T}$ attached two-to-one onto the ends of the cylinder.*

The famous *Kadison–Kaplansky Conjecture* is this: if G is a torsion-free group then $C^*_r(G)$ contains no non-trivial idempotent. We can verify this explicitly for our crystallographic group G. Indeed, a non-trivial idempotent in $C^*_r(G)$ must be a function of the form 3.19 whose value is a rank-one projection. Look at the boundary circle given by $t = 0$, $w = 1$ in this parameterization. The only projections of this sort are

$$\frac{1}{2}\begin{pmatrix} 1 & u \\ u^{-1} & 1 \end{pmatrix}$$

where $u^2 = z$; and simple topology shows that there is no *continuous* determination of such a projection around the whole circle $\mathbb{T} = \{z : |z| = 1\}$.

3.6 The reduced C^*-algebra of the free group

Let G be a discrete group. The *canonical trace* on $C^*_r(G))$ is the vector state τ associated with $[e] \in \ell^2(G)$; that is,

$$\tau(a) = \langle \lambda(a)[e], [e] \rangle.$$

The associated positive definite function is equal to 1 at the identity and 0 at every other group element. Notice that we may also write

$$\tau(a) = \langle \lambda(a)[g], [g] \rangle$$

for any fixed $g \in G$ (the right-hand side agrees with τ on group elements, and it is continuous, so it agrees with τ everywhere. Otherwise stated, $\tau([g]^* a[g]) = \tau(a)$, so $\tau(a[g]) = \tau([g]a)$ and by linear extension we deduce

(3.23) PROPOSITION: *The state τ is a* trace*, that is,* $\tau(aa') = \tau(a'a)$ *for every* $a, a' \in C^*_r(G)$.

The trace τ is *faithful*, meaning that if $a \in C^*_r(G)$ is positive and $\tau(a) = 0$, then $a = 0$. Indeed, this is a consequence of the fact that the regular representation (a faithful representation) is the GNS representation associated to τ. To prove it from first principles, use the Cauchy–Schwarz inequality to estimate the matrix coefficients of $\lambda(a)$:

$$|\langle \lambda(a)[g], [g'] \rangle| \leqslant \langle \lambda(a)g, g \rangle^{\frac{1}{2}} \langle \lambda(a)g', g' \rangle^{\frac{1}{2}} = \tau(a).$$

We are going to use the faithful trace τ as a key tool in analyzing the reduced group C^*-algebra $C^*_r(F_2)$. Notice that since the free group is non-amenable, this is different from the full C^*-algebra $C^*(F_2)$. We are going to distinguish these by C^*-algebraic properties. Specifically, we will prove that $C^*_r(F_2)$ is *simple*, which means that it has no non-trivial ideals. This is very, very different from $C^*(F_2)$: let G be any finite group with two generators, then according to the universal property of $C^*(F_2)$ there is a surjective $*$-homomorphism $C^*(F_2) \to C^*(G)$, and of course the kernel of this $*$-homomorphism is a non-trivial ideal.

Here is the key lemma. Denote by u, v the unitaries in $C^*r(F_2)$ corresponding to the generators x, y of the group.

(3.24) LEMMA: *Let $a \in C_r^*(F_2)$. Then the limit*

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} u^i v^j a v^{-j} u^{-i}$$

exists in the norm of $C_r^*(F_2)$ and equals $\tau(a)1$.

Before proving this lemma let's see how we can apply it.

(3.25) THEOREM: *The C^*-algebra $C_r^*(F_2)$ has no non-trivial ideals (it is simple).*

(3.26) REMARK: As we earlier pointed out (corollary 1.5), maximal ideals in C^*-algebras are closed. Thus it makes no difference whether we say *topologically simple* (no nontrivial closed ideals) or *algebraically simple* (no nontrivial ideals, closed or not).

PROOF: Let J be a nonzero ideal and let $a \in J$ be positive and nonzero. Then the norm limit

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} u^i v^j a v^{-j} u^{-i}$$

belongs to J. But this limit equals $\tau(a)1$ by Lemma 3.24, and $\tau(a) \neq 0$ by the faithfulness of the trace. Consequently $1 \in J$ so J is the whole C^*-algebra. ■

(3.27) THEOREM: *The trace on $C_r^*(F_2)$ is unique.*

PROOF: If τ' is another trace, apply it to the left side of the formula in Lemma 3.24 to find that $\tau'(a) = \tau(a)\tau'(1) = \tau(a)$. ■

Let us now begin the proof of Lemma 3.24. The basic observation that is needed is this: if Hilbert space operators A and B have orthogonal ranges, then $\|A+B\|^2 \leqslant \|A\|^2 + \|B\|^2$. We are going to argue then that if $a = [g]$, $g \neq e$, then the ranges of the terms in the sum

$$\sum_{i=1}^{m} \sum_{j=1}^{n} u^i v^j a v^{-j} u^{-i}$$

are 'essentially' orthogonal, so that the norm of the sum is of order $O((mn)^{\frac{1}{2}})$, and thus becomes zero in the limit after we divide by mn.

Here is a more precise statement of the idea in the form that is needed.

(3.28) LEMMA: *Let $H = H_1 \oplus H_2$ be a Hilbert space equipped with an orthogonal direct sum decomposition. Let $T \in \mathfrak{B}(H)$ be an operator mapping H_2 into H_1,* *and let $U_1, \dots, U_n \in \mathfrak{B}(H)$ be unitary operators such that all the compositions $U_j^* U_i$, $j \neq i$, map H_1 into H_2. Then*

$$\Big\|\sum_{i=1}^{n} U_i T U_i^*\Big\| \leqslant 2\sqrt{n}\|T\|.$$

PROOF: Start with a special case: assume that T maps H (and not just H_2) into H_1. Write

$$\Big\|\sum_{i=1}^{n} U_i T U_i^*\Big\|^2 = \Big\|T + \sum_{i=2}^{n} U_1^* U_i T U_i^* U_1\Big\|^2.$$

The first term T in the sum above has range contained in H_1, whereas the remaining sum has range contained in H_2 (since $U_1^* U_i$ maps H_1 into H_2. Thus the ranges of the two terms are orthogonal and we get

$$\Big\|\sum_{i=1}^{n} U_i T U_i^*\Big\| \leqslant \|T\|^2 + \Big\|\sum_{i=2}^{n} U_i T U_i^*\Big\|^2.$$

By induction we get

$$\Big\|\sum_{i=1}^{n} U_i T U_i^*\Big\|^2 \leqslant n\|T\|^2$$

giving the result (without the factor 2) in the special case. The general case follows since any operator T that maps H_2 into H_1 can be written as $T = T' + (T'')^*$, where $\|T'\| \leqslant \|T\|$, $\|T''\| \leqslant \|T\|$, and both T' and T'' map H into H_1 (to see this, put $T' = TP_2$ and $T'' = P_1 T^*$, where P_1 and P_2 are the orthogonal projections of H onto H_1 and H_2 respectively. ■

Now we will use this to prove Lemma 3.24. It will be enough to prove that if $a = [g]$, then the average $\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} u^i v^j a v^{-j} u^{-i}$ converges in norm to 1 if $g = e$ and to 0 otherwise. For simplicity work with one sum at a time, so consider the expression

$$S_n(a) = \frac{1}{n}\sum_{j=1}^{n} v^j a v^{-j}.$$

Clearly, $S_n([g]) = [g]$ if g is a power of y. Otherwise, I claim, $\|S_n([g])\| \leqslant 2n^{-\frac{1}{2}}$. To see this decompose the Hilbert space $H = \ell^2(G)$ as $H_1 \oplus H_2$, where H_1 is the subspace of ℓ^2 spanned by those free group elements represented by reduced words that begin with a non-zero power of x (positive or negative), and similarly H_2 is spanned by those free group elements represented by reduced words that begin with a non-zero power of y (positive or negative), together with the identity. Suppose that g is not a power of y. Then the reduced word for g contains at least one x and by multiplying on the left and right by suitable powers of y (an operation which commutes with S_n and does not change the norm) we may assume that g begins and ends with a power of x. Now we apply Lemma 3.28 with T the operation of left multiplication by g and U_i the operation of left multiplication by v^i. Once we have checked that these elements indeed map H_2 and H_1 in the way that Lemma 3.28 applies, the result follows immediately.

To complete the proof of Lemma 3.24 all we need to do is to reiterate the above argument to show that

$$\|\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} u^i v^j a v^{-j} u^{-i}\| \leqslant \frac{4}{\min\{\sqrt{m},\sqrt{n}\}}$$

whenever $a = [g]$ and g is not the identity.

3.7 The Kadison–Kaplansky conjecture for F_2

In this section we are going to prove that the reduced C^*-algebra $C^*_r(G)$ for $G = F_2$ has no non-trivial projections. Thus $C^*_r(G)$ is an example of a simple, unital C^*-algebra with no non-trivial projections. Such examples had long been sought in C^*-algebra theory. This is not the first such example (that was constructed by Blackadar in the seventies using the theory of AF algebras), but it is the ‘most natural’.

We begin with some abstract ideas which lie at the root of K-homology theory for C^*-algebras. They are developments of a basic notion of Atiyah (1970) to provide a ‘function-analytic’ setting for the theory of elliptic pseudodifferential operators.

(3.29) DEFINITION: *Let A be a C^*-algebra. A* Fredholm module M *over A consists of the following data: two representations ρ_0, ρ_1 of A on Hilbert spaces H_0 and H_1, together with a unitary operator $U\colon H_0 \to H_1$ that ‘almost intertwines’ the representations, in the sense that $U\rho_0(a) - \rho_1(a)U$ is a compact operator from H_0 to H_1 for all $a \in A$.*

It is often convenient to summarize the information contained in a Fredholm module in a ‘supersymmetric’ format: we form $H = H_0 \oplus H_1$, which we consider as a ‘super’ Hilbert space (that is, the direct sum decomposition, which may be represented by the matrix $\gamma = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, is part of the data), we let $\rho = \rho_0 \oplus \rho_1$, and we put

$$F = \begin{pmatrix} 0 & U^* \\ U & 0 \end{pmatrix},$$

which is a self-adjoint operator, odd with respect to the grading (that is, anticommuting with γ), having $F^2 = 1$, and commuting modulo compacts with $\rho(a)$ for each $a \in A$. Thus we may say that a Fredholm module is given by a triple $M = (\rho, H, F)$.

(3.30) LEMMA: *Suppose that a Fredholm module over the C^*-algebra A is given. Then for each projection $e \in A$ the operator $U_e = \rho_1(e)U\rho_0(e)$ is Fredholm, when considered as an operator from the Hilbert space $\rho_0(e)H_0$ to the Hilbert space $\rho_1(e)H_1$.*

PROOF: We have

$$U_eU_e^* = \rho_1(e)U\rho_0(e)U^*\rho_1(e) \sim \rho_1(e)UU^*\rho_1(e) = \rho_1(e)$$

where we have used $\sim$ to denote equality modulo the compact operators. Similarly $U_e^*U_e \sim \rho_0(e)$. Thus U_e is unitary modulo the compacts. ■

It follows that each Fredholm module M on A gives rise to an integer-valued invariant

$$\mathrm{Index}_M \colon \mathrm{Proj}(A) \to \mathbb{Z}$$

defined on the space of projections in A.

Let $M = (\rho, H, F)$ be a Fredholm module over a C^*-algebra A. Its *domain of summability* is the subset

$$(A) = \{a \in A : [F, \rho(a)] \text{ is of trace class}\}$$

of A (here $[x, y]$ denotes the commutator $xy - yx$). One sees easily that $\mathcal{A}$ is a $*$-subalgebra of A. In fact, it is a Banach algebra under the norm

$$\|a\|_1 = \|a\| + \mathrm{Trace}\,|\rho(a)|$$

which is in general stronger than the norm of A itself.

(3.31) LEMMA: *Let M be a Fredholm module over A, with domain of summability $\mathcal{A}$. The linear functional*

$$\tau_M(a) = \tfrac{1}{2}\,\mathrm{Trace}(\gamma F[F, a])$$

is a trace[11] on $\mathcal{A}$.

[11] That is to say $\tau(ab) = \tau(ba)$ for $a, b \in \mathcal{A}$.

PROOF: We write (suppressing for simplicity explicit mention of the representation ρ)

$$\begin{aligned}
\tau_M(ab) &= \tfrac{1}{2}\operatorname{Trace}(\gamma F a[F,b] + \gamma F[F,a]b) \\
&= \tfrac{1}{2}\operatorname{Trace}(\gamma F a[F,b] - \gamma[F,a]Fb) \\
&= \tfrac{1}{2}\operatorname{Trace}(\gamma F a[F,b] - Fb\gamma[F,a]) \\
&= \tfrac{1}{2}\operatorname{Trace}(\gamma F a[F,b] + \gamma Fb[F,a])
\end{aligned}$$

and the last expression is now manifestly symmetric in a and b. (We used the identity $F[F,a] + [F,a]F = [F^2,a] = 0$, the symmetry property of the trace, and the fact that γ anticommutes with F and commutes with b.) ■

We have now used our Fredholm module M to construct an integer-valued index invariant of projections in A, and a complex-valued trace invariant of general elements of $\mathcal{A}$. These invariants agree where both are defined:

(3.32) PROPOSITION: *Let M be a Fredholm module over a C^*-algebra A and let e be a projection in the domain of summability $\mathcal{A}$ of M. Then $\tau_M(e) = -\operatorname{Index}_M(e)$. In particular, $\tau_M(e)$ is an integer.*

PROOF: We will need to make use of a standard index formula for Fredholm operators, namely the following: suppose that P is a Fredholm operator and that Q is a 'parametrix' or approximate index for P which has the property that $1 - QP$ and $1 - PQ$ are trace-class operators. Then

$$\operatorname{Index}(P) = \operatorname{Trace}(1 - QP) - \operatorname{Trace}(1 - PQ).$$

One can prove this formula by first showing that any two choices for the parametrix Q differ by a trace-class operator, next using the symmetry property of the trace to show that the right-hand side does not depend on the choice of Q, and finally manufacturing an explicit Q for which $1 - QP$ is the orthogonal projection onto the kernel of P and $1 - PQ$ is the orthogonal projection onto the kernel of P^*.

Let us grant this index formula. Use the computation in the proof of the preceding lemma to write

$$\tau_M(e) = \tau_M(e^2) = \operatorname{Trace}(\gamma F\rho(e)[F,\rho(e)]).$$

Putting

$$\rho(e) = \begin{pmatrix} e_0 & 0 \\ 0 & e_1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & U^* \\ U & 0 \end{pmatrix}$$

we get

$$[F,\rho(e)]F = F\rho(e)F - \rho(e) = \begin{pmatrix} U^*e_1U - e_0 & 0 \\ 0 & Ue_0U^* - e_1 \end{pmatrix}.$$

Thus

$$\tau_M(e) = \mathrm{Trace}(\gamma\rho(e)[F,\rho(e)]F\rho(e)) = \mathrm{Trace}\begin{pmatrix} U_e^*U_e - e_0 & 0 \\ 0 & e_1 - U_eU_e^* \end{pmatrix}.$$

Since U_e^* is a parametrix for U_e the index formula now gives

$$\tau_M(e) = -\,\mathrm{Index}(U_e) = -\,\mathrm{Index}_M(e)$$

as asserted. ■

(3.33) DEFINITION: *We will say that a Fredholm module over a unital C^*-algebra A is* summable *if its domain of summability is a dense, unital subalgebra of A.*

(3.34) THEOREM: *Let A be a unital C^*-algebra. Suppose that there exists a summable Fredholm module M over A, for which the associated trace τ_M on the dense subalgebra $\mathcal{A} \subseteq A$ is positive, faithful, and has $\tau_M(1) = 1$. Then A has no non-trivial projections.*

PROOF: Suppose first that e is a projection in $\mathcal{A}$. Since $0 \leqslant e \leqslant 1$, $0 \leqslant \tau(e) \leqslant 1$; since e is a projection, $\tau(e)$ is an integer by Proposition 3.32. Thus $\tau(e)$ equals zero or one. Since τ is faithful, if $\tau(e) = 0$, then $e = 0$ as an operator; if $\tau(e) = 1$ then $\tau(1-e) = 0$ so $1 - e = 0$ and $e = 1$.

To complete the proof we need only show that $\mathrm{Proj}\,\mathcal{A}$ is dense in $\mathrm{Proj}\,A$. This is a consequence of two facts. The first is that $\mathcal{A}$ is *inverse closed*: if $a \in \mathcal{A}$ is invertible in A, then its inverse belongs to $\mathcal{A}$. This follows from the well-known identity

$$[F, a^{-1}] = -a^{-1}[F,a]a^{-1}.$$

Thus an element $a \in \mathcal{A}$ has the same spectrum when considered as an element of $\mathcal{A}$ as it does when considered as an element of A.

The second key fact, or technique, is the *holomorphic functional calculus* for Banach algebras. This states that if a is a member of a Banach algebra $\mathcal{A}$, and if $f\colon \Omega \to \mathbb{C}$ is a holomorphic function, for some open $\Omega \subseteq \mathbb{C}$ that contains $\mathrm{sp}(a)$, then one can define $f(a) \in \mathcal{A}$ in such a way that the usual laws of the functional calculus hold: $f \mapsto f(a)$ is a homomorphism and the definition of $f(a)$ by holomorphic functional calculus agrees with the usual one if f is a rational function with poles off the spectrum. Moreover, the *continuous functional calculus* for self-adjoint (or normal) elements of a C^*-algebra, which we have been using throughout the course, agrees with this holomorphic functional calculus in the instances in which both are defined.

Let then $e \in A$ be a projection, let ε be given with $0 < \varepsilon < 1$, and let $a \in \mathcal{A}$ be a selfadjoint element with $\|a - e\| < \varepsilon/2$. The spectrum of a (in A and therefore in

$\mathcal{A}$) is contained in the subset $(-\frac{1}{2}\varepsilon, \frac{1}{2}\varepsilon) \cup (1-\frac{1}{2}\varepsilon, 1+\frac{1}{2}\varepsilon)$ of $\mathbb{R}$. Hence the function f defined by setting $f(z) = 0$ if $\Re z < \frac{1}{2}$ and $f(z) = 1$ if $\Re z > \frac{1}{2}$ is holomorphic on a neighborhood of the spectrum of a. So $f(a) \in \mathcal{A}$ and, by the properties of the functional calculus, $f(a)$ is a projection. Finally,

$$\|a - f(a)\| \leqslant \varepsilon/2$$

using the norm properties of the *continuous* functional calculus, since $|f(\lambda) - \lambda| < \varepsilon/2$ for all $\lambda \in \mathrm{sp}(a)$. It follows that $\|e - f(a)\| < \varepsilon$, as required. ■

So much for abstraction. Following an idea of Cuntz, we will now *construct* a Fredholm module over $A = C^*_r(F_2)$ that meets the criteria of Theorem 3.34. This Fredholm module makes use of the geometry of the *tree* associated to the free group $G = F_2$, which is shown in the figure.

The tree is an infinite graph (1-dimensional simplicial complex) which is equipped with a natural action of G by translation. Let V denote the set of vertices of the tree, E the set of edges. Considered as a representation of G, the Hilbert space $\ell^2(V)$ is a copy of the regular representation; the Hilbert space $\ell^2(E)$ is the direct sum of *two* copies of the regular representation, one copy being made up by the edges in the 'x-direction' and the other by the edges in the 'y-direction'.

We construct a Fredholm module as follows. First we describe the Hilbert spaces:

$$H_0 = \ell^2(V), \qquad H_1 = \ell^2(E) \oplus \mathbb{C}.$$

The representations of $C^*_r(G)$ on H_0 and H_1 are the regular ones, and we use the zero representation on the additional copy of $\mathbb{C}$. We define the unitary $U\colon H_0 \to H_1$ by first picking a 'root' vertex of the tree (labeled e in the figure). For each vertex $v \neq e$ we define

$$U[v] = ([l_v], 0),$$

where l_v is the unique edge of the tree originating from v and pointing in the direction of e. For the vertex e itself we define

$$U[e] = (0, 1),$$

thus mapping to the basis vector of the additional copy of $\mathbb{C}$. It is clear that U is unitary. Moreover, if g is a group element, then $\lambda(g)U\lambda(g^{-1})$ differs from U only in that the special rôle of the vertex e is now played by ge. This affects the definition of $\ell(v)$ only for those finitely many vertices v that lie along the geodesic segment from e to ge. Thus $\lambda(g)U\lambda(g^{-1}) - U$ is of finite rank. It follows that U commutes modulo finite rank operators with elements of $\mathbb{C}[G]$, and hence by an approximation argument that U commutes modulo compact operators with elements of $C_r^*(G)$. We have shown that our data define a summable Fredholm module.

To complete the proof it is necessary to identify the trace τ_M on the domain of summability. We need only compute $\tau_M([g])$ for group elements g. If $g = e$ is the identity, we easily calculate $\tau_M([e]) = 1$. (For instance this follows from Proposition 3.32 applied to the projection $[e] = 1$. If g is not the identity, then to compute $\tau_M([g]) = \frac{1}{2}\,\mathrm{Trace}(\gamma F[F,[g]])$, we must compute traces of operators such as

$$\lambda(g) - U^*\lambda(g)U$$

on H_0. Represent this operator by an infinite matrix relative to the standard basis of H_0. The diagonal entries of the matrix are all zeros, so the trace is zero. It follows that

$$\tau_m(\sum a_g[g]) = a_e$$

and thus τ_M agrees with the canonical trace on $C_r^*(F_2)$, which we know to have the properties listed in Theorem 3.34.

This completes the proof of

(3.35) THEOREM: *The C^*-algebra $C_r^*(F_2)$ has no non-trivial projections.*

4 More Examples of C^*-Algebras

4.1 The Cuntz Algebras

Let us recollect that an *isometry* in a unital C^*-algebra A is an element w such that $w^*w = 1$. It then follows that ww^* is a projection, which we sometimes call the *range projection* of the isometry. Any isometry has norm $\leqslant 1$. Any isometry on a finite-dimensional Hilbert space is a unitary, but on infinite-dimensional Hilbert spaces there exist isometries that are not unitaries, the most famous example being the unilateral shift which we have discussed earlier in our work on Toeplitz operators.

(4.1) DEFINITION: *Let* $n = 2, 3, 4, \ldots$. *The* Cuntz algebra $\mathcal{O}_n$ *is the universal* C^**-algebra generated by* n *isometries* $s_1, \ldots, s_n$ *subject to the single relation* $\sum_{i=1}^n s_i s_i^* = 1$.

For the definition of the universal C^*-algebra defined by generators and relations we refer back to Remark 2.13. Since the projections $s_i s_i^*$ sum to 1, they must be mutually orthogonal.[12] Thus we have

$$s_i^* s_j = \delta_{ij} 1. \tag{4.2}$$

(4.3) REMARK: One can produce an explicit Hilbert space representation of the Cuntz algebra, by taking for generators the isometries $S_1, \ldots, S_n$ on the Hilbert space $H = \ell^2(\mathbb{N})$ defined by

$$S_k(e_j) = e_{nj+k-1}$$

where $e_0, e_1, \ldots$ are the standard basis vectors for H. We will see later that the algebra generated by these isometries actually is isomorphic to $\mathcal{O}_n$.

Let us use the term *word* for a finite product of s's and their adjoints, something like $s_1 s_2^2 s_3 s_5^*$. Because of the relations that we just noted, every word can be written in a *reduced form* in which all the terms with adjoints appear to the right of all those without. If μ is a list whose members come from $\{1, 2, \ldots, n\}$, we use the shorthand notation s_μ to denote the word $s_{\mu_1} s_{\mu_2} \cdots s_{\mu_k}$. Thus any word

[12] Proof — suppose that $p_1 + \cdots + p_n = 1$ where the p_i are self-adjoint projections. Multiplying on the left and on the right by p_1 we get

$$p_1 p_2 p_1 + p_1 p_3 p_1 + \cdots + p_1 p_n p_1 = 0.$$

The operators here are all positive and sum to zero, so they are all zero. In particular $p_1 p_2 p_1 = (p_2 p_1)^*(p_2 p_1) = 0$, and so $p_2 p_1 = 0$. Similarly all the other products $p_i p_j$ are zero for $i \neq j$.

can be written in reduced form as $s_\mu s_\nu^*$. We use the notation $|\mu|$ for the number of members of the list μ, and we call the number $|\mu| - |\nu|$ the *weight* of the word $s_\mu s_n u^*$. The linear span of the words is a dense subalgebra of $\mathcal{O}_n$.

The product of a word of weight l and a word of weight m is a word of weight $l + m$, or else zero. Thus the words of weight zero span a subalgebra of the Cuntz algebra. We are going to identify this subalgebra.

(4.4) LEMMA: *Let A_n^k be the subspace of $\mathcal{O}_n$ spanned by the words $s_\mu s_\nu^*$ where $|\mu| = |\nu| = k$. Then A_n^k is a subalgebra, and it is isomorphic to the full matrix algebra $M_{n^k}(\mathbb{C})$.*

PROOF: How does one identify a matrix algebra? A m^2-dimensional $*$-algebra over $\mathbb{C}$ is a matrix algebra if and only if it is spanned by a *system of matrix units,* which is a basis e_{IJ}, where $I, J = 1, \ldots, m$, with respect to which the multiplication law of the algebra is

$$e_{IJ}e_{KL} = \delta_{JK}e_{IL}$$

and the adjunction law is

$$e_{IJ}^* = e_{JI}.$$

(Of course, e_{IJ} corresponds to the matrix with 1 in the (I, J)-position and zeroes elsewhere.) Now, in A_n^k, the n^{2k} words $s_\mu s_\nu^*$ where $|\mu| = |\nu| = k$ are linearly independent (for instance because their images in the explicit representation of Remark 4.3 are linearly independent). Moreover by repeated application of Equation 4.2 we find that

$$s_\mu s_\nu^* s_{\mu'} s_{\nu'}^* = \delta_{\nu\mu'} s_\mu s_{\nu'}^*$$

so that the $s_\mu s_\nu^*$ comprise a system of matrix units. ∎

The fundamental relation

$$1 = s_1 s_1^* + \cdots + s_n s_n^*$$

expresses the identity operator (which we regard as the generator of $A_n^0 = \mathbb{C}1$) as a linear combination of generators of A_n^1. Generalizing this observation we have inclusions $A_n^{k-1} \subseteq A_n^k$ of algebras. In terms of the representations of A_n^{k-1} and A_n^k as matrix algebras these inclusions amount to the n-fold inflation maps

$$T \mapsto \begin{pmatrix} T & & \\ & \ddots & \\ & & T \end{pmatrix}.$$

It follows that the C^*-subalgebra A_n of $\mathcal{O}_n$ defined as the closed linear span of the words of weight zero is the direct limit of matrix algebras

$$\mathbb{C} \to M_n(\mathbb{C}) \to M_{n^2}(\mathbb{C}) \to M_{n^3}(\mathbb{C}) \to \cdots$$

which is called the *uniformly hyperfinite* (UHF) algebra of *type* n^∞. We will discuss UHF algebras, and more general inductive limits of matrix algebras, later in the course; we won't need any of their theory in our discussion of the Cuntz algebra for now.

(4.5) DEFINITION: *Let A be a C^*-algebra and B a C^*-subalgebra of A. A* conditional expectation *of A onto B is a positive linear map $\Phi\colon A \to B$ which is the identity on B. (*Positive *means that Φ carries positive elements to positive elements.) The expectation Φ is* faithful *if $a > 0$ implies $\Phi(a) > 0$.*

Our analysis of the Cuntz algebra will be based on a certain faithful conditional expectation of $\mathcal{O}_n$ onto A_n. Rather as we did with the free group C^*-algebra and its canonical trace, we will represent this expectation in a certain sense as a limit of inner endomorphisms, and we will use this representation to show that $\mathcal{O}_n$ is simple.

Let $z \in \mathbb{T}$ be a complex number of modulus one. The elements $zs_1, \ldots, zs_n$ of $\mathcal{O}_n$ are isometries which satisfy the defining relation for the Cuntz algebra, so by the universal property there is an automorphism

$$\theta_z\colon \mathcal{O}_n \to \mathcal{O}_n$$

which sends s_i to zs_i for each i. One has $\theta_z(s_\mu s_\nu^*) = z^{|\mu|-|\nu|} s_\mu s_\nu^*$, so that A_n is precisely the fixed-point algebra for the one-parameter group of automorphisms θ_z. It is easy to see that $z \mapsto \theta_z(a)$ is continuous for each $a \in \mathcal{O}_n$.

(4.6) LEMMA: *With notation as above, the map Φ defined by*

$$\Phi(a) = \frac{1}{2\pi} \int_0^{2\pi} \theta_{e^{it}}(a)\, dt$$

is a faithful conditional expectation of $\mathcal{O}_n$ onto A_n.

The proof is easy. We will now show how to approximate Φ by inner endomorphisms:

(4.7) LEMMA: *For each positive k one can find an isometry $w_k \in \mathcal{O}_n$ with the property that*

$$\Phi(a) = w_k^* a w_k$$

for every a in the span of those words $s_\mu s_\nu^*$ where $|\mu|, |\nu| < k$.

PROOF: Let us begin with the following observation. Suppose that $m \geqslant k$, that $x = s_1^m s_2$, and that $y = s_\mu s_\nu^*$ is a word where $|\mu|, |\nu| \leqslant k$. Then $x^* y x = 0$ in all cases except when y is of the form $s_1^\ell s_1^{*\ell}$, when $x^* y x = 1$. Indeed, in order that $x^* s_\mu$ not equal zero it is necessary that μ be of the form $(1, 1, 1, \ldots)$ and in order that $s_\nu^* x$ not equal zero it is necessary that ν be of the form $(1, 1, 1, \ldots)$. Thus we need only consider $y = s_1^\ell s_1^{*\ell'}$ and direct computation yields the stated result.

Now put $x = s_1^{2k} s_2$ and let $w = \sum s_\gamma x s_\gamma^*$, where the sum runs over all words s_γ of length k. Notice that w is an isometry. Indeed, we have

$$w^* w = \sum_{\gamma,\delta} s_\delta x^* s_\delta^* s_\gamma x s_\gamma^* = \sum_\gamma s_\gamma x^* x s_\gamma^* = \sum_\gamma s_\gamma s_\gamma^* = 1.$$

Let us consider what is $w^* y w$ for words $y = s_\mu s_\nu^*$ with $|\mu|, |\nu| \leqslant k$. If $|\mu| \neq |\nu|$ then our previous observation shows that each of the terms in the double sum that makes up $w^* y w$ is equal to zero, so the whole sum is zero. Suppose now that $|\mu| = |\nu| = k$.[13] Then y is a matrix unit $s_\mu s_\nu^*$. We have

$$wy = \sum_\gamma s_\gamma x s_\gamma^* y = s_\mu x s_\nu^*,$$

since only the term with $\gamma = \mu$ contributes, and similarly

$$yw = s_\mu x s_\nu^*.$$

Thus w commutes with y, so $w^* y w = y$. We have shown that (on the set of words with $|\mu|, |\nu| \leqslant k$), the map $y \mapsto w^* y w$ fixes those words of weight zero and annihilates those words of nonzero weight. That is to say, $w^* y w = \Phi(y)$ on those words. ■

(4.8) THEOREM: *Let $a \in \mathcal{O}_n$ be nonzero. Then there are elements $x, y \in \mathcal{O}_n$ such that $xay = 1$.*

As we shall see, this property is expressed by saying that $\mathcal{O}_n$ is a *purely infinite* C^*-algebra.

(4.9) COROLLARY: *The Cuntz algebra $\mathcal{O}_n$ is simple. In particular, therefore, the concrete C^*-algebra of operators on ℓ^2 described in Remark 4.3 is isomorphic to $\mathcal{O}_n$.*

PROOF: To begin the proof, notice that each word $a = s_\mu s_\nu^*$ has the stated property; just take $x = s_\mu^*$ and $y = s_\nu$. The proof is an approximation argument to

[13] Remember that the terms with $|\mu| = |\nu| < k$ are included in the linear span of those with $||mu| = |\nu| = k$, so we do not have to consider them separately.

use this property of the words in the generators. We need to be careful because it is not at all obvious that if a', a'' have the desired property then $a = a' + a''$ will have it too.

Since $a \neq 0$, the faithfulness of the expectation Φ gives $\Phi(a^*a) > 0$, and by a suitable normalization we may assume that $\Phi(a^*a)$ is of norm one. Approximate a^*a by a finite linear combination of words b well enough that $\|a^*a - b\| < \frac{1}{4}$. In particular we have $\|\Phi(b)\| = \lambda > \frac{3}{4}$.

Now $\Phi(b)$ is a positive element of some matrix algebra over $\mathbb{C}$, and has norm $\lambda > \frac{3}{4}$. The norm of a positive element of a matrix algebra is just its greatest eigenvalue. So there must be some one-dimensional eigenprojection e for $\Phi(b)$, in the matrix algebra, for which $e\Phi(b) = \Phi(b)e = \lambda e > \frac{3}{4}e$. We can find a unitary u in the matrix algebra which conjugates e to the first matrix unit, that is

$$u^*eu = s_1^m s_1^{*m}.$$

It is now simple to check that if we put $z = \lambda^{-\frac{1}{2}} eus_1^m$, then $z^*\Phi(b)z = 1$.

By the preceding lemma, $\Phi(b) = w^*bw$ for some isometry $w \in \mathcal{O}_n$. Put $v = wz$; then $v^*bv = 1$. Moreover, $\|v\| > \frac{2}{\sqrt{3}}$.

Now we compute

$$\|1 - v^*a^*av\| \leqslant \|v\|^2\|b - a^*a\| < \tfrac{1}{3},$$

and so v^*a^*av is invertible. Let $y = v(v^*a^*av)^{-\frac{1}{2}}$ and let $x = y^*a^*$. Then

$$xay = y^*a^*ay = 1$$

as required. ■

4.2 Real Rank Zero

(4.10) DEFINITION: *A projection p in a C^*-algebra A is called* infinite *if there is a partial isometry $w \in A$ with $w^*w = p$ and $ww^* = q$ with $q < p$. A C^*-algebra A is called* infinite *if it contains an infinite projection.*

Clearly the Cuntz algebras are infinite (the identity is an infinite projection). The same applies for instance to the Toeplitz algebra $\mathfrak{T}$. We are going to show, however, that the Cuntz algebras are 'more infinite' than the Toeplitz algebra. Indeed, $\mathfrak{T}$ contains the finite C^*-subalgebra $\mathfrak{K}$, but we will see that the analogous statement is not true for $\mathcal{O}_n$. To find out what *is* the analogous statement we need the notion of hereditary subalgebra.

(4.11) DEFINITION: *Let A be a C^*-algebra. A C^*-subalgebra B of A is* hereditary *if whenever $b \in B$ is positive, and $0 \leqslant a \leqslant b$, then $a \in B$ also.*

It is a consequence of the Cohen factorization theorem that *every ideal is a hereditary subalgebra*. Other examples of hereditary subalgebras include the subalgebra pAp where p is a projection in A, or more generally $\overline{xAx}$ whenever x is a positive element of A (we will prove this below). By a general theorem of Effros, each hereditary subalgebra is of the form $\mathfrak{l} \cap \mathfrak{l}^*$, where $\mathfrak{l}$ is a closed left ideal.

(4.12) LEMMA: *Let A be a unital C^*-algebra and let $x \in A$ be positive. Then $\overline{xAx}$ is a hereditary C^*-subalgebra.*

PROOF: Clearly it is a C^*-subalgebra. To prove that it is hereditary, let $0 \leqslant b \leqslant xax$. Then by the first part of the Cohen factorization theorem we have $b^{\frac{1}{2}} = c(xax)^{\frac{1}{4}}$ and so

$$b = (xax)^{\frac{1}{4}} c^* c (xax)^{\frac{1}{4}}.$$

Using the Stone-Weierstrass theorem, approximate $(xax)^{\frac{1}{4}}$ uniformly by polynomials in xax, to find that $b \in \overline{xAx}$. A slight additional approximation argument handles the general case. ■

(4.13) DEFINITION: *A C^*-algebra A is* purely infinite *if each hereditary C^*-subalgebra of A is infinite.*

Clearly the Toeplitz algebra is not purely infinite. However, we do have

(4.14) THEOREM: *The Cuntz algebras $\mathcal{O}_n$ are purely infinite.*

PROOF: The proof will depend only on the conclusion of Theorem 4.8, and in fact this conclusion is *equivalent* to the property of being purely infinite, at least for simple C^*-algebras. However, we will not prove the whole of this latter assertion, but only the easier direction.

Let B be a proper hereditary subalgebra of $A = \mathcal{O}_n$ and let $b \in B$ be a non-zero positive element. Remark that it is impossible that b is invertible in A, for then $1 \leqslant \|b^{-1}\| b \in B$ and the hereditary property now tells us that $B = A$. I claim now that we can find $z \in A$ with $z^* b z = 1$. Indeed, we can find x, y with $x b^{\frac{1}{2}} y = 1$, hence

$$1 \leqslant y^* b^{\frac{1}{2}} x^* x b^{\frac{1}{2}} y \leqslant \|x\|^2 y^* b y.$$

So $y^* b y$ is invertible, and we can take $z = y(y^* b y)^{-\frac{1}{2}}$. Having found z put $s = b^{\frac{1}{2}} z$. Then s is a proper isometry in A, with range projection

$$p = ss^* = b^{\frac{1}{2}} z^* z b^{\frac{1}{2}} \leqslant \|z\|^2 b$$

belonging to the hereditary subalgebra B. I now claim that p is an infinite projection *in* B. To see this notice first that $sp = psp$ belongs to B. Now

$p = (sp)^*(sp)$, whereas $q = (sp)(sp)^* = sps^*$ is a proper sub-projection of p.[14] ■

We will see that the Cuntz algebras are 'universal models' for the behavior of infinite simple C^*-algebras, in a certain rather weak sense.

(4.15) LEMMA: *Let A be a simple unital C^*-algebra, and let $a \in A$ be positive and non-zero. Then there exist finitely many elements $x_1, \ldots, x_n \in A$ such that $\sum_{i=1}^n x_i^* a x_i$.*

PROOF: Because 1 belongs to the ideal algebraically generated by a, there exist $y_1, \ldots, y_k$ and $z_1, \ldots, z_k$ in A such that

$$y_1 a z_1 + \cdots + y_k a z_k = 1.$$

Note the inequality

$$yaz + z^* a y^* \leqslant y a y^* + z^* a z$$

which follows from the positivity of $(y + z^*)a(y^* + z)$. We obtain from this

$$b = \sum_{i=1}^{k} y_k a y_k^* + z_k^* a z_k \geqslant 2.$$

With $n = 2k$ put $x_i = y_i^* b^{\frac{1}{2}}$ for $1 \leqslant i \leqslant k$ and $x_i = z_{i-n} b^{\frac{1}{2}}$ for $k + 1 \leqslant i \leqslant n$. ■

(4.16) PROPOSITION: *Let A be an infinite simple (unital) C^*-algebra. Then A contains isometries t_i, $i = 1, 2, \ldots$, such that the $t_i t_i^*$ are pairwise orthogonal projections and $\sum_{i=1}^n t_i t_i^* < 1$ for all n.*

PROOF: The projection 1 is infinite so there is an isometry $s \in A$ with $p = ss^*$ a proper projection. By Lemma 4.15 there are $x_1, \ldots, x_k \in A$ such that $\sum_{i=1}^k x_i^*(1-p)x_i = 1$. Put

$$t_1 = (1-p)x_1 + s(1-p)x_2 + \cdots + s^{k-1}(1-p)x_k.$$

Since the projections $s^i(1-p)$ are mutually orthogonal we get

$$t_1^* t_1 = \sum x_i^*(1-p)x_i = 1.$$

Thus $t_1 t_1^*$ is a projection and, since we easily compute that $t_1^* s^k = 0$, we have

$$t_1 t_1^* \leqslant 1 - s^k s^{*k}.$$

[14] Let s be a proper isometry, that is, $p = ss^* < 1$. Then $s^n(s^*)^n$ is a proper sub-projection of $s^{n-1}(s^*)^{n-1}$ for each n. This follows from the fact that the projection $s^n(1-p)s^{*n}$ is non-zero; it is equal to xx^*, where $x = s^n(1-p)$ and we note that $s^{*n}x = 1 - p \neq 0$.

We can now easily construct the higher t's by induction as $t_i = s^{k(i-1)}t_1$. Then

$$t_i t_i^* \leqslant s^{k(i-1)}(s^*)^{k(i-1)} - s^{ki}(s^*)^{ki}$$

and so we obtain the desired property. ■

One says that a unital C^*-algebra A is *properly infinite* if it contains isometries t_1, t_2 such that $t_1t_1^* + t_2t_2^* \leqslant 1$. Thus we have shown that a simple unital C^*-algebra is properly infinite. In fact we have shown more:

(4.17) COROLLARY: *Every simple infinite unital C^*-algebra A has $\mathcal{O}_n$ as a subquotient (that is, as a quotient of a subalgebra).*

PROOF: Let $t_1, \ldots, t_n$ be isometries as in the preceding proposition, let B be the unital subalgebra of A generated by $t_1, \ldots, t_n$, and let J be the ideal in B generated by $q = 1 - \sum t_i t_i^*$. Notice that since q is non-invertible and is in the center of B, the ideal J is proper. In the quotient algebra B/J the images of the isometries t_i satisfy the relations for the Cuntz algebra, hence $B/J \cong \mathcal{O}_n$ since $\mathcal{O}_n$ is simple. ■

Here is a slight generalization.

(4.18) LEMMA: *Let A be a simple unital C^*-algebra and let p be an infinite projection in A. Then any other projection q in A is equivalent to a subprojection of p (that is, there is a partial isometry t with $tt^* = q$ and t^*t a subprojection of p).*

PROOF: Use Lemma 4.15 to find $z_1, \ldots, z_n$ such that $q = \sum_{i=1}^n z_i^* p z_i$. Apply Proposition 4.16 in the hereditary subalgebra pAp to find partial isometries t_i with $t_i^* t_i = p$ and $\sum t_i t_i^* < p$. Put $t = \sum z_i^* p t_i^*$. Then $tt^* = q$, and so t is a partial isometry; moreover, $t^*t = pt^*tp$ is a subprojection of p. ■

(4.19) DEFINITION: *A unital C^*-algebra A has* real rank zero *if every selfadjoint element of A is a limit of* invertible *selfadjoint elements.*

This is a disconnectedness condition. For instance, $C(X)$ has real rank zero if X is the Cantor set, but not if $X = [0, 1]$.

(4.20) THEOREM: *Each purely infinite simple C^*-algebra A has real rank zero.*

PROOF: Let a be a self-adjoint element of A and let $\varepsilon > 0$ be given. We are going to find an invertible within 2ε of a. Let $f(\lambda)$ and $g(\lambda)$ be the functions whose graphs are shown in the figure: $f(\lambda) = \max(\lambda - \varepsilon, \min(0, \lambda + \varepsilon))$ and $g(\lambda) = \max(\varepsilon - |\lambda|, 0)$. Since A is purely infinite, there is an infinite projection p in the hereditary subalgebra $\overline{g(a)Ag(a)}$. By lemma 4.18, there is a partial isometry s in that subalgebra such that $ss^* = p$ and $s^*s = 1 - p$. The projections $p_0 = 1 - p$,

$p_1 = q$, and $p_2 = q - p$ are mutually orthogonal and sum to 1, and with respect to the decomposition provided by these projections we have

$$f(a) = \begin{bmatrix} f(a) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad s = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where we have denoted by 1 the isomorphism from p_1 to p_0 provided by s. It follows that

$$f(a) + \varepsilon(s + s^* + p_2) = \begin{bmatrix} f(a) & \varepsilon & 0 \\ \varepsilon & 0 & 0 \\ 0 & 0 & \varepsilon \end{bmatrix}$$

is invertible, and it lies within 2ε of a. ■

4.3 UHF Algebras

In this section we shall study C^*-algebras (and associated von Neumann algebras) that are constructed as inductive limits of matrix algebras. An important example is the algebra $A_n \subseteq \mathcal{O}_n$ which was used in our earlier discussion of Cuntz algebras.

(4.21) LEMMA: *Let $M_m(\mathbb{C})$ and $M_n(\mathbb{C})$ be matrix algebras. There exists a unital $*$-homomorphism $M_m(\mathbb{C}) \to M_n(\mathbb{C})$ if and only if $m|n$; and, when this condition is satisfied, all such $*$-homomorphisms are unitarily equivalent to that given by*

$$T \mapsto \begin{pmatrix} T & & \\ & \ddots & \\ & & T \end{pmatrix}$$

with T repeated n/m times down the diagonal and zeroes elsewhere.

PROOF: Let $\alpha\colon M_m(\mathbb{C}) \to M_n(\mathbb{C})$ be a $*$-homomorphism of the given sort. It is injective because $M_m(\mathbb{C})$ is simple. Let e_{ij} be a set of matrix units in $M_m(\mathbb{C})$. Then $\alpha(e_{11})$ is a projection in $M_n(\mathbb{C})$; let $v_1, \ldots, v_d$ be an orthonormal basis for its range. For every $T \in M_m(\mathbb{C})$, $e_{11}Te_{11} = \lambda_T e_{11}$ where $\lambda_T \in \mathbb{C}$. It follows that for $r \neq s$,

$$\langle \alpha(T)v_r, v_s \rangle = \lambda_T \langle v_r, v_s \rangle = 0,$$

so that the $\alpha(M_m(\mathbb{C}))$-cyclic subspaces generated by $v_1, \ldots, v_d$ are orthogonal to one another. (We already used this argument — see the proof of lemma 2.37.) Each such cyclic subspace is spanned as a vector space by the m vectors

$$v_r = \alpha(e_{11})v_r,\ \alpha(e_{12})v_r, \ldots,\ \alpha(e_{1n})v_r$$

which one checks are orthonormal. Moreover, $M_m(\mathbb{C})$ acts on these m vectors in the standard way. We have shown therefore that $\mathbb{C}^n$ has an orthonormal basis comprising d sets of m vectors, on each of which sets $M_m(\mathbb{C})$ acts in the standard way. This is what was required. ■

Suppose now that $k_1|k_2|k_3|\cdots$ is an increasing sequence of natural numbers, each of which divides the next. By the lemma, there is associated to this sequence a unique (up to unitary equivalence) sequence of matrix algebras and unital $*$-homomorphisms

$$M_{k_1}(\mathbb{C}) \xrightarrow{\alpha_1} M_{k_2}(\mathbb{C}) \xrightarrow{\alpha_2} M_{k_3}(\mathbb{C}) \longrightarrow \cdots.$$

Let $\mathcal{A}$ denote the (algebraic) inductive limit of this sequence. (Reminder about what this means: Think of unions. More formally, the elements of $\mathcal{A}$ are equivalence classes of sequences $\{T_i\}$, $T_i \in M_{k_i}(\mathbb{C})$, which are required to satisfy $\alpha_i(T_i) = T_{i+1}$ for all but finitely many i, and where two sequences are considered to be equivalent if they differ only in finitely many places. These may be added, multiplied, normed (remember that the α_i are isometric inclusions!), and so on, by pointwise operations.) The algebra $\mathcal{A}$ is a normed algebra which satisfies all the C^*-axioms except that it need not be complete. Its completion is a C^*-algebra which is called the *uniformly hyperfinite* (UHF) *algebra* (or *Glimm algebra*) determined by the inductive system of matrix algebras.

(4.22) PROPOSITION: *All UHF algebras are simple, separable, unital, and have a unique trace.*

PROOF: Obviously they are separable and unital. Let A be a UHF algebra and let $\alpha\colon A \to B$ be a unital $*$-homomorphism to another C^*-algebra. Since each matrix subalgebra $M_{k_i}(\mathbb{C})$ of A is simple, the restriction of α to it is an injective $*$-homomorphism and is therefore isometric. Thus α is isometric on the dense subalgebra $\mathcal{A}$ of A, whence by continuity α is isometric on the whole of A and in particular has no kernel. Thus, A is simple.

For each matrix algebra $M_{k_i}(\mathbb{C})$ let $\tau_{k_i}\colon M_{k_i}(\mathbb{C}) \to \mathbb{C}$ denote the normalized trace, that is

$$\tau_{k_i}(T) = \frac{1}{k_i}\,\mathrm{Trace}(T).$$

The τ_{k_i} then are consistent with the embeddings α_i so they pass to the inductive limit to give a trace τ on the dense subalgebra $\mathcal{A}$ of A. Moreover, τ is norm continuous on $\mathcal{A}$ (since the τ_{k_i} are all of norm one on the matrix algebras $M_{k_i}(\mathbb{C})$) and so extends by continuity to a trace, indeed a tracial state, on A. The uniqueness of the trace on matrix algebras[15] shows that every other trace must agree, up to normalization of course, with this one. ■

Suppose that $k_1|k_2|\cdots$ is the sequence of orders of matrix algebras making up a UHF algebra. For each prime p there is a natural number or infinity n_p defined to be the supremum of the number of times p divides k_i, as $i \to \infty$. The formal product $\prod p_r^{n_{p_r}}$ is called the *supernatural number* associated to the UHF algebra.

EXAMPLE: The UHF algebra associated to the sequence of embeddings

$$M_2(\mathbb{C}) \qquad M_4(\mathbb{C}) \qquad M_8(\mathbb{C}) \qquad \cdots$$

has supernatural number 2^∞. This example is particularly important; it is called the *CAR algebra* (we will see why in a moment) or the *Fermion algebra*.

(4.23) LEMMA: *Let p and q be projections in a C^*-algebra A with $\|p-q\| < 1$. Then $\tau(p) = \tau(q)$ for any trace τ on A. In fact, p and q are unitarily equivalent.*

PROOF: Define $x = qp + (1-q)(1-p)$. Then $xp = qx$, and a simple calculation shows that

$$x - 1 = 2qp - q - p = (2q-1)(p-q).$$

Thus $\|x-1\| < 1$ and so x is invertible. We may define a unitary $u = x(x^*x)^{-\frac{1}{2}}$ by the functional calculus. Now x^*x commutes with p, and so

$$up = xp(x^*x)^{-\frac{1}{2}} = qx(x^*x)^{-\frac{1}{2}} = qu$$

as required. ■

(4.24) THEOREM: *Two UHF algebras are isomorphic if and only if they have the same supernatural number.*

PROOF: First we will show that two isomorphic Glimm algebras have the same supernatural number. Let the two algebras A and B be obtained from generating sequences $k_1|k_2|k_3|\cdots$ and $l_1|l_2|l_3|\cdots$ respectively. Let $\alpha\colon A \to B$ be an isomorphism. Notice that α takes the unique trace τ_A on A to the unique trace τ_B on B.

We will show that for each i there is a j such that k_i divides l_j. Indeed, let p denote the matrix unit e_{11} in $M_{k_i}(\mathbb{C}) \subseteq A$. Then $\alpha(p)$ is a projection in B. By density there is $x = x^* \in M_{l_j}(\mathbb{C}) \subseteq B$ such that $\|x - \alpha(p)\| < \frac{1}{20}$. Simple estimates give $\|x^2 - x\| < \frac{3}{16}$. Thus the spectrum of x is contained in

[15] And how do you prove *that*? Compute with matrix units to show that every trace-zero finite matrix lies in the span of commutators.

$[-\frac{1}{4}, \frac{1}{4}] \cup [\frac{3}{4}, \frac{5}{4}]$ and the spectral projection q of x corresponding to $\lambda > \frac{1}{2}$ has $\|q - x\| < \frac{1}{4}$. Notice that $q \in M_{l_j}(\mathbb{C})$ and that, in particular, $\|q - \alpha(p)\| < 1$. By the previous lemma and the uniqueness of the trace

$$\tau_B(q) = \tau_B(\alpha(p)) = \tau_A(p) = \frac{1}{k_i}.$$

Since the normalized trace of every projection in $M_{l_j}(\mathbb{C})$ is a multiple of $1/l_j$, we conclude that k_i divides l_j, as required.

Conversely, if A and B have the same supernatural number, then by passing to subsequences (which doesn't change the direct limit) we can arrange so that $k_i | l_i | k_{i+1}$. Then construct inductively embeddings of matrix algebras as in the diagram below

$$\begin{array}{cccc} M_{k_1}(\mathbb{C}) & M_{k_2}(\mathbb{C}) & M_{k_3}(\mathbb{C}) & \cdots \\ \\ M_{l_1}(\mathbb{C}) & M_{l_2}(\mathbb{C}) & M_{l_3}(\mathbb{C}) & \cdots \end{array}$$

twisting by appropriate unitary equivalences to make the diagram commute exactly. We obtain a $*$-isomorphism between dense subalgebras of A and B, which extends automatically to a $*$-isomorphism from A to B. ■

Notice as a corollary that there are uncountably many isomorphism classes of UHF algebras.

EXAMPLE: The Fermion algebra (CAR algebra) arises in quantum field theory. The second quantization of a fermion field leads to the following problem: given a Hilbert space H, find a C^*-algebra A generated by operators c_v, $v \in H$, such that $v \mapsto c_v$ is linear and isometric and

$$c_v c_w + c_w c_v = 0, \quad c_v^* c_w + c_w c_v^* = \langle w, v \rangle 1.$$

These are called the *canonical anticommutation relations*. Compare the definition of a Clifford algebra. We will show that the C^*-algebra A arising in this way is the unique UHF algebra with supernatural number 2^∞.

Let v be a unit vector on H. Then c_v has square zero, and $c_v^* c_v + c_v c_v^* = 1$. Multiplying through by $c_v^* c_v$ we find that $c_v^* c_v$ is a projection, call it e_v, and that c_v is a partial isometry with range projection e_v and domain projection $1 - e_v$. We can therefore identify the C^*-subalgebra of A generated by c_v: it is a copy of $M_2(\mathbb{C})$, generated by matrix units $e_{21} = c_v$, $e_{11} = e_v$, $e_{22} = 1 - e_v$, $e_{12} = c_v^*$.

The idea now is to apply the process inductively to the members $v_1, v_2, \ldots$ of an orthonormal basis for the Hilbert space H. Let $A_1, A_2, \ldots$ be the subalgebras of A (all isomorphic to $M_2(\mathbb{C})$) generated by $c_{v_1}, c_{v_2}, \ldots$. If the algebras A_i, A_j all commuted with one another for $i \neq j$ then we would be done, because the subalgebra spanned by $A_1, \ldots, A_n$ would then be the tensor product $M_2(\mathbb{C}) \otimes \cdots \otimes M_2(\mathbb{C}) = M_{2^n}(\mathbb{C})$ and we would have represented A as an inductive limit of matrix algebras. Unfortunately the A_i's do not commute; but they do 'supercommute' when they are considered as superalgebras in an appropriate way. Rather than developing a large machinery for discussing superalgebras, we will inductively modify the algebras A_j to algebras B_j which are each individually isomorphic to 2×2 matrix algebras, which do commute, and which have the property that the subalgebra of A spanned by $A_1, \ldots, A_n$ is equal to the subalgebra of A spanned by $B_1, \ldots, B_n$. This, then, will complete the proof.

Let γ_i be the *grading operator* for A_i, which is to say the operator corresponding to the 2×2 matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Notice that γ_i is self-adjoint, has square 1 and commutes with c_{v_j} for $j \neq i$, while it anticommutes with c_{v_i}; in particular γ_i commutes with γ_j. Let

$$B_1 = A_1 = C^*(c_{v_1}), \quad B_2 = C^*(c_{v_2}\gamma_1), \ldots \quad B_n = C^*(c_{v_n}\gamma_1 \cdots \gamma_{n-1}).$$

By the same argument as for the A_i above, each B_i is isomorphic to $M_2(\mathbb{C})$. Moreover the generators of the algebras B_i all commute. Finally, it is clear that the subalgebra spanned by $B_1, \ldots, B_n$ is the same as the subalgebra spanned by $A_1, \ldots, A_n$. The proof is therefore complete.

4.4 UHF Algebras and von Neumann Factors

We are going to use UHF algebras to construct interesting examples of von Neumann algebras. (This is something of a reversal of history: Glimm's UHF-algebra construction is explicitly patterned after a construction of Murray and von Neumann, that of the so-called hyperfinite II_1 factor.) Let H be a Hilbert space. Recall that a *von Neumann algebra* of operators on H is a unital, weakly closed $*$-subalgebra $M \subseteq \mathfrak{B}(H)$. Its commutant

$$M' = \{T \in \mathfrak{B}(H) : TS = ST \;\forall\, S \in M\}$$

is also a von Neumann algebra.

(4.25) DEFINITION: *If the center*

$$\mathfrak{Z}(M) = M \cap M'$$

consists only of scalar multiples of the identity, then the von Neumann alegbra M is called a factor.

There is a sophisticated version of the spectral theorem which expresses every von Neumann algebra as a 'direct integral' of factors.

We are going to manufacture examples of factors by the following procedure: take a C^*-algebra A (in our examples, a UHF algebra) and a state σ of A. By way of the GNS construction, form a representation $\rho_\sigma \colon A \to \mathfrak{B}(H_\sigma)$ and let M_σ be the von Neumann algebra generated by the representation, that is $M_\sigma = \rho_\sigma[A]''$. If M_σ is a factor, then we call σ a *factorial state*. As we shall see, many different examples of factors can be generated from the factorial states on one and the same C^*-algebra.

By way of a warm-up let us see that UHF C^*-algebras have trivial center.

(4.26) LEMMA: *Let A be a UHF algebra; then $\mathfrak{Z}(A) = \mathbb{C}1$.*

PROOF: The proof applies to any unital C^*-algebra with a unique faithful trace, τ. Let $z \in \mathfrak{Z}(A)$. Then the linear functional $a \mapsto \tau(az)$ is a trace on A, so by uniqueness it is equal to a scalar multiple of τ, say $\lambda\tau$. But then

$$\tau(a(z - \lambda 1)) = 0$$

for all $a \in A$, in particular for $a = (z - \lambda 1)^*$, so by faithfulness $z - \lambda 1 = 0$ and z is a multiple of the identity. ■

Of course this does *not* prove that various von Neumann algebra completions of A must have trivial center. We need more work to see that! We will discuss a special class of states on UHF algebras, the *product states*.

Recall that a UHF algebra A is a completed inductive limit of matrix algebras

$$M_{k_1}(\mathbb{C}) \subseteq M_{k_2}(\mathbb{C}) \subseteq M_{k_3}(\mathbb{C}) \subseteq \cdots .$$

Putting $d_1 = k_1$, $d_{i+1} = k_{i+1}/k_i$ for $i \geqslant 1$, we can write the inductive sequence equivalently as

$$M_{d_1}(\mathbb{C}) \subseteq M_{d_1}(\mathbb{C}) \otimes M_{d_2}(\mathbb{C}) \subseteq M_{d_1}(\mathbb{C}) \otimes M_{d_2}(\mathbb{C}) \otimes M_{d_3}(\mathbb{C}) \subseteq \cdots .$$

Notice that $M_{d_{i+1}}(\mathbb{C})$ is the commutant of $M_{k_i}(\mathbb{C})$ inside $M_{k_{i+1}}(\mathbb{C})$. For each i, let σ_i be a state of $M_{d_i}(\mathbb{C})$. (We will discuss the explicit form of these states in a moment.) Then one can easily define a state σ on the algebraic inductive limit $\mathcal{A}$ by

$$\sigma(a_1 \otimes a_2 \otimes \cdots) = \sigma_1(a_1)\sigma_2(a_2) \cdots .$$

Note that all but finitely many of the a_i are equal to one. When each of the σ_i are the normalized trace, this is exactly the construction we performed in the last lecture to obtain τ. The functional σ has norm one, so it extends by continuity to a state on the UHF algebra A. Such a state is called a *product state*.

(4.27) LEMMA: *Let A be a UHF algebra and let σ be a product state on it. Let $M = M_{k_i}(\mathbb{C})$ be one of the matrix algebras in an inductive sequence defining A. If $x \in M$ and $y \in M'$ (by which I mean the commutant of M in A), then $\sigma(xy) = \sigma(x)\sigma(y)$.*

PROOF: The result is apparent if $y \in M_{k_j}(\mathbb{C})$ for some $j \geqslant i$. For then the commutation condition allows us to write $x = X \otimes 1$, $y = 1 \otimes Y$ relative to the tensor product decomposition of the matrix algebra, and the result follows from the definition of a product state. To complete the proof we must therefore show that any $y \in A$ that commutes with $M_{k_i}(\mathbb{C})$ can be approximated in norm by elements $y_j \in M_{k_j}(\mathbb{C})$ that commute with $M_{k_i}(\mathbb{C})$. To do this, first find a sequence $y'_j \in M_{k_j}(\mathbb{C})$ with $y'_j \to y$ in norm. Then define y_j by averaging y'_j over the unitary group of $M_{k_i}(\mathbb{C})$:

$$y_j = \int_{U(k_i)} u^* y'_j u \, du$$

where du denotes Haar measure on the compact group $U(k_i)$. Then $y_j \in M_{k_j}(\mathbb{C})$ and commutes with $U(k_i)$ and hence with the whole of $M_{k_i}(\mathbb{C})$ (every matrix is a linear combination of unitaries). Moreover, since

$$y = \int_{U(k_i)} u^* y u \, du$$

we see that $\|y - y_j\| \leqslant \|y - y'_j\| \to 0$. This completes the proof. ■

Recall that a state σ is *pure* if it is an extreme point of the space of states. It is equivalent to say that if φ is a positive linear functional with $\varphi \leqslant \sigma$, then φ is a scalar multiple of σ. Let $M = M_r(\mathbb{C})$ be a finite matrix algebra; then the states

$$\sigma_q(T) = T_{qq}, \qquad q = 1, \dots, r$$

are pure states of M. To see this we can argue as follows, taking $q = 1$ for simplicity of notation: if $\varphi \leqslant \sigma_q$ and φ is positive, then $\varphi(e_{ii}) = 0$ for $i \geqslant 2$; therefore $\varphi e_[ij^* e_{ij}) = 0$ for $i \geqslant 2$; therefore by Cauchy-Schwarz $\varphi(e_{ij}) = 0$ for $i \geqslant 2$, and symmetrically for $j \geqslant 2$. Thus $\varphi = \varphi(1)\sigma_1$.

(4.28) LEMMA: *Let $A = M_{d_1}(\mathbb{C}) \otimes M_{d_2}(\mathbb{C}) \otimes \cdots$ be a UHF algebra and let $\sigma = \sigma_{q_1} \otimes \sigma_{q_2} \otimes \cdots$ be a product state in which each of the σ_{q_i} is a pure state of $M_{d_i}(\mathbb{C})$ of the sort described above. Then σ is a pure state of A.*

PROOF: Let $\varphi \leqslant \sigma$ and prove by induction on i that $\varphi = \varphi(1)\sigma$ when restricted to $M_{k_i}(\mathbb{C}) \subseteq A$, using the argument sketched above which proves that the σ_q are pure states of the corresponding matrix algebras. ■

Let us call the state described above the *pure product state* $\sigma_{\mathbf{q}}$ associated to the sequence of indices $\mathbf{q} = (q_1, q_2, \ldots)$. We are going to investigate when the irreducible representations associated to two pure product states are equivalent. We need the following lemma:

(4.29) LEMMA: *Let A be a unital C^*-algebra and let σ, σ' be pure states of A. Then the irreducible representations associated to σ and σ' are unitarily equivalent if and only if there is a unitary $u \in A$ fow which $\sigma(a) = \sigma'(uau^*)$.*

In other words, unitarily equivalent irreducible representations are 'internally' unitarily equivalent. The proof follows from a version of Kadison's transitivity theorem; it will be set as an exercise.

(4.30) PROPOSITION: *Two pure product states $\sigma_{\mathbf{q}}$ and $\sigma_{\mathbf{q}'}$ define unitarily equivalent irreducible representations if and only if the sequences $\mathbf{q}$ and $\mathbf{q}'$ agree except for finitely many terms.*

(4.31) COROLLARY: *Let A be a UHF algebra. Then the unitary dual $\widehat{A}$ is uncountable, but has the indiscrete topology.*

Indeed, the Proposition shows that a UHF algebra has uncountably many inequivalent unitary representations; but since it is simple the primitive ideal space is trivial, and hence $\widehat{A}$ has the indiscrete topology by Theorem 2.31.

PROOF: If the sequences $\mathbf{q}$ and $\mathbf{q}'$ agree for subscripts $i > n$, say, then the associated pure states define two different irreducible representations of $M_{k_n}(\mathbb{C})$. Any two irreducible representations of a full matrix algebra are equivalent, so there is a unitary $u \in M_{k_n}(\mathbb{C})$ such that $\sigma_{\mathbf{q}}(a) = \sigma\mathbf{q}'(u^*au)$ for all $a \in M_{k_n}(\mathbb{C})$. This equality extends to all $a \in \mathcal{A}$ by construction, and then it extends to A by continuity. So the associated irreducible representations of A are unitarily equivalent.

Conversely, suppose that $\sigma_{\mathbf{q}}(a) = \sigma\mathbf{q}'(u^*au)$ for some unitary u. Find an element $x \in M_{k_n}(\mathbb{C})$ (n sufficiently large) with $\|u - x\| < \frac{1}{2}$. Suppose for a contradiction that $q_j = 1$, $q'_j = 2$ for some $k > n$, and let $e \in M_{k_j}(\mathbb{C})$ be of the form $1 \otimes e_{11}$, where e_{11} is the matrix unit in $M_{d_j}(\mathbb{C})$. Now $\sigma'(e) = 0$, so $\sigma'(x^*ex) = \sigma'(x^*xe) = \sigma'(x^*x)\sigma'(e) = 0$ using the multiplicative property of product states and the fact that e commutes with x. Now write

$$1 = \sigma(e) = \sigma'(u^*eu) \leqslant |\sigma'((u^* - x^*)eu)| + |\sigma'(x^*e(u - x))| + |\sigma'(x^*ex)| < 1$$

to obtain a contradiction. ■

Irreducible representations are (of course!) factorial. However UHF algebras also possess factorial representations that are not irreducible, In fact we have

(4.32) PROPOSITION: *Each product state of a UHF algebra is factorial.*

PROOF: Let σ be a product state of the UHF algebra A and let $(\rho_\sigma, H_\sigma, u_\sigma)$ be a cyclic representation associated with σ via the GNS construction. Suppose that $z \in \mathfrak{Z}(\rho_\sigma[A]'')$. By the Kaplansky Density Theorem 1.46, there is a sequence y_j in A with $y_j \to z$ weakly and $\|y_j\| \leqslant \|z\|$ for all j. Fix n, and let z_j be obtained from y_j be averaging over the unitary group of $M_{k_n}(\mathbb{C}) \subseteq A$, as in the proof of Lemma 4.27. For each fixed such unitary u we have

$$\rho(uy_ju^*) \to \rho(u)z(u^*) = z$$

in the weak topology, boundedly in norm; so by applying the Dominated Convergence Theorem we see that $\rho(z_j) \to z$ weakly. Since z_j commutes with $M_{k_n}(\mathbb{C})$ we have for $x, y \in M_{k_n}(\mathbb{C})$,

$$\langle z[x], [y]\rangle = \lim\langle \rho(z_j)[x], [y]\rangle = \lim \sigma(y^*z_jx) = \sigma(y^*x)\sigma(z) = \langle [x], [y]\rangle\sigma(z)$$

using Lemma 4.27. Since this holds for all n, it follows that $z = \sigma(z)1$ is a scalar multiple of the identity. Thus $\rho_\sigma A''$ is a factor. ■

4.5 The Classification of Factors

We now know how to build a great variety of factors from product states of UHF algebras. But how are these factors to be distinguished? In this section we will briefly outline the classification of factors that was developed by Murray and von Neumann.

(4.33) DEFINITION: *Let M be a von Neumann algebra. A* weight *on M is a function $\varphi\colon M_+ \to [0, \infty]$ (notice that the value ∞ is permitted) which is positive-linear in the sense that*

$$\varphi(\alpha_1x_1 + \alpha_2x_2) = \alpha_1\varphi(x_1) + \alpha_2\varphi(x_2)$$

for $\alpha_1, \alpha_2 \in \mathbb{R}^+$ and $x_1, x_2 \in M_+$.

The canonical example is the usual trace on $\mathfrak{B}(H)$. We say that φ is *semi-finite* if the domain

$$M_+^\varphi = \{x \in M_+ : \varphi(x) < \infty\}$$

is weakly dense in M_+, and that φ is *normal* if

$$\lim \varphi(x_\lambda) = \varphi(\lim x_\lambda)$$

whenever x_λ is a monotone increasing net in M that is bounded above (and hence weakly convergent). If $\varphi(u^*xu) = \varphi(x)$ for all $x \in M_+$ and unitary u we say that φ is a *tracial weight*, or sometimes just a *trace* (so long as we remember that it does not have to be bounded).

bbbb

	Finite faithful normal trace	Semifinite faithful normal trace	No faithful normal trace
Minimal projection	Type I_n	Type I_∞	Impossible
No minimal projection	Type II_1	Type II_∞	Type III

5 Hilbert Modules

This section represents a discussion which was not delivered in the course. I want to give a brief introduction to the theory of *Hilbert modules* over C^*-algebras, and to some of their applications.

5.1 Basic Definitions

Let A be a C^*-algebra. A *Hilbert module* over A is a right A-module M equipped with an A-valued $\mathbb{C}$-sesquilinear 'inner product'

$$\langle \cdot, \cdot \rangle M \times M \to A$$

satisfying the following axioms analogous to the usual ones for a Hilbert space:

(i) $\langle x, ya + y'a' \rangle = \langle x, y \rangle a + \langle x, y' \rangle a'$, for all $x, y, y' \in M$ and $a, a' \in A$;

(ii) $\langle x, y \rangle = \langle y, x \rangle^*$;

(iii) $\langle x, x \rangle \geqslant 0$ (the inequality in terms of the order on A_{sa});

(iv) If $\langle x, x \rangle = 0$ then $x = 0$;

(v) M is complete in the norm $\|x\|_M = \|\langle x, x \rangle\|_A^{\frac{1}{2}}$ (we will prove in a moment that, given (i) through (iv), this really is a norm).

Obviously, A is a Hilbert module over itself, with inner product $\langle x, y \rangle = x^* y$. The basic commutative examples are sections of bundles. Indeed, let $A = C(X)$ and let V be a Hermitian vector bundle over the compact space X. Let M be the space of continuous sections of V, and let $\langle \rangle \colon M \times M \to A$ be the fiberwise inner product. Then M is a Hilbert A-module. Later, we will see that all *finitely generated* Hilbert A-modules are of this form.

(5.1) REMARK: General Hilbert modules over $C(X)$ can be thought of as made up of sections of 'generalized bundles' over X. Indeed, suppose that M is a Hilbert $C(X)$-module. Then, for each fixed $p \in X$, $\langle x, y \rangle_p = \langle x, y \rangle(p)$ gives a positive semidefinite sesquilinear form on M (a 'pre-inner-product') and completing we obtain a Hilbert space M_p. Thus $p \mapsto M_p$ assigns to each point of X a Hilbert space and M provides a space of 'continuous sections' of $p \mapsto M_p$, whose pointwise inner products are continuous and whose restrictions to each fiber are dense in that fiber. This data is said to describe a *continuous field of Hilbert spaces*. Conversely, the sections of a continuous field of Hilbert spaces form a Hilbert $C(X)$-module. For an example of such a continuous field that is not a vector bundle, consider the space $M = C_0(0,1]$ as a Hilbert module over $A = C[0,1]$. This space is the space of sections of a field of Hilbert spaces with fiber 0 over 0 and $\mathbb{C}$ over the other points of the interval. Note that M is not finitely generated over A.

Return to the proof that the norm $\|x\|_M$ defined above really is a norm. As in the classical case this will follow from a Cauchy–Schwarz inequality.

(5.2) LEMMA: *Let M be a right A-module satisfying (i)–(iii) above and suppose that $x, y \in M$. Then we have the inequality (in A)*

$$\langle x, y\rangle^* \langle x, y\rangle \leqslant \|\langle x, x\rangle\| \langle y, y\rangle.$$

Consequently, with the definition of $\|\cdot\|_M$ in (v) above we have

$$\|\langle x, y\rangle\|_A \leqslant \|x\|_M \|y\|_M,$$

and it follows easily that $\|\cdot\|_M$ is indeed a norm.

PROOF: Imitate the usual proof, as follows: consider the inequalities in A_{sa}

$$\begin{aligned} 0 &\leqslant \langle xa + y, xa + y\rangle \\ &= a^* \langle x, x\rangle a + a^* \langle x, y\rangle + \langle y, x\rangle a + \langle y, y\rangle \\ &\leqslant \|\langle x, x\rangle\| a^* a + a^* \langle x, y\rangle + \langle y, x\rangle a + \langle y, y\rangle \end{aligned}$$

where we have used Proposition 1.22. Putting $a = \lambda \langle x, y\rangle$, $\lambda \in \mathbb{C}$, we get

$$\left(\lambda^2 \|\langle x, x\rangle\| + 2\lambda\right) \langle x, y\rangle^* \langle x, y\rangle + \langle y, y\rangle \geqslant 0.$$

If $\langle x, x\rangle = 0$, then taking λ large and negative in this inequality we see that $\langle x, y\rangle = 0$ also. Otherwise, put $\lambda = -\|\langle x, x\rangle\|^{-1}$ to get Cauchy-Schwarz. ■

Note that the proof of Cauchy-Schwarz only requires (i)–(iii). It follows that given an A-module M satisfying (i)–(iii), the set N of vectors of norm zero forms a submodule, and dividing by that N we obtain a module M/N satisfying (i)–(iv). Then by completing we may obtain a Hilbert module. This is the analog for Hilbert modules of a construction used in the GNS process.

EXAMPLE: An important example of a Hilbert A-module is the 'universal' module H_A which is comprised of sequences $\{a_n\}$ of elements of A such that $\sum a_n^* a_n$ converges in A. Using the Cauchy-Schwarz inequality it is not hard to show that this is a Hilbert module. Note the convergence condition carefully however: it is not equivalent to say that $\sum \|a_n\|^2$ converges (a series of positive elements of a C^*-algebra can converge in norm without converging absolutely, for instance the series $\sum \frac{1}{n} e_n$, where e_n is the orthogonal projection onto the n'th basis vector, converges in norm but not absolutely in $\mathfrak{K}(\ell^2)$.

The above example is a special case of a general notion of infinite direct sum of Hilbert modules.

(5.3) LEMMA: *Let M be a Hilbert A-module. Then MA is dense in M. In fact $M\langle M, M\rangle$ is dense in M, where $\langle M, M\rangle$ denotes the subspace of A spanned by inner products.*

Notice that $\langle M, M\rangle$ need not be dense in A (obvious examples). We say that M is *full* if this is so.

PROOF: The closure of $\langle M, M\rangle$ is a C^*-ideal J in A. Let u_α be an approximate unit for J. Then for $x \in M$

$$\langle x - x_\alpha, x - x_\alpha\rangle = \langle x, x\rangle - u_\alpha\langle x, x\rangle - \langle x, x\rangle u_\alpha + u_\alpha\langle x, x\rangle u_\alpha \to 0$$

and thus $x_\alpha \to x$, giving the result. ■

(5.4) REMARK: As well as the ordinary norm on a Hilbert A-module M, we may define an 'A-valued norm' $|x| = \langle x, x\rangle^{\frac{1}{2}}$ (the square root being implemented by functional calculus in A). From Cauchy–Schwarz one easily deduces the inequalities

$$|\langle x, y\rangle| \leqslant \|x\||y|, \quad |xa| \leqslant \|x\||a|.$$

However the A-valued norm shares the odd properties of the absolute value on a C^*-algebra, e.g. it isn't true always that $|x + y| \leqslant |x| + |y|$.

In the theory of Hilbert space a critical role is played by the existence of orthogonal complements to closed subspaces. If M is a Hilbert space (or Hilbert module) and N a subspace, the *orthogonal* subspace is

$$N^\perp = \{y \in M : \langle x, y\rangle = 0\ \forall x \in N\}.$$

The *Projection Theorem* states that if M is a Hilbert *space* then $N^\perp$ is *complementary* to N; each $x \in M$ can be decomposed uniquely into a component in N and a component in $N^\perp$. The Projection Theorem however fails for Hilbert modules. For instance if $A = C[0, 1]$, $M = A$, and $N = C_0(0, 1]$ considered as a Hilbert A-module, then $N^\perp = \{0\}$ and we have no direct sum decomposition of M into $N \oplus N^\perp$. This is the fundamental difficulty of Hilbert module theory.

(5.5) DEFINITION: *Let M and N be Hilbert A-modules. An* adjointable map *from M to N is a linear map $T\colon M \to N$ for which there exists an* adjoint *$T^*\colon N \to M$, necessarily unique, satisfying*

$$\langle Tx, y\rangle = \langle x, T^*y\rangle, \quad x \in M,\ y \in N.$$

The set of adjointable maps will be denoted $\mathfrak{B}(M, N$, or $\mathfrak{B}(M)$ if $M = N$.

Because of the failure of the Projection Theorem, it need not be the case that a norm bounded linear map $M \to N$ is adjointable. On the other hand, an adjointable map must be bounded (a Uniform Boundedness Principle argument).

(5.6) PROPOSITION: *Let M be a Hilbert A-module. Then $\mathfrak{B}(M)$ is a C^*-algebra.*

PROOF: $\mathfrak{B}(M)$ is clearly a normed $*$-algebra. The C^*-identity can be proved by mimicking the Hilbert space proof: for $x \in M$, $\|x\| = 1$,

$$\|Tx\|^2 = \langle Tx, Tx\rangle = \langle x, T^*Tx\rangle \leqslant \|T^*T\|.$$

The supremum of the left side over all possible x is $\|T\|^2$, giving the result. It follows as usual that $\|T\| = \|T^*\|$ for all $T \in \mathfrak{B}(M)$, and so we see that $\mathfrak{B}(M)$ is a closed subspace of the Banach algebra of all bounded $\mathbb{C}$-linear maps $M \to M$, and therefore is complete. ■

It is just a restatement of the definition to say that if $T \in \mathfrak{B}(M, N)$ and $x \in M$ then $\|Tx\| \leqslant \|T\|\|x\|$. In fact, however, there is a similar inequality for the A-valued norm (think of T as operating 'fiberwise' on sections of a vector bundle.) To prove it, we use states of A to reduce to the scalar case — a device which could also have been used in some earlier arguments.

(5.7) PROPOSITION: *Let M and N be Hilbert A-modules, and let $T\colon M \to N$ be an adjointable operator. Then*

$$|Tx| \leqslant \|T\||x|$$

for all $x \in M$.

PROOF: Let σ be a state of A. We have $\sigma(|Tx|^2) = \sigma(\langle T^*Tx, x\rangle)$. A simple induction using the Cauchy-Schwarz inequality for the positive sesquilinear form $\sigma(\langle\cdot,\cdot\rangle)$ gives

$$\sigma(\langle T^*Tx, x\rangle) \leqslant \sigma(\langle (T^*T)^{2^n}x, x\rangle)^{2^{-n}}\sigma(|x|^2)^{1-2^{-n}}.$$

As $n \to \infty$ we obtain

$$\sigma(\langle T^*Tx, x\rangle) \leqslant \|T\|^2\sigma(|x|^2)$$

and since this holds for all states σ we get $|Tx|^2 \leqslant \|T\|^2|x|^2$, whence the result. ■

Let E and F be Hilbert A-modules. A *rank one operator* from E to F is a linear map $E \to F$ of the form

$$\theta_{x,y}(z) = x\langle y, z\rangle, \quad x \in F, y \in E.$$

It is adjointable, with adjoint $\theta_{y,x}$. The closed linear span of rank one operators is denoted $\mathfrak{K}(E, F)$ and called the space of *compact* operators from E to F. (Warning: They need not be compact in the sense of Banach space theory!) If $E = F$, the subspace $\mathfrak{K}(E)$ of compact operators in $\mathfrak{B}(E)$ is a C^*-ideal.

EXAMPLE: If A is a unital C^*-algebra, and we consider A as a Hilbert module over itself, the it is easy to see that $\mathfrak{B}(A) = \mathfrak{K}(A) = A$. If A is non-unital then $\mathfrak{B}(A)$ is much bigger than A (in fact it is the so-called *multiplier algebra* of A, see later), but it is still the case that $\mathfrak{K}(A) = A$. To see this, choose an approximate unit $\{u_\lambda\}$ for A and define an isomorphism $A \to \mathfrak{K}A$ by sending $a \in A$ to $\lim \theta_{a,u_\lambda}$, having first checked of course that the net appearing on the right is Cauchy.

Note that $\mathfrak{B}(E)$ is not a von Neumann algebra in general.

(5.8) DEFINITION: *The* strict topology *on* $\mathfrak{B}(E)$ *is the topology induced by the seminorms* $T \mapsto \|Tx\|$, $T \mapsto \|T^*y\|$ *for all* $x, y \in E$. *(Essentially the strong-*$*$ *topology.)*

Let E be a Hilbert A-module and let u_λ be an approximate unit for $\mathfrak{K}(E)$. If $x \in E\langle E, E\rangle$ then x can be written as a linear combination of images of rank one operators and so it is simple to see that $u_\lambda x \to x$. From Lemma 5.3 we conclude that $u_\lambda x \to x$ for all $x \in E$. Therefore $Tu_\lambda \to T$ strictly for all $T \in \mathfrak{B}(E)$ and we conclude:

(5.9) PROPOSITION: *Let E be a Hilbert A-module. The unit ball of $\mathfrak{K}(E)$ is strictly dense in the unit ball of $\mathfrak{B}(E)$.*

6 Exercises

6.1 Exercises Sep 8, due Sep 15

EXERCISE 6.1: Let A be the *disk algebra*, that is the Banach algebra (with pointwise operations and supremum norm) of continuous functions on the closed unit disk $\overline{\mathbb{D}}$ in the complex plane which are holomorphic in the interior of $\mathbb{D}$. Show that the operation

$$f^*(z) = \overline{f(\bar{z})}$$

defines an involution on A which makes it into a commutative Banach $*$-algebra in which not every selfadjoint element has real spectrum.

EXERCISE 6.2: Show that if λ is an isolated point in the spectrum of a normal operator T then λ is an eigenvalue of T. Is this true if T is not normal?

EXERCISE 6.3: Let X be a locally compact Hausdorff space, and let $C_b(X)$ denote the algebra of bounded, continuous, complex-valued functions on X. Show that $C_b(X)$ is a commutative C^*-algebra. Deduce that $C_b(X) = C(Z)$, where Z is a compact Hausdorff space containing X as a dense open subset (a *compactification* of X). Prove moreover that Z is the universal compactification, in the sense that if Z' is another compactification of X then there is a commutative diagram

$$\begin{array}{cc} X & Z \\ & \\ & Z' \end{array}$$

where the vertical arrow is a uniquely determined surjection. (The space Z is called the *Stone–Čech compactification* of X.)

EXERCISE 6.4: Let $A = C(X)$ be a commutative and unital C^*-algebra. If Z is a closed subset of X then the set of those functions in A whose restriction to Z is zero constitutes an ideal in A. Prove that every ideal arises in this way.

EXERCISE 6.5: Let A be a C^*-algebra.

(a) Let $x, a \in A$ with $a \geqslant xx^* \geqslant 0$. Show that the elements

$$\left[\tfrac{1}{j} + a\right]^{-\frac{1}{2}} a^{\frac{1}{4}} x, \qquad j = 1, 2, \ldots$$

form a Cauchy sequence in A which converges to an element $b \in A$ with $x = a^{\frac{1}{4}} b$ and $\|a\|^{\frac{1}{4}} \geqslant \|b\|$.

Now let $\pi\colon A \to \mathfrak{B}(H)$ be a representation.

(b) Show that for any $v \in \overline{\pi[A]H}$ and any $\varepsilon > 0$, one can find x in the unit ball of A such that $\|v - \pi(x)v\| < \varepsilon$. (Use an approximate unit.)

(c) Given $v \in \overline{\pi[A]H}$, use part (b) above to define by induction a sequence $\{x_n\}$ of elements in the unit ball of A and a sequence $\{v_n\}$ of elements of $\overline{\pi[A]H}$ such that $v_1 = v$ and

$$v_n = v_{n-1} - \pi(x_{n-1})v_{n-1}, \quad \|v_n\| \leqslant 4^{-n}.$$

for $n > 1$.

(d) Show that $\sum \pi(x_n)v_n$ converges to v.

(e) Let $a = \sum 4^{-n} x_n x_n^*$. Using part (a) above write $2^{-n}x_n = a^{\frac{1}{4}} b_n$ with $\|b_n\|$ bounded. Show that

$$v = \pi(a^{\frac{1}{4}}) \sum \pi(b_n)\,(2^n v_n)$$

where the series converges in norm, and deduce that $v \in \pi[A]H$.

Deduce that $\pi[A]H$ is always closed in H.

6.2 Exercises Sep 22, due Sep 29

EXERCISE 6.6: Let A be a C^*-algebra. Show that the two by two matrix algebra $M_2(A)$ can be made into a C^*-algebra also. (Hint: Use the Gelfand–Naimark Representation Theorem.)

EXERCISE 6.7: Show that every irreducible representation of a commutative C^*-algebra is one-dimensional.

EXERCISE 6.8: Let T be a normal operator on a Hilbert space H and let E be the associated spectral measure. Show that the projection $E(D)$ corresponding to the open unit disc D has range equal to the subspace

$$\{v \in H : \lim_{n\to\infty} \|T^n v\| = 0\}.$$

Deduce that if a bounded operator S commutes with T, then it commutes with $E(D)$. Argue further that S commutes with $E(U)$ for every open disk $U \subset \mathbb{C}$. Deduce finally that S commutes with T^* (*Fuglede's Theorem*).

EXERCISE 6.9: Let $\mathfrak{F}$ be the vector space of finite-rank operators from H to H', where H and H' are Hilbert spaces. Show that an inner product may be defined on $\mathfrak{F}$ by

$$\langle S, T\rangle_{HS} = \mathrm{Trace}(S^*T)$$

where the trace of a finite-rank operator is the sum of its diagonal matrix entries relative to an orthonormal basis. This inner product is called the *Hilbert–Schmidt* inner product.

Let $\|\cdot\|_{HS}$ denote the corresponding norm. Prove that

$$\|T\| \leqslant \|T\|_{HS}$$

for any $T \in \mathfrak{F}$. Deduce that the completion of $\mathfrak{F}$ with respect to the Hilbert–Schmidt inner product may be identified with a certain space of bounded operators from H to H'. These are called the *Hilbert–Schmidt operators*.

Prove that every Hilbert–Schmidt operator is compact, and give an example of a compact operator which is not Hilbert–Schmidt. If either H or H' is of finite dimension n, prove that every bounded operator is Hilbert–Schmidt and that $\|T\|_{HS} \leqslant n^{\frac{1}{2}}\|T\|$.

EXERCISE 6.10: An operator $T \in \mathfrak{B}(H)$ is said to be of *trace class* if it can be written $T = T_1T_2$, where T_1 and T_2 are Hilbert–Schmidt operators. Show that if this is so, then the number

$$\mathrm{Trace}(T) = \langle T_1^*, T_2\rangle_{HS}$$

depends only on T, and that

$$\mathrm{Trace}(ST) = \mathrm{Trace}(TS)$$

for all bounded operators S.

Show that $T \mapsto \mathrm{Trace}\,|T|$ defines a norm on the space of trace-class operators which makes it into a Banach space, and that the dual of this Banach space may be identified with $\mathfrak{B}(H)$. Deduce that *every* von Neumann algebra is isomorphic as a Banach space to the dual of some Banach space.

Here is some discussion of the last exercise, which could be put into the text of the notes later. We take up the story at the beginning of the second paragraph of the last exercise above.

(6.11) LEMMA: *Suppose that* $S \in \mathfrak{B}(H)$ *and that* T *is of trace class. Then* $|\,\mathrm{Trace}(ST)| \leqslant \|S\|\,\mathrm{Trace}\,|T|$.

PROOF: Let $T = V|T|$ be the polar decomposition. Think of $\mathrm{Trace}(ST)$ as the Hilbert-Schmidt inner product of $SV|T|^{\frac{1}{2}}$ and $|T|^{\frac{1}{2}}$, and use the Cauchy-Schwarz inequality to get

$$|\,\mathrm{Trace}(ST)|^2 \leqslant \mathrm{Trace}(|T|^{\frac{1}{2}}V^*S^*SV|T|^{\frac{1}{2}})\,\mathrm{Trace}(|T|) \leqslant \|S\|^2\,\mathrm{Trace}(|T|)^2$$

where in the second step we used the inequality of positive operators $|T|^{\frac{1}{2}}V^*S^*SV|T|^{\frac{1}{2}} \leqslant \|S^*S\||T|^{\frac{1}{2}}V^*V|T|^{\frac{1}{2}} = \|S^*S\||T|$. ■

(6.12) LEMMA: *If* S *and* T *are trace-class then* $\mathrm{Trace}(|S+T|) \leqslant \mathrm{Trace}\,|S| + \mathrm{Trace}\,|T|$.

PROOF: Let $S + T = V|S+T|$ be a polar decomposition. Then

$$\mathrm{Trace}\,|S+T| = \mathrm{Trace}(V^*S) + \mathrm{Trace}(V^*T) \leqslant \mathrm{Trace}\,|S| + \mathrm{Trace}\,|T|$$

where we used the previous lemma for the second step. ■

This shows that the trace-norm is a norm. Since by spectral theory $\mathrm{Trace}\,|T|$ is the sum of the eigenvalues of $|T|$, whereas $\|T\|$ is the supremum of these eigenvalues, it is clear that $\mathrm{Trace}\,|T| \geqslant \|T\|$. It now follows that the space $\mathfrak{L}^1(H)$ of trace-class operators on H is a Banach space of compact operators, just as in the Hilbert-Schmidt case above (by the way, the space of Hilbert-Schmidt operators is sometimes denoted $\mathfrak{L}^2(H)$.)

A map Φ from $\mathfrak{B}(H)$ to $\mathfrak{L}^1(H)^*$ is defined by assigning to a bounded operator S the linear functional

$$\Phi(S) : T \mapsto \mathrm{Trace}(ST)$$

on $\mathfrak{L}^1(H)$. It is easy to check that this is a continuous, isometric, linear map. We need to see that it is also surjective, which is to say that every linear functional on $\mathfrak{L}^1(H)$ is of this sort. Thus let $\varphi \in \mathfrak{L}^1(H)^*$. For every finite-rank projection P there is a trace-class (indeed finite-rank) operator $S_P = S_P P$ with $\|S_P\| \leqslant \|\varphi\|$ such that

$$\varphi(PT) = \mathrm{Trace}(S_P T)$$

(for instance one can use the Riesz representation theorem in the Hilbert space $\mathfrak{L}^2(H)$ to see this). The S_P form a net parameterized by the space of finite rank projections. Since closed balls in $\mathfrak{B}(H)$ are weakly compact, there is a subnet of the S_P that converges weakly to a bounded operator S, with $\|S\| \leqslant \|phi\|$. Now it is easy to verify that for any trace-class operator T the functional $X \mapsto \mathrm{Trace}(XT)$ is weakly continuous on bounded subsets of $\mathfrak{B}(H)$. Consequently,

$$\mathrm{Trace}(ST) = \lim \mathrm{Trace}(S_P T) = \lim \varphi(PT) = \varphi(T)$$

since $PT \to T$ in trace norm.

The *ultraweak topology* on $\mathfrak{B}(H)$ is the weak-$*$ topology that it inherits from its representation as the dual of $\mathfrak{L}^1(H)$. Confusingly, the ultraweak topology is *stronger* than the weak topology. We have already noted above that the weak and ultraweak topologies agree on bounded subsets of $\mathfrak{B}(H)$. Since according to Kaplansky's density theorem a C^*-subalgebra of $\mathfrak{B}(H)$ is a von Neumann algebra iff its unit ball is weakly closed, we see:

(6.13) PROPOSITION: *A C^*-subalgebra $M \subseteq \mathfrak{B}(H)$ is a von Neumann algebra iff it is ultraweakly closed.* ■

(6.14) COROLLARY: *Every von Neumann algebra is isomorphic, as a Banach space, to the dual of a Banach space.*

PROOF: Let $M \subseteq \mathfrak{B}(H)$ be a von Neumann algebra. Let $N \leqslant \mathfrak{L}^1(H)$ be its preannihilator, that is, $N = \{T \in \mathfrak{L}^1(H) : \mathrm{Trace}(ST) = 0 \,\forall\, S \in M\}$. N is a closed subspace in a Banach space, and is a M is weak-star closed subspace of the dual. By duality theory in Banach spaces, M is the dual of the quotient space $\mathfrak{L}^1(H)/N$. ■

6.3 Exercises Oct 6, due Oct 20

EXERCISE 6.15: Show that every finite-dimensional C^*-algebra has a faithful representation on a finite-dimensional Hilbert space.

Solution: By the nature of the GNS construction, the GNS representation associated to any state on a finite-dimensional C^*-algebra A is itself a finite-dimensional representation. The intersection of the kernels of all the GNS representations of A is zero; hence (by dimension counting) the intersection of finitely many of these kernels is zero. It follows that there are finitely many GNS representations (each finite-dimensional) whose direct sum is faithful.

EXERCISE 6.16: Let ρ_1, ρ_2 be irreducible representations of a C^*-algebra A on Hilbert spaces H_1, H_2. Show that $(\rho_1 \oplus \rho_2)[A]' = \rho_1[A]' \oplus \rho_2[A]'$ if and only if ρ_1 and ρ_2 are *not* unitarily equivalent.

Solution: In general $(\rho_1 \oplus \rho_2)[A]'$ consists of matrices

$$\begin{bmatrix} W & X \\ Y & Z \end{bmatrix}$$

where $W \in \rho_1[A]'$, $Z \in \rho_2[A]'$, and X and Y are intertwining operators for ρ_1 and ρ_2, which is to say that

$$\rho_1(a)X = X\rho_2(a), \quad \rho_2(a)Y = Y\rho_1(a).$$

Since ρ_1 and ρ_2 are irreducible, W and Z are multiples of the identity. Moreover, X^*X and XX^* are multiples of the identity on H_2 and H_1 respectively, say $X^*X = \lambda I_2$, $XX^* = \lambda I_1$. If $\lambda \neq 0$, then $\lambda > 0$ by positivity and $\lambda^{-\frac{1}{2}}X$ is a unitary equivalence between ρ_1 and ρ_2.

EXERCISE 6.17: Show that $\mathfrak{T}$ is the universal C^*-algebra generated by an isometry.

Solution: Suppose that W is an isometry on a Hilbert space H, so that $W^*W = I$. Let $K \leqslant H$ be the closed subspace $\operatorname{Ker} W^*$. For $n > m \geqslant 0$ the subspaces W^nK and W^mK are orthogonal, since for $u, v \in K$

$$\langle W^n u, W^m v\rangle = \langle W^{n-m}u, v\rangle = \langle W^{n-m-1}u, W^*v\rangle = 0.$$

Thus we have an orthogonal decomposition

$$H = L \oplus L^\perp, \qquad L = K \oplus WK \oplus W^2K \oplus \cdots .$$

The operators W and W^* map L to itself and thus also map $L^\perp$ to itself. The restriction of W to L is unitarily equivalent to the direct sum of $p = \dim K$ copies of the unilateral shift, and the restriction of W to $L\perp$ is an isometry with zero kernel, that is a unitary. We have proved **Wold's theorem**: any isometry on a Hilbert space is unitarily equivalent to a direct sum of copies of the unilateral shift, plus a unitary. We write $W \equiv V^p \oplus U$.

Now we must define a $*$-homomorphism from the Toeplitz algebra $\mathfrak{T}$ to the C^*-algebra generated by W, which sends V to W. From the Wold decomposition of W above we see

$$C^*(W) = \mathfrak{T}^p \oplus C(\operatorname{sp} U).$$

where $\operatorname{sp} U \subseteq S^1$ is the spectrum of the unitary part U of W. Our $*$-homomorphism is the direct sum of p copies of the identity map $\mathfrak{T} \to \mathfrak{T}$ together with the map

$$\mathfrak{T} \to C(S^1) \to C(\operatorname{sp} U)$$

where the first arrow of the display is the symbol map of the Toeplitz exact sequence and the second is the restriction map. It is clear that this $*$-homomorphism has the desired property $V \mapsto W$, and since the Toeplitz algebra is generated by V, the $*$-homomorphism is uniquely determined by this property.

EXERCISE 6.18: Use the spectral theorem to prove *Stone's theorem:* if π is a strongly continuous unitary representation of the group $\mathbb{R}$, then there is a spectral measure E on the Borel subsets of $\hat{\mathbb{R}}$ such that

$$\pi(t) = \int_{\hat{\mathbb{R}}} e^{i\xi t}\, dE(\xi).$$

(Here $\hat{R}$ just denotes another copy of $\mathbb{R}$, one should think of it as the dual group in the sense of Pontrjagin's theory of locally compact abelian groups.)

Solution: By the general theory that we have discussed, a representation π of the group $\mathbb{R}$ is the same thing as a non-degenerate representation of the C^*-algebra $C^*(\mathbb{R})$, which in turn can be identified with the commutative C^*-algebra $C_0(\hat{R})$. According to the Spectral Theorem, given any non-degenerate representation ρ of a commutative C^*-algebra $C_0(X)$ on a Hilbert space H, there is a spectral measure E on X (with values in the projections of H) such that

$$\rho(f) = \int_X f(x)\, dE(x)$$

and substituting in the explicit form of the duality between $\mathbb{R}$ and $\hat{\mathbb{R}}$ we obtain Stone's theorem as given above.

EXERCISE 6.19: Let A denote the maximal C^*-algebra of the free group F_2, and let $\rho\colon A \to \mathfrak{B}(H)$ be the universal representation. Let B be the algebra of continuous operator-valued functions $T\colon [0,1] \to \mathfrak{B}(H)$ such that $T(0)$ is a multiple of the identity. Show that there exists a $*$-homomorphism

$$\alpha\colon A \to B$$

whose composite with evaluation at $t = 1$ is the representation ρ. Deduce that A contains no projections other than 0 and 1.

Solution: Let $u, v \in A$ be the elements corresponding to the generators of the free group and let $U = \rho(u)$, $V = \rho(v)$ be the corresponding operators on H. By the Borel functional calculus there exist selfadjoint operators X and Y on H such that

$$U = \exp(2\pi i X), \quad V = \exp(2\pi i Y).$$

Let U' and V' be the unitary elements of B that are given by the continuous operator-valued functions

$$U'(t) = \exp(2\pi i t X), \quad V'(t) = \exp(2\pi i t Y).$$

Notice that $U'(0) = I$, $U'(1) = U$, and similarly for V'. By the universal property of the free group C^*-algebra there exists a $*$-homomorphism $\alpha\colon A \to B$ taking u to U' and v to V', and at $t = 1$, α is the representation ρ. Now suppose that $p \in A$ is a projection. Then $\alpha(p) = P' \in B$ is a projection-valued function. In particular the scalar $P'(0)$ must be either 0 or 1. But the point 0 is isolated in the space of projections (because every nonzero projection has norm 1) and, dually, the point 1 is isolated in the space of projections also. Thus P' is a constant function with value 0 or 1 and, evaluating at $t = 1$, we find that $\rho(p)$ equals 0 or 1, as required.

6.4 Exercises Oct 27, due Nov 3

EXERCISE 6.20: Let A be a unital C^*-algebra, and let $S = S(A)$ be its state space. Show that the convex hull of $S \cup (-S)$ is the space of all real linear functionals on A of norm $\leqslant 1$. (Hint: Use the Hahn-Banach Theorem as in the proof of Lemma 2.34.)

Solution — The space of real linear functionals on A is the dual of the real Banach space A_{sa}. The convex hull K of $S \cup (-S)$ is a weak-$*$ closed convex set in A^*_{sa}. Suppose, if possible, that ψ is a real linear functional of norm 1 not belonging to K. Then by the Hahn–Banach theorem there exists $a \in A_{\mathrm{sa}}$ and $\alpha \in \mathbb{R}$ with $\psi(a) > \alpha$ and $\varphi(a) \leqslant \alpha$ for all $\varphi \in K$. The latter condition implies in particular that $|\sigma(a)| \leqslant \alpha$ for all $\sigma \in S$. But $\|a\| = \sup\{|\sigma(a)| : \sigma \in S\}$ by Lemma 2.10 and the remark following, so we get $\psi(a) > \|a\|$, a contradiction.

EXERCISE 6.21: Show that the following conditions are equivalent for positive linear functionals φ and ψ on a (unital) C^*-algebra A:

(a) $\|\varphi - \psi\| = \|\varphi\| + \|\psi\|$;

(b) For each $\varepsilon > 0$ there is a positive a in the unit ball of A with $\varphi(1 - a) < \varepsilon$ and $\psi(a) < \varepsilon$.

(One then says that φ and ψ are *orthogonal*. What does this condition say when $A = C(X)$?)

Solution — (When $A = C(X)$ this says that the measures defined by φ and ψ have essentially disjoint supports.)

Suppose (a). Then given $\varepsilon > 0$ there is a selfadjoint a of norm at most 1 such that

$$\varphi(a) - \psi(a) + \varepsilon \geqslant \|\varphi - \psi\| = \varphi(1) + \psi(1).$$

Therefore

$$\varphi(1 - a) + \psi(1 + a) < \varepsilon.$$

Choose $z = \frac{1}{2}(1 + a)$.

Suppose (b). We reverse the above argument. We have

$$\|\varphi-\psi\| \leqslant \|\varphi\|+\|\psi\| = \varphi(1)+\psi(1) \leqslant \varphi(2z-1)+\psi(1-2z)+4\varepsilon \leqslant \|\varphi-\psi\|+4\varepsilon$$

using the fact that $\|2z - 1\| \leqslant 1$. Let $\varepsilon \to 0$ to obtain the desired conclusion.

EXERCISE 6.22: Show that every real linear functional φ on a unital C^*-algebra A can be decomposed as $\varphi_+ - \varphi_-$ where $\varphi_\pm$ are mutually orthogonal positive linear functionals.

Can you show the uniqueness of this decomposition? (This is true, but is harder than existence.)

Solution — Suppose that φ has norm one. According to the first exercise we may write φ as a convex combination $\lambda\sigma_1 - (1-\lambda)\sigma_2$ where σ_1, σ_2 are states and $\lambda \in [0,1]$. Put $\varphi_+ = \lambda\sigma_1$ and $\varphi_- = -(1-\lambda)\sigma_2$. We have

$$\|\varphi_+\| + \|\varphi_-\| = \lambda + (1-\lambda) = 1 = \|\varphi\| = \|\varphi_+ - \varphi_-\|,$$

so the decomposition is orthogonal. For the uniqueness see p.45 of Pedersen's book (this is a theorem of Grothendieck).

EXERCISE 6.23: Let G be a discrete group. Suppose that the trivial one-dimensional representation of G extends to a $*$-homomorphism $\varphi\colon C^*_r(G) \to \mathbb{C}$. Show that then G is amenable. (Hint: Using the Hahn-Banach theorem, extend φ to a state of $\mathfrak{B}(\ell^2(G))$, and then restrict to a state of $\ell^\infty(G)$, embedded in $\mathfrak{B}(\ell^2(G))$ as multiplication operators. Show that this process produces an invariant mean on G.)

Solution — Let σ be the state of $\mathfrak{B}(\ell^2(G))$ produced by the process described in the hint. We want to prove that $\sigma(M_{L_g f}) = \sigma(M_f)$, where M_f denotes the multiplication operator by the ℓ^∞ function f. Observing that

$$M_{L_g f} = \lambda(g) M_f \lambda(g)^*,$$

we see that it will suffice to show that $\sigma(T) = \sigma(T\lambda(g))$ for all $g \in G$.

This is a general fact: if A is a unital C^*-algebra with a state σ, and $u \in A$ is unitary such that $\sigma(u) = 1$, then $\sigma(au) = \sigma(a)$ for all $a \in A$. To see this note that

$$\sigma((1-u^*)(1-u)) = 2 - \sigma(u) - \sigma(u^*) = 0$$

so that by Cauchy-Schwarz

$$|\sigma(a(1-u))|^2 \leqslant \sigma(a^*a)\sigma((1-u)^*(1-u)) = 0$$

as required.

EXERCISE 6.24: Let

$$1 \to K \to G \to H \to 1$$

be a short exact sequence of groups. Show that if H and K are amenable, then G is amenable.

Solution — Let $\mathfrak{m}_K$ and $\mathfrak{m}_H$ be invariant means for K and H respectively. Define a unital positive linear map $\Phi\colon \ell^\infty(G) \to \ell^\infty(H)$ by

$$\Phi(f)(h) = \mathfrak{m}_K(k \mapsto f(ks(h))$$

where $s\colon H \to G$ is a set-theoretic section of the quotient map. Because of the invariance of the mean on K, Φ is well-defined independent of the choice of s. Now define an invariant mean on G by

$$f \mapsto \mathfrak{m}_H(\Phi(f)).$$

6.5 Exercises Nov 15, due Nov 27

EXERCISE 6.25: Let G be the free group on two generators and let $a \in \mathbb{C}[G]$ be an element of the algebraic group ring. Let A_a be the subalgebra of $\mathfrak{B}(\ell^2(G))$ generated (algebraically) by $C_r^*(G)$ together with the orthogonal projection P_a onto the kernel of the operator of (left) convolution by a. Show that the Fredholm module over $C_r^*(G)$ that we defined in class extends to a summable Fredholm module over the larger algebra A_a. Deduce that if $a \neq 0$ then the operation of left convolution by a is injective on $\ell^2(G)$ and has dense range. (This is the essential idea of Linnell's proof of the Atiyah conjecture for G.)

Solution (outline). The Fredholm module M over $C_r^*(G)$ that we have defined has the following strong commutation property: if $a \in \mathbb{C}[G]$ then the commutator $[a, F]$ is an operator of *finite rank*. We will show that if $[a, F]$ has finite rank then so does $[P_a, F]$. By replacing a by a^*a we see that it is enough to assume that $a \geqslant 0$.

In this case we may write $P_a = \lim_{\varepsilon \to 0} \varepsilon(a + \varepsilon)^{-1}$, with the limit in the strong operator topology (this follows from the spectral theorem). Consequently,

$$[P_a, F] = \lim_{\varepsilon \to 0} -\varepsilon(a + \varepsilon)^{-1}[a, F](a + \varepsilon)^{-1}$$

with the limit in the strong operator topology. If $[a, F]$ has rank k, the operators appearing in this limit all have rank $\leqslant k$; and since the collection of operators of rank $\leqslant k$ is strongly closed, it follows that $[P_a, F]$ has rank $\leqslant k$. This then establishes that F defines a Fredholm module over the larger algebra A_a. Moreover, the projection P_a lies in the domain of summability of this module.

From our earlier results it now follows that $\tau(P_a)$ is an integer. Notice that τ extends to a faithful trace on A_a, indeed it extends to a faithful trace on the group von Neumann algebra of A. We have $\tau(P_a) = 0$ or $= 1$; in the former case $P_a = 0$ by faithfulness, so a has no kernel as an operator on ℓ^2 and is therefore injective; in the latter case $P_a = 1$, so the kernel of a is the whole of ℓ^2 and thus $a = 0$.

EXERCISE 6.26: Prove that the Toeplitz algebra is not properly infinite.

Solution: If P is a projection in $\mathfrak{T}$, then its symbol $\sigma(P)$ is a projection in $C(S^1)$, and is therefore equal either to 0 or 1. If $\sigma(P) = 0$ then P is compact, and if $\sigma(P) = 1$ then $1 - P$ is compact (call such projections 'cocompact'). No compact projection is equivalent to 1, and two cocompact projections cannot be orthogonal, so it is impossible to write 1 as the sum of two orthogonal projections both of which are equivalent to 1.

EXERCISE 6.27: Show that if A is a simple C^*-algebra, and $a \in A$ is positive and non-zero, then there exist $x_1, \ldots, x_n \in A$ such that $x_1^* a x_1 + \cdots + x_n^* a x_n = 1$.

Solution: Given in the text.

EXERCISE 6.28: Show that $\mathcal{O}_3$ can be embedded as a subalgebra of $\mathcal{O}_2$ (consider the subalgebra generated by s_1, s_2s_1, s_2^2). Generalizing this, show that every Cuntz algebra appears as a subalgebra of $\mathcal{O}_2$.

Solution: Let $t_1 = s_1$, $t_2 = s_2s_1$, $t_3 = s_2^2$ in $\mathcal{O}_2$. These are isometries. We compute

$$\sum_{i=1}^{3} t_i t_i^* = s_1s_1^* + s_2(s_1s_1^* + s_2s_2^*)s_2^* = 1,$$

using (twice) the identity $s_1s_1^* + s_2s_2^* = 1$. Thus t_1, t_2, t_3 generate a copy of $\mathcal{O}_3$ within $\mathcal{O}_2$.

A straighforward generalization of the argument shows that

$$t_1 = s_1, t_2 = s_2s_1, \quad \ldots \quad t_{n-1} = s_2^{n-2}s_1, t_n = s_2^{n-1}$$

generate a copy of $\mathcal{O}_n$ inside $\mathcal{O}_2$.

EXERCISE 6.29: Prove that $M_2(\mathcal{O}_2) \cong \mathcal{O}_2$. Prove that $M_3(\mathcal{O}_2) \cong \mathcal{O}_2$. Generalize, if you dare.

Solution: Here is the solution in the case of $M_2(\mathcal{O}_2)$. Let

$$t_1 = \begin{pmatrix} s_1 & s_2 \\ 0 & 0 \end{pmatrix}, \quad t_2 = \begin{pmatrix} 0 & 0 \\ s_1 & s_2 \end{pmatrix}.$$

Then we have by calculation $t_1^*t_1 = t_2^*t_2 = I$ and $t_it_j^* = e_{ij}$ where e_{ij} are the matrix units. Since t_1, t_2 are isometries and $t_1t_1^* + t_2t_2^* = 1$, it follows that t_1, t_2 generate a copy of $\mathcal{O}_2$ inside $M_2(\mathcal{O}_2)$. Since the C^*-algebra generated by t_1 and t_2 contains all the matrix units, as well as matrices with s_1 and s_2 as entries, it contains every matrix whose entries are either s_1, s_2, 0, or 1, and therefore it is the whole of $M_2(\mathcal{O}_2)$. Thus $M_2(\mathcal{O}_2)$ is isomorphic to $\mathcal{O}_2$.

For $M_3(\mathcal{O}_2)$ we similarly consider the 3×3 matrices

$$t_1 = \begin{pmatrix} s_1 & s_2s_1 & s_2^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad t_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & s_1 & s_2 \end{pmatrix}.$$

These are isometries such that $t_1t_1^* + t_2t_2^* = 1$, so they generate a copy of $\mathcal{O}_2$ inside $M_3(\mathcal{O}_2)$. Arguing as above, in order to show that the C^*-algebra that they generate is the whole of $M_3(\mathcal{O}_2)$ it will suffice to show that it contains every matrix unit; in fact it will suffice to show that it contains e_{21} and e_{31}, since these two matrix units generate $M_3(\mathbb{C})$. But in fact we have

$$e_{21} = t_2t_1t_1^*, \quad e_{31} = t_2^2t_1^*$$

and so we are done.

6.6 Exercises Dec 1, due Dec 8

EXERCISE 6.30: Extend the proof of the Kadison transitivity theorem to show that if ρ is an irreducible representation of a C^*-algebra A on a Hilbert space H, and if K is a finite-dimensional subspace of H, then for each self-adjoint operator T on K there exists a self-adjoint $a \in A$ such that $\rho(a)_{|K} = T$. Prove the same statement also with the word 'unitary' in place of 'self-adjoint'.

Solution: For the self-adjoint case, the ordinary version of Kadison transitivity produces *some* $a \in A$ agreeing with T on K; to make it self-adjoint, replace a by $(a + a^*)/2$. For the unitary case, write $T = \exp(iS)$ with S self-adjoint. Use the first part to produce a self-adjoint b agreeing with S on K, then define $a = \exp(ib)$.

EXERCISE 6.31: Using the previous exercise, prove that if σ_1, σ_2 are pure states of a unital C^*-algebra A, and if the corresponding representations of A are equivalent, then there exists a unitary $u \in A$. such that $\sigma_1(a) = \sigma_2(u^*au)$.

Solution: Suppose that the GNS representations ρ_1, ρ_2 are unitarily equivalent. Using the unitary, identify the Hilbert spaces of the two representations so that we are in the following situation: we are given an irreducible representation $\rho\colon A \to \mathfrak{B}(H)$ and two vector states

$$\sigma_i(a) = \langle \rho(a)v_i, v_i \rangle$$

for unit cyclic vectors v_1, v_2. On the finite-dimensional subspace spanned by v_1 and v_2 there exists a unitary operator U such that $U(v_2) = v_1$. By the previous result, there exists a unitary $u \in A$ such that $\rho(u)$ agrees with U on the subspace spanned by v_1, v_2. Then

$$\sigma_2(u^*au) = \langle \rho(u)^*\rho(a)\rho(u)v_2, v_2 \rangle = \langle \rho(a)\rho(u)v_2, \rho(u)v_2 \rangle = \langle \rho(a)v_1, v_1 \rangle = \sigma_1(a).$$