Geometry and Relativity

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Contents

Lecture 1
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The plan for this lecture course is to understand Einstein's theory of gravity. This will involve a journey starting from Newtonian gravity, passing through some differential geometry and ending up in the realm of black holes and the Global Positioning System.

One of the basic consequences of Einstein's General Theory of Relativity is that clocks will run at differing speeds depending upon the ambient gravitational field. For instance, consider firstly yourself standing upon the surface of the earth and secondly a satellite orbiting high above the earth. The two frames of reference are subject to different gravitational fields and so, according to the theory, experience a tiny difference in the passage of time.

The Global Positioning System involves just such a situation. The system comprises twenty four orbiting satellites, each equipped with a clock, and broadcasting time signals to us on the earth. If we accurately measure the difference between the incoming signal and the time on our own clock then, knowing the speed of light, we can determine how far away the source of the signal is. If we do this with several different satellites, we obtain several different distances, which we can use to determine our position on the surface of the earth.

However, Einstein's theory predicts that the difference in gravitational field between our frame of reference and that of the satellite will introduce an error in these time measurements of the order of a few hundred nanoseconds. This sounds small, but multiplying by the speed of light yields a positional error of the order of a few hundred feet. This is not what you would want if you were trying to land an aeroplane on a foggy runway.

When the GPS system was first introduced, this general relativistic effect was unconfirmed, and the engineers were unsure whether to account for it in the computational software. As a safe solution, a software switch was incorporated, which when turned off would ignore the relativistic effect. The system was started up with the switch off and, lo and behold, an error in test measurements was detected exactly matching that predicted by Einstein's theory. The switch was activated, and our GPS system is now as accurate as one could hope.

1.1   Newtonian Gravity

In order to understand Einstein's theory of gravity, we should begin by understanding Newton's gravity.

Newton's theory of gravity has, built into it, a mystery. We must think of Galileo's experiment at the Leaning Tower of Pisa in the seventeenth century.

At this point, John climbs upon a chair with two bags of bagels of differing weight an differing composition — one of plain bagels, the other of onion bagels. Both are dropped, and an observant and fair-minded student noted that both struck the ground at exactly the same time, at least within the limits of experimental error.

Galileo observed this using balls of differing masses and composition and concluded thus:

> The variation of speed in air between balls of gold, lead, copper,

porphyry is so slight that in a fall of 100 cubits a ball of gold would surely not outstrip one of copper by as much as four fingers. Having observed this, I came to the conclusion in a medium totally void of resistance, all bodies would at the same speed.

This strongly refuted the previously held theory of gravity proposed by Aristotle, who thought that " ... the downward movement of a mass of gold or lead ... is quicke in proportion to its size."

Herein lies the mystery: why should it be that gravitational acceleration is independent of the mass and composition of a falling body?

Well, there is a philosophical reason why it should be independent of mass. For imagine two cannon balls, one weighing $2lb$ and the other $4lb$. The $4lb$ ball could be considered as being two $2lb$ masses glued together, both of which must accelerate at the same rate at the $2lb$ ball. But this does not answer question about different materials. For instance, if I had a cannonball made of lead and another cannonball made of gold — I'd be rich!!

Note also that this fact about independence on composition is *not* true of electrostatic forces. Electrons, protons, and neutrons all react differently to an imposed electrostatic field.

Newton further confirmed Galileo's experiment with pendulum bobs made of different materials. These days, the most accurate experiments are Eötvös-type experiments. These involve using a sensitive torsion balance with equal weights of differing materials — say gold and aluminium — at each end. If each were affected differently by gravity then one would notice a 24-hour periodicity to the torsion balance's motion as the earth travels around the sun. No such motion is detected, and these experiments confirm material independence to an accuracy of within 1 part in 10^11. It has been said that this is the most important null results ever.

We could conceivably argue the independence of gravitational acceleration using Newton's Laws as follows. Newton proposed an inverse square law for the gravitational attraction of masses.

Newton's law of Gravity:

$$F = -GMm/r^2$$

He also proposed several well-known laws of motion, including the following.

Newton's law of motion:

$$F = ma$$

Combining these principles yields that acceleration is independent of the value of m. But this is a cheat, since the values of m in the above two equations are *assumed* to be equal, whereas actually one represents gravitational mass — which one could perversely call "gravitostatic charge" — and the other represents inertial mass. Why should these be the same?

Einstein gave a wonderful answer to this question, and essentially, the purpose of this course is to explain what this answer was.

Essentially, Einstein's answer is:

Gravity is an illusion.

Thus it is no wonder that gravity affects everything in the same way, because gravity doesn't *really* affect anything.

Well, obviously this is a ridiculous statement. Less provocatively, Einstein's answer could be rephrased as *gravity is geometry*. His key idea could be brutally paraphrased as follows: the action of falling should be completely natural. Indeed we would all be falling right now if the floor was not holding us up. So we should rewrite physics in such a way that falling is natural, and staying upright is the activity that needs to be explained.

1.2   The gravitational field generated by a point mass

We will now reformulate Newton's inverse square law of gravity in terms of potentials and partial differential equations.

Instead of having two particles with masses M and m, let us think of only one mass M. The quantity m will represent a test mass. Let a particle of mass M be located at the origin of $\mathbb{R}^3$. Newton's law says that a test particle at position $\mathbf{r}$ will experience an acceleration of

$$\mathbf{F} = \frac{GM\mathbf{r}}{r^3},$$

Where r is the length of the position vector $\mathbf{r}$. This quantity $\mathbf{F}$ is a vector field. It is called the *gravitational field*.

One can show that it is a conservative field. This implies, in particular, that

$$\mathbf{F} = -\nabla\Phi,$$

for some potential function Φ. In fact, we have $\Phi = -GM/r$, as we will now compute. The definition of $\nabla\Phi$ is

$$\nabla\Phi = (\frac{\partial\Phi}{\partial x}, \frac{\partial\Phi}{\partial y}, \frac{\partial\Phi}{\partial y}).$$

If we take $\Phi(x, y, z) = -GM/(x^2 + y^2 + z^2)^{1/2}$, then,

$$\frac{\partial\Phi}{\partial x} = \frac{-\frac{1}{2}.2x(-GM)}{(x^2 + y^2 + z^2)^{3/2}},$$

which is what we claimed.

1.3   Gauss' flux theorem

We want to reformulate this gravitational law in terms of partial differential equations. The earth is actually not a point particle, but is made up of distributed matter.

Suppose we have a region in space, Ω with boundary surface $\partial\Omega$, and suppose we have a field $\mathbf{F}$ in this region. Then, associated with this data, we have the quantity called the flux of $\mathbf{F}$ through $\partial\Omega$, which is given by

$$\text{Flux} = \iint_{\partial\Omega} \mathbf{F} \cdot \mathbf{n} \, d\sigma.$$

The easiest way to think of this is, for the moment, to think of $\mathbf{F}$ as representing the velocity of some fluid in the region Ω. Then the above flux represents the nett quantity of fluid exiting the region Ω.

Gauss' Flux Law for Gravity says,

$$[\text{Flux of gravitational field through } \partial\Omega]$$
$$= -4\pi G \times [\text{total amount of matter enclosed by } \Omega].$$

When we compare the gravitational theories of Newton and Einstein we will see many similarities. For instance, the Newtonian theory has a potential. Einstein also has a potential — in fact ten of them, since the potential becomes a 4×4 symmetric matrix. Newton has a law of motion, given by a second order differential equation. So does Einstein. However, there are also many significant differences, which we'll come across in due course.

Lecture 2
(Friday, 5 September 2003)

Imagine we are situated in a sealed chamber, which is towed up into space, far above the surface of the earth. The chamber is equipped with sufficient supplies of air, and coffee and bagels, so that we can survive up there. Now we let this chamber drop earthwards, from a sufficient height that we can finish this class before impact. What do we feel as we drop?

Suggestions come from the class: . . . weightlessness . . . the lack of any normal force from the floor . . .

We are all accelerating at the same rate. Why? Because gravitational acceleration is independent of the mass and composition of the object. In this freely-falling room — at least until we hit the ground — it appears to us that there is no gravitational force on the surrounding objects at all.

Now let us imagine we tow the room even further into outer space that there are no objects close enough to give any noticeable gravitational force. Now we are supposedly experiencing "genuine" weightlessness. But now suppose we attach a hook to the roof of the chamber, and attach a rope to this hook, and attack the rope to a tow-ball on the back USS Enterprise, and that Captain James T Kirk tows us upwards under impulse power at an acceleration of exactly $9.8ms^{-2}$. We are accelerating upwards, but in our chamber we feel as if we are being pulled down towards the floor. We feel as if we are being acted upon by a gravitational force.

Thus, in our chamber, we have no way of distinguishing the situations of an ambient gravitational force and an acceleration. This can be summarized as the Equivalence Principle.

Equivalence Principle:

Gravity is locally indistinguishable from acceleration of the frame of reference.

To quote Einstein:

> . . . we arrive at a very satisfactory interpretation of this law of experience, if we assume the the systems K [stationary in a gravitational field] and K' [uniformly accelerating, as our chamber in outer space] are physically exactly equivalent, that is, if we assume that we may just as well regard the system K as being in a space free from gravitational fields, if we then regard K as uniformly accelerated. This assumption of exact physical equivalence makes it impossible for us to speak of the absolute acceleration of the system of reference, just as the usual theory of relativity[1] forbids us to talk of th absolute velocity of a system; and it makes the equal falling of all bodies in a gravitational field seem a matter of course.

This is, more or less, the main idea of General Relativity.

Let us now develop this idea by looking more carefully at this Equivalence Principle from a less local point of view.

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[1]which we will study in due course

Imagine a very long thin room, again situated high above the surface of the earth. Two people stand at the distant opposite ends of this room, and I stand in the middle. As before, we now let the room plummet earthwards. However, since the room is so wide, the action of the gravitational field on my distant companions is not parallel. It appears to me as if they are drifting slowly towards me in the middle. This is a non-local behaviour of the gravitational field.

Similarly, we can consider an enormously tall, thin room standing on it's end, with people located at different heights within the room. I am located half-way up this tower. We let the room drop. From my point of view, the others appear to be accelerating away from me — those toward the bottom are accelerated more rapidly than I am because they are closer to the gravitationally attracting body. Those nearer the top accelerate more slowly than I do.

Imagining ourselves situated in the centre of a huge spherical room, with people scattered about throughout, we observe a motion as indicated in the diagramme below. In particular, this thought experiment explains the existence of the tides, as well as their twice-daily periodicity. The tides are caused by the gravitational attraction of the oceans by the moon. The earth and the moon are a freely falling system, with no interaction other than that due to gravity. The oceans are thus affected by the gravitational variations as just described to produce two rotating bulges — one towards the moon and one opposite it.

These drift effects — or *tide effects* — depend upon small variations of the gravitational field.

Let us quantify this. In the diagram below, we want to compute the spatial variation in the gravitation field, given by $\mathbf{F}(x + h) - \mathbf{F}(x)$, where h is an infinitesimally small displacement. According to Newtonian Theory,

$$\mathbf{F} = \frac{-GM}{r^3} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Notice that, by differentiating $r^2 = x^2 + y^2 + z^2$ implicitly, we get

$$2r \frac{\partial r}{\partial x} = 2x,$$

and so

$$\frac{\partial r}{\partial x} = \frac{x}{r}.$$

Therefore,

$$\frac{\partial}{\partial x} \left(\frac{x}{r^3} \right) = \frac{1}{r^3} - \frac{3x^2}{r^5} = \frac{r^2 - 3x^2}{r^5}.$$

Similarly,

$$\frac{\partial}{\partial y} \left(\frac{x}{r^3} \right) = \frac{-3xy}{r^5}.$$

Continuing in this way, we conclude that

$$D\mathbf{F} = \frac{-GM}{r^5} \begin{pmatrix} r^2 - 3x^2 & -3xy & -3xz \\ -3xy & r^2 - 3y^2 & -3yz \\ -3xz & -3yz & r^2 - 3x^2 \end{pmatrix}$$

This 3×3 matrix is that matrix which, when multiplied by a vector representing a small change in position, yields the corresponding small variation in gravitational field strength. For instance, at position $(x_0, 0, 0)$,

$$DF = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that these calculations demonstrate that the tidal effects vary according to an inverse cube law, rather than an inverse square law. This explains why the earth's tides are dominated by the location of the moon, rather than the sun. (See Exercise 2, Homework Set 1.)

We also note that from this calculation, we see

$$\text{div}\mathbf{F} = \text{Trace}D\mathbf{F} = \mathbf{0}.$$

This fact is pertinent to Gauss' Flux Law of Gravity. Recall that last time we stated Gauss' Flux Law in the following form.

Gauss' Flux Law:

The flux of the gravitational field through the boundary $\partial\Omega$ of a region Ω in space is equal to $-4\pi G \times [\text{total mass enclosed by}\Omega]$.

We can now indicate a proof of this. We begin by considering the case where the gravitational field is due to a point mass.

Proof for the case of a point source of mass M. Case (i): The source is outside Ω.

By the Divergence Theorem,

$$\iint_{\partial\Omega} \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_{\Omega} \text{div}\mathbf{F}\, dx\, dy\, dz,$$

where $\mathbf{n}$ is the outward unit normal vector to the surface $\partial\Omega$. But we just observed that $\text{div}F = 0$ everywhere away from the mass.

Case (ii): The source lies inside Ω.

Surround the mass by a tiny ball of radius r, which lies entirely inside the region Ω. Now apply the previous case to the region Ω with this ball removed. We see that the total flux through the surfaces of this punctured region is zero, which is equivalent to saying that the flux through the surface of Ω is the same as the flux through the surface of the tiny sphere. So now we are reduced to considering the case where Ω is a sphere of radius r, centred at the mass source.

For this case we can do a direct calculation. The field is everywhere normal to the surface of the ball, and we have

$$\mathbf{F} \cdot \mathbf{n} = \frac{-GM}{r^2} \qquad \text{(constant)}.$$

So,

$$\begin{aligned} \iint \mathbf{F} \cdot \mathbf{n}\, d\sigma &= \frac{-GM}{r^2} \times [\text{area of sphere}] \\ &= \frac{-GM}{r^2} 4\pi r^2 \\ &= -4\pi GM. \end{aligned}$$

This confirms Gauss' Flux law in the case of a point mass. $\qquad\qquad\square$

For the general case, we need to appeal to the Principle of Superposition. This is not satisfied by Einstein's General Relativity, but it is satisfied by Newtonian gravity — indeed you could take it as a postulate.

Principle of Superposition:

The gravitational field arising due to the influence of a number of bodies is equal to the sum of the gravitational fields which would occur due to each body in isolation.

This implies that Gauss' Flux Law holds for all mass distributions. For, we can observe that both of the quantities mentioned in the Flux Law are linear in the field F. Thus, if we consider the contributions of each element of matter, the corresponding quantities will sum on both sides of the equation, and equality will be maintained.

We can thus restate Gauss' Flux Law by applying this Principle of Superposition, as follows: Gauss' Flux Law states that

$$\frac{-1}{4\pi G} \iint_{\partial\Omega} \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}\sigma$$

is equal to the total mass contained. In the case of a distributed mass with density ρ, this gives

$$\iiint_{\Omega} \rho \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = \iint_{\partial\Omega} \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}\sigma = \iiint_{\Omega} \mathrm{div}\mathbf{F} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z.$$

Since this is true for any region Ω, the integrands on the left and right must be equal. Thus Newton's Law of Gravity can be succinctly restated as

$$\nabla \cdot \mathbf{F} = -4\pi G\rho.$$

In terms of the gravitational potential, we obtain the "Poisson Equation" formulation of Newton's Law of Gravity.

Newton's Law of Gravity:

$$\nabla^2 \Phi = 4\pi G\rho. \tag{2.1}$$

Remark. In empty space, this reduces to the "Laplace Equation",

$$\nabla^2 \Phi = 0. \tag{2.2}$$

Lecture 3

(Mon 8 September 2003)

3.1 Newton vs Einstein

Let us compare and contrast the gravitational theories of Newton and Einstein. We have seen that in Newton's theory, there is a second order differential equation which gives the gravitational potential in terms of the gravitational source (Equation 2.1). What about Einstein?

Einstein, too, has a second order differential equation giving the potential in terms of the source. However, there are two important differences. The first is that we actually have ten differential equations, since they will come from the entries of a symmetric 4×4 matrix. But also, there is a non-linearity in Einstein's potential equations, which gives rise to new phenomena.

As well as having a law for computing potentials, we need to have an equation of motion. In Newton's theory, this comes from his well-known law of motion, which becomes

$$\ddot{\mathbf{r}} = -\nabla\Phi(\mathbf{r}).$$

In Einstein's theory we also have a second order ODE for the equation of motion. However, while in Newton's theory the equation of motion is separate from the field law, in general relativity the equation of motion can be deduced from the law of gravity. Newton's gravitational theory does not incorporate time ("timeless"?), while Einstein's does ("timely"!).

One of the consequences of Newton's gravitational theory making no mention of time is that the gravitational field produced by a configuration of masses will immediately readjust itself throughout the universe as the masses move. We get instantaneous action at a distance.

This cannot happen within the structure of relativity. No information should be able to be faster than the speed of light. Time is inextricably woven into Einstein's gravitational equations. This has a lot of very interesting consequences. Let us mention a few.

(i) <u>Automatic conservation of the source.</u>

Let us pretend that there is only one kind of matter in the universe and that it cannot be created or destroyed. Now if we imagine a region Ω in space, then the only way the amount of matter inside the region Ω can change is by matter passing through the boundary. From the divergence theorem:

$$\mathrm{div}\mathbf{J} + \frac{\partial\rho}{\partial t} = 0,$$

where $\mathbf{J}$ represents the mass flow-field.

If we were doing classical gravitational theory, this fact would have to be added as a separate postulate. However, in Einstein's theory there is a certain symmetry in the equations from which this and other consequences can be deduced. In particular, conservation of energy and conservation of momentum are corollaries of the theory of gravity. This is analogous to the deduction of the conservation of charge from Maxwell's Equations.

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(ii)   <u>Equation of motion.</u>

We can consider a test particle as being a "glitch" in the gravitational field. Then, when we study the laws of Einstein's Gravity, we see that this "glitch" keeps its shape, and moves in a way which is described by the usual second order differential equation. Thus, this too becomes fused with the theory of gravity.

(iii)   <u>Gravitational waves.</u>

Einstein's theory predicts that gravitational waves can propagate through space, although this phenomenon is yet to be detected.

3.2   Geometry

Beginning with the ordinary Euclidean geometry that we all know and love — straight lines, angles, and so on — the kind of geometry that is relevant to Einstein's gravity is obtain by a twofold process of generalization.

	Space $\longrightarrow$	Space-time
Global	Euclidean Geometry	Minkowskian Geometry (special relativity)
$\downarrow$		
Local	Riemannian Geometry	General Relativity

Riemannian geometry is what one gets when one considers space which, in small regions, looks like Euclidean space, but on the large scale is glued together in a way that is possibly quite different from the ordinary space $\mathbb{R}^n$. This leads to effects like the tidal effects we considered last lecture.

A standard example of interesting Riemannian geometry is a sphere, considered as a two-dimensional surface. When we draw a picture of such a surface we usually imagine it to be sitting inside some larger Euclidean space, in this case $\mathbb{R}^3$, and it is tempting to use facts about that ambient space in analyzing the structure of the surface. This is *extrinsic geometry*. But we are interested only in *intrinsic* features of the geometry — those features which one can determine without leaving the surface itself, *ie*, without using features of any supposed ambient space.

To illustrate this, consider this puzzle: A man walks one mile south, then one mile east, then one mile north and ends up where he started. He is then eaten by a bear. What colour was the bear?

Where can the man be? This question has a well-known answer — the north pole — and an infinite family of less-well-known answers — a sequence of latitudes which are slightly more than one mile north of the south pole. However, since there are no bears at the south pole, we know he must be at the north pole, and the bear is white.

But what is more interesting, at least to us, is that the man has been able to determine something intrinsic about the surface on which he is, or was, living. In the few moments before the bear's jaw close around him, this man can realize that this journey could not have taken place on the Euclidean plane. He has

ascertained something about the curvature of his world, without ever leaving it's surface.

Now let us follow the above diagram in the horizontal direction. In the transition from Euclidean to Minkowskian geometry, we replace *points* with *events*, which are described by both position *and* time. For instance, Bill Clinton eating the first french-fry of his lunch in the White House on a particular day constitutes an event. Now we consider the universe to be comprised of events. We are naturally led to examine the geometry, not of space, but of space-time.

We obtain general relativity by combining these two notions. We need to generate a global version of special relativity. This is the purpose of quite a lot of the rest of this course. We kick off with groups and group actions.

3.3   Groups and Actions

Definition 3.1. A *group* is a set G, together with a binary operation, $\cdot$, (usually just denoted by juxtaposition), such that

- There is an identity $e \in G$, with

$$eg = ge = g \quad \text{for all } g \in G.$$

- There are inverses: For all $g \in G$, there is an element $g^{-1} \in G$ such that

$$gg^{-1} = g^{-1}g = e.$$

- Associativity holds: For all $g_1, g_2, g_3 \in G$,

$$g_1(g_2 g_3) = (g_1 g_2)g_3$$

This definition is best understood with a few examples.

Example 3.2. The real numbers, $G = \mathbb{R}$, with operation $+$, form a group.

Example 3.3. The rotation group, $G = \{\text{rotations of 3D-space}\}$, denoted by $SO(3)$, is a group with the operation of composition.

To describe this example physically, if I take some object and rotate it by $90°$ about the x-axis, and then by $90°$ about the y-axis, the effect is the same as if I rotated it by $90°$ about the z-axis. Furthermore, associativity holds, there is an identity, namely rotation by $0°$, and there are inverses (because I can undo any rotation by rotating in the opposite direction by the same amount). Thus we have a group. It is a group in which we will be quite interested.

Example 3.4. The set of invertible $n \times n$ matrices, which is denoted by $GL(n)$, is a group with the operation of matrix multiplication.

Definition 3.5. An *action* of a group G on a set X is a map

$$G \times X \to X$$

denoted by

$$(g, x) \mapsto g \cdot x,$$

or simply by juxtaposition, such that

- $e.x = x$ for all $x \in X$,

- $(g_1.g_2).x = g_1.(g_2.x)$ for all $x \in X, g_1, g_2 \in G$.

In words, this means that that each element of the group is acting as a transformation of the space X — moving points of the space around — in a way which is compatible with the group structure.

Example 3.6. The group of rotations $SO(3)$ acts on three dimensional space is a natural way, namely, if you take any point $\mathbf{x}$ in space, we define the image of the point under the action of a rotation g to be the point to which $\mathbf{x}$ is moved when we rotate our space by g.

Lecture 4
(Wed 10 September 2003)

According to Felix Klein, geometry is really just the study of certain group actions. This viewpoint is rather extreme, but certainly group actions will be fundamental to our considerations of geometry.

Last time we had the definition of a group and a group action on a space. When considering a group action, we should think of each element of the group as being a transformation of the space.

From a physical standpoint, the transformations arise when we compare the viewpoints of two different observers. For suppose we have two observers, a Frenchman and an American, examining the events of the world. The Frenchman will describe events in terms of some chosen system of x, y and z coordinates, probably centred at Paris, while the American's viewpoint will involve a very different set of coordinate axes. If the two are to be able to communicate, we need some kind of dictionary will allow us to convert positional descriptions in terms of the American's x, y and z coordinate system, to descriptions in the Frenchman's coordinates, and vice versa.

The collection of all possible coordinate transformations between all possible observers of all conceivable nationalities forms a group.

Definition 4.1. An *invariant* of a group action (of a group G on a space X) is an "object" that is not changed by the action of any g in G.

The vagueness of the term "object" is necessary because we do not wish to constrain ourselves to the exact form of such an object, but consideration of the ensuing selection of examples should be enough to clarify the definition.

However, before turning to examples, note that we expect that physics itself (or more precisely, the laws of physics) should be an invariant of some group of coordinate transformations. For we expect that the laws of physics should be described in the same for any observer in his or her chosen system of coordinates. This general principle is what eventually gives rise to Einstein's theory of relativity.

Now, some examples of invariants.

Example 4.2. A function f on the space X is invariant if

$$f(gx) = f(x),$$

for all $x \in X$ and all $g \in G$.

For an example of this, consider the group $G = O(n)$ of orthogonal $n \times n$-matrices acting on n-dimensional space $\mathbb{R}^n$. This group $O(n)$ is, by definition, the group of $n \times n$-matrices M, satisfying

$$MM^t = M^t M = I.$$

Here M^t denotes the transpose of the matrix M, obtained by interchanging the rows and columns of M.

Familiar examples of orthogonal matrices are the rotation matrices in $O(2)$, given by

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

One can easily check that multiplying this matrix by its transpose yields the identity matrix.

Letting the orthogonal $n \times n$-matrices act on the space $\mathbb{R}^n$ by usual matrix-vector multiplication yields an action of the group $O(n)$ on n-dimensional space.

Proposition 4.3. *The "distance to the origin" function is an invariant of this group action.*

Proof. Let $f(\mathbf{x})$ denote the distance of the vector

$$\mathbf{x} = \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix}$$

from the origin. We can write

$$f(\mathbf{x})^2 = \mathbf{x}^t \mathbf{x}.$$

Then,

$$\begin{aligned} f(M\mathbf{x})^2 &= (M\mathbf{x})^t (M\mathbf{x}) \\ &= \mathbf{x}^t M^t M \mathbf{x} \\ &= \mathbf{x}^t \mathbf{x}. \end{aligned}$$

$\square$

Proposition 4.4. *The "distance between two points" function is an invariant of the $O(n)$ action.*

Remark. This example indicates why we were vague about the invariant "objects" in Definition 4.1. Our invariant here is not a function on the action space $\mathbb{R}^n$, but a function on $\mathbb{R}^n \times \mathbb{R}^n$. The exact nature of other possible invariants is limited only by the imagination.

We can show something stronger than the above propositions, namely, that the invariance of the functions of Propositions 4.3 and 4.4 characterizes the group $O(n)$, as the following theorem shows.

Theorem 4.5. *If G is a group acting on $\mathbb{R}^n$, which fixes the origin and leaves invariant the distance between pairs of points, then $G \subseteq O(n)$, ie, G is a subgroup of $O(n)$.*

Proof. Let $T \in G$. The main fact we have to show is that T is linear, *ie*,

$$T(\lambda \mathbf{x} + \mu \mathbf{y}) = \lambda T(\mathbf{x}) + \mu T(\mathbf{y}),$$

for any vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and any real numbers λ and μ. Linearity implies that T is represented by a matrix.

Using $\|\mathbf{x}\|$ to denote the distance to the origin of a vector $\mathbf{x}$, we know that

$$\|T\mathbf{x}\|^2 = \|\mathbf{x}\|^2,$$

and

$$\|T\mathbf{x} - T\mathbf{y}\|^2 = \|\mathbf{x} - \mathbf{y}\|^2.$$

To inspire the next trick, we go back to the dawn of computing and ask how the first computer, ENIAC, did multiplication. ENIAC had a table of squares hard-wired into it, which it employed, along with the identity

$$xy = \frac{1}{2}(x^2 + y^2 - (x - y)^2).$$

This anecdote illustrates that if you know how to add, subtract, and compute squares, then you can multiply. We pull a similar stunt now.

Write

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^t \mathbf{y}.$$

Then you can check that

$$\mathbf{x} \cdot \mathbf{y} = \frac{1}{2}(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - \|\mathbf{x} - \mathbf{y}\|^2).$$

This is known as the *polarization identity*. Since the terms of the right hand side are invariants, we have

$$T\mathbf{x} \cdot T\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$$

for all vectors $\mathbf{x}$ and $\mathbf{y}$.

We will now prove by example that T is linear. Consider the case of proving that $T(2\mathbf{x}) = 2T(\mathbf{x})$. For any choice of a vector $\mathbf{z} \in \mathbb{R}^n$, the linearity of the dot product, and its invariance under T, show that

$$(T(2\mathbf{x}) - 2T(\mathbf{x})) \cdot T(z) = 2\mathbf{x} \cdot \mathbf{z} - 2(\mathbf{x} \cdot \mathbf{z}) = 0.$$

So we've shown that $T(2\mathbf{x}) - 2T(\mathbf{x})$ dotted with anything is zero. But since T belongs to a group of transformations, it must be one-to-one and onto — it has an inverse — and thus $T(\mathbf{z})$ could be anything. The only vector which is perpendicular to everything in $\mathbb{R}^n$ is $\mathbf{0}$, and we obtain the desired conclusion.

This argument can easily be generalized by applying it to the quantity

$$T(\lambda \mathbf{x} + \mu \mathbf{y}) - \lambda T(\mathbf{x}) + \mu T(\mathbf{y})$$

instead. We therefore have proven linearity, and the transformation T is represented by some matrix M.

But now, for any $\mathbf{x}$ and $\mathbf{y}$,

$$\mathbf{x}^t \mathbf{y} = (M\mathbf{x})^t (M\mathbf{y}) = \mathbf{x}^t M^t M \mathbf{y},$$

so

$$\mathbf{x}^t (I - M^t M)\mathbf{y} = 0.$$

If we take $\mathbf{x}$ and $\mathbf{y}$ each to be on of the standard basis vectors for $\mathbb{R}^n$, the left-hand side of this last equation gives the corresponding matrix coefficient of M, and we see that $I - M^t M = 0$. Therefore, M is a element of $O(n)$. $\qquad\square$

Finally, let us consider the special case of the group $SO(3)$. To define this group, note that if M is an orthogonal $n \times n$ matrix, then by the properties of the determinant (and noting in particular that $\det(M) = \det(M^t)$),

$$\begin{aligned}
(\det(M))^2 &= \det(M)\det(M^t) \\
&= \det(MM^t) \\
&= 1.
\end{aligned}$$

So $\det(M) = \pm 1$. Those with determinant $+1$ form a subgroup of the group $O(n)$, which we call $SO(n)$.

Theorem 4.6. *The group $SO(3)$ is the group of rotations of 3-dimensional space (which fix the origin).*

To make sense of this theorem, we need a definition of a rotation. We will define a rotation of 3-dimensional space to be a transformation of $\mathbb{R}^3$ which fixes some line ("axis") and in planes perpendicular to that line, rotates points through some angle θ.

Proof. The matrix of a rotation, when written with respect to a well-chosen coordinate system, is

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

One can check that this matrix has $M^t M = M M^t = I$ and $\det(M) = 1$. This proves the easy part.

The hard part is to show that every matrix $M \in SO(3)$ has an axis, *ie*, that there is some nonzero vector $\mathbf{v} \in \mathbb{R}^3$ such that $M\mathbf{v} = \mathbf{v}$. Such a $\mathbf{v}$ is an eigenvector with eigenvalue 1. Observe that,

$$\begin{aligned} \det(M - I) &= \det(M - I)\det(M^t) \\ &= \det(I - M^t) \\ &= \det(I^t - M^t) \\ &= \det(I - M) \end{aligned}$$

But since we are dealing with 3×3 matrices,

$$\det(I - M) = (-1)^3 \det(M - I).$$

So $\det(I - M)$ is zero. Thus $I - M$ is singular, and hence there is some nonzero vector $\mathbf{v}$ such that

$$(I - M)\mathbf{v} = \mathbf{0},$$

which yields

$$M\mathbf{v} = \mathbf{v}.$$

This shows the existence of a fixed axis. We leave the problem of showing that it is a rotation in the perpendicular planes as an exercise. $\square$

Lecture 5

(Fri 12 September 2003)

We have been discussing group actions. In particular we were looking at the group $SO(3)$ of rotations acting on three dimensional space. We can also consider the group $O(3)$ which includes both rotations and reflections (in planes through the origin). You may wish to convince yourself that this too is a group.

These groups are important groups of motions of Euclidean space. But they are not *quite* the correct ones to be considering from the point of view of physics. Remember, from last time, that those things that are left invariant by our groups of transformations are those that are important for physics. The groups $O(3)$ and $SO(3)$ leave invariant the origin. So if the universe contained a distinguished point, with a sign there that read, "here lies the centre of the universe[2]," then we could expect $SO(3)$ and $O(3)$ to be the groups relevant to our physical theory. However, since there is no such point, we must also consider the translations.

Definition 5.1.

- A *translation* of $\mathbb{R}^n$ is a mapping

$$\mathbf{x} \mapsto \mathbf{x} + \mathbf{v},$$

  where $\mathbf{v}$ is some fixed vector.

- The group $\mathrm{Iso}(\mathbb{R}^3)$ of *isometries* of $\mathbb{R}^3$ is the group generated by $O(3)$ and the translations.

- Similarly, $\mathrm{Iso}^+(\mathbb{R}^3)$ is the group generated by $SO(3)$ and the translations.

The word *generated* in the above definition means that we consider the group of all possible compositions of the generating elements described. So the group $\mathrm{Iso}^+(\mathbb{R}^3)$ consists of all the rotations, and all the translations, and all the things you get by first rotating then translating, and all the things you get by first translating then rotating, and all the things you get by first rotating then translating then rotating again, and so on *ad infinitum* or *ad nauseum*, whichever happens first. Fortunately, the situation in this particular case turns out to be quite simple — all the elements of the group reduce to just a composition of a rotation and a translation, but that is not important right now.

The word *isometry* was defined implicitly above. In fact, the definition of an isometry is a transformation of space which leaves the "distance between two points" function invariant. The reader may wish to prove that the isometries of $\mathbb{R}^3$ by this definition are precisely the elements of $\mathrm{Iso}(\mathbb{R}^3)$ described above.

It is the group $\mathrm{Iso}^+(\mathbb{R}^3)$ which is relevant to physics, at least in terms of spatial symmetries. There are some fancy words physicist use to describe this: the invariance of physics under rotations in called *isotropy*, while the invariance under translations is called *homogeneity*.

---

[2]Those who have been classically educated will know this point by the name ὀμφαλος.

5.1 Invariance of the Laplace operator

Recall that we had the Poisson equation for describing gravity:

$$\nabla^2 \phi = 4\pi G\rho.$$

If our belief in the invariance of physics is correct, this should be invariant under the above group action.

Theorem 5.2. *The* Laplace operator

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

is invariant under the action of $O(3)$.

What does this mean? If $T \in O(3)$ and f is a function on $\mathbb{R}^3$, we can define f^T by

$$f^T(\mathbf{p}) = f(T\mathbf{p}).$$

Essentially, this definition says that, given a function f on three dimensional space, we move it around[3] by the rotation T to the function f^T. What we are claiming in the theorem is that $\nabla^2(f^T) = (\nabla^2 f)^T$. This invariance is necessary for the invariance of the laws of physics, since we need the Poisson Equation to hold good in any chosen set of orthogonal coordinates, *ie,*

$$\nabla^2(f^T) = 4\pi G\rho^T.$$

We will prove Theorem 5.2 twice — first by brute force, and then we will see why it is advantageous to introduce some clever notation to facilitate the proof.

Proof. <u>Idea</u>: Use the chain rule (a lot!).

Supppose the transformation $T \in O(3)$ is given by the matrix

$$T = \begin{pmatrix} l & m & n \\ p & q & r \\ s & t & u \end{pmatrix}.$$

Then

$$f^T(x, y, z) = f(lx + my + nz, px + qy + rz, sx + ty + uz).$$

For a shorthand, let us write $\varphi(x, y, z)$ for this function. What we want to do is compute $\nabla^2\varphi$ in terms of $\nabla^2 f$.

Firstly,

$$\frac{\partial \varphi}{\partial x} = l\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial y} + s\frac{\partial f}{\partial z},$$

which we will abbreviate as

$$\frac{\partial \varphi}{\partial x} = lf_{,1} + pf_{,2} + sf_{,3}.$$

Similarly,

$$\frac{\partial \varphi}{\partial y} = mf_{,1} + qf_{,2} + tf_{,3},$$

---

[3]To be technically precise, we are actually moving f by the transformation T^{-1}.

and
$$\frac{\partial \varphi}{\partial z} = nf_{,1} + rf_{,2} + uf_{,3}, .$$

Let us note at this point that $f_{,1}$, $f_{,2}$ and $f_{,3}$ are functions, which in the above equations should be evaluated at the point $(lx+my+nz, px+qy+rz, sx+ty+rz)$. So when we differentiate again, we get

$$\begin{aligned}
\frac{\partial^2 \varphi}{\partial x^2} &=& l(lf_{,11} + pf_{,21} + sf_{,31}) \\
&& +p(lf_{,12} + pf_{,22} + sf_{,32}) \\
&& +s(lf_{,13} + pf_{,23} + sf_{,33}) \\
&=& l^2 f_{,11} + p^2 f_{,22} + s^2 f_{,33} \\
&& +2lpf_{,12} + 2lsf_{,13} + 2psf_{,23}.
\end{aligned}$$

where $f_{,ij}$ denotes the second partial derivative of f with respect to the ith and jth coordinates. Similarly,

$$\begin{aligned}
\frac{\partial^2 \varphi}{\partial y^2} &=& m^2 f_{,11} + q^2 f_{,22} + t^2 f_{,33} \\
&& +2mqf_{,12} + 2mtf_{,13} + 2qtf_{,23},
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2 \varphi}{\partial z^2} &=& n^2 f_{,11} + r^2 f_{,22} + u^2 f_{,33} \\
&& +2nrf_{,12} + 2nuf_{,13} + 2ruf_{,23},
\end{aligned}$$

Now we sum these, and obtain a huge mess. But a miracle occurs — we remember what kind of matrix we are dealing with (that is not the miracle!). T is an orthogonal matrix, which means that

$$\begin{pmatrix} l & m & n \\ p & q & r \\ s & t & u \end{pmatrix} \begin{pmatrix} l & p & s \\ m & q & t \\ n & r & u \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

If we write out the nine different equations that this matrix equation implies, and compare what we get to the coefficients in the huge sum we just computed, we get

$$\begin{aligned}
\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} &=& 1.f_{,11} + 0.f_{,12} + 0.f_{,13} \\
&& +0.f_{,21} + 1.f_{,22} + 0.f_{,23} \\
&& +0.f_{,31} + 0.f_{,32} + 1.f_{,33} \\
&=& \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.
\end{aligned}$$

Hence, this huge calculation has ultimately proven the invariance of the Laplace operator. $\square$

General relativity requires an enormous number of computations of the above form. It is clear that if we had to do all of them in gory detail as we just did, we'd never get anything done. So we need some new notation which will expedite calculations of this kind.

We set up a more compact notation as follows. Suppose we have

- a function f on space,

- two coordinate systems $(x^1, \ldots, x^n)$ and $(y^1, \ldots, y^n)$, which are related by an orthogonal transformation.

We write the (i, j)-matrix entry of the transformation T as a^i_j. The equation for the transformation from x to y coordinates becomes

$$x^i = a^i_j y^j. \tag{5.1}$$

In Equation 5.1, we are using *Einstein's summation convention*, which says that, in any expression, if we see two terms with the same indexing symbol, with one upper and one lower — for instance the index j on the right-hand side of the equation above — then we must sum over all values of that index. This is not mathematics, it is just laziness. Einstein noticed, when he was working on this theory, that he regularly came across sums of the above form, and sooner or later decided that he couldn't be bothered to write in the symbol $\sum_j$ any more.

Equation 5.1 is a short-hand representation for three different equations,

$$
\begin{aligned}
x^1 &= a^1_1 y^1 + a^1_2 y^2 + a^1_3 y^3, \\
x^2 &= a^2_1 y^1 + a^2_2 y^2 + a^2_3 y^3, \\
x^3 &= a^3_1 y^1 + a^3_2 y^2 + a^3_3 y^3,
\end{aligned}
$$

which correspond to the ordinary rules of matrix-vector multiplication.

The condition of orthogonality of a matrix $T = (a^i_j)$ is written, according to this notational convention, as

$$a^i_j a^k_j = \delta^{ik}, \tag{5.2}$$

where δ^{ik} is the *Kronecker delta*, another convenient shorthand:

$$
\delta^{ik} = \begin{cases} 1, & \text{if } i = k, \\ 0, & \text{if } i \neq k. \end{cases}
$$

Strictly speaking, Equation 5.2 is not grammatically correct, since the summation over the index j is represented by two *lower* j indices. If we wish to be fussy about this point, we can rewrite the equation as

$$\delta^{jl} a^i_j a^k_l = \delta^{ik}.$$

Einstein's notation makes the proof of Theorem 5.2 much simpler, as we will now see.

Proof of Theorem 5.2 in Einstein notation. In our new notation, the operator ∇^2 is given in the two coordinate systems by

$$\nabla^2_x f = \delta^{ij} \frac{\partial^2 f}{\partial x^i \partial x^j}$$

and

$$\nabla^2_y f = \delta^{ij} \frac{\partial^2 f}{\partial y^i \partial y^j}. \tag{5.3}$$

Our notation is also perfect for expressing the chain rule, since we see

$$\frac{\partial f}{\partial y^k} = \frac{\partial f}{\partial x^i}\frac{\partial x^i}{\partial y^k} = \frac{\partial f}{\partial x^i}a_k^i,$$

where a_k^i are the matrix entries of our orthogonal transformation T. Applying the chain rule again,

$$\frac{\partial^2 f}{\partial y^k \partial y^l} = \frac{\partial^2 f}{\partial x^i \partial x^j}a_k^i a_l^j.$$

Thus Equation 5.3 becomes

$$\nabla_y^2 f = \delta^{kl} a_k^i a_l^j \frac{\partial^2 f}{\partial x^i \partial x^j}.$$

But

$$\delta^{kl} a_k^i a_l^j = \delta^{ij},$$

and thus

$$\nabla_y^2 f = \nabla_x^2 f.$$

$\square$

Lecture 6

(Mon 15 September 2003)

6.1 Symmetries of Space-time

In the last couple of lectures we've been talking about the symmetries of space. In particular, we talked about the way that the groups $SO(3)$ or $O(3)$ act on the three dimensional space in which we seem to be living, and we discussed how we expect the laws of physics to be invariant under this group action. This is the isotropy of physics, the lack of any preferred directions. We also talked about translations — the action of $\mathbb{R}^3$ on $\mathbb{R}^3$ — whereby we get the homogeneity of physics. However, since physics involves motion, it is clear that time, as well as space, is important.

Physics takes place in *spacetime*. Points in spacetime are *events*. For instance, my wedding is an event: it took place in a particular place, and at a particular time. This is how we will describe our universe.

Thinking in this way, physical objects are not points any longer. A physical point-object which is stationary (with respect to the spatial coordinates) corresponds to a line through spacetime which is parallel to the time axis. A large object, such as you or I, is some sort of tube through space time.

This manner of thinking is convenient for solving certain problems. For instance, in your school-years you may have been tormented with problems along the following lines: a train leaves Chicago at noon, travelling towards Indianapolis at 60mph. Another train leaves Indianapolis one hour later for Chicago, with a speed of 50mph. At what time do the two trains pass?

The trains' trajectories correspond to lines through spacetime, which could be drawn on a space-time diagramme. The intersection of the two lines is a point in spacetime — an *event*. It is the event of the meeting of the two trains.

<u>Picture</u>: spacetime diagramme of the two trains.

In the preceding lectures we saw that space has certain symmetries. Spacetime, too, has symmetries. What are these symmetries? For the time-being we are only interested in classical physics — no relativity yet.

(i)   <u>The symmetries of space</u>: Leaving time alone and changing the space coordinates by rotations and/or translations does not affect our description of physics.

(ii) <u>The symmetries of time</u>: Leaving space alone and translating in the time coordinate should also not change our physical laws. A time translation corresponds to a new choice of the starting of your chosen universal clock. Whether one believes the world was created 6,000 years ago or 4,500 million years ago makes no difference to the way physics works now (at least, barring any fundamental philosophical differences.)

Additionally, classical mechanics is independent of time reflection. Running the universe forward or backward we see the same physical laws governing its evolution. In fact, explaining why we see time travel in one direction only is a big puzzle in classical physics.

(iii) <u>Galilean symmetries</u>: Galileo was the first to notice that physics also has certain symmetries which combine space and time transformations.

> Shut yourself up with some friend in the main cabin below decks on some large ship, and have with you there some flies, butterflies, and other small flying animals.
>
> Have a large bowl of water with some fish in it; hang up a bottle that empties drop by drop into a wide vessel beneath it.
>
> With the ship standing still, observe carefully how the little animals fly with equal speed to all sides of the cabin.
>
> The fish swim indifferently in all directions; the drops fall into the vessel beneath; and, in throwing something to your friend, you need to throw it no more strongly in one direction than another, the distances being equal; jumping with your feet together, you pass equal spaces in every direction.
>
> When you have observed all of these things carefully (though there is no doubt that when the ship is standing still everything must happen this way), have the ship proceed with any speed you like, so long as the motion is uniform and not fluctuating this way and that.
>
> You will discover not the least change in all the effects named, nor could you tell from any of them whether the ship was moving or standing still.
>
> In jumping, you will pass on the floor the same spaces as before, nor will you make larger jumps toward the stern than towards the prow even though the ship is moving quite rapidly, despite the fact that during the time that you are in the air the floor under you will be going in a direction opposite to your jump.
>
> In throwing something to your companion, you will need no more force to get it to him whether he is in the direction of the bow or the stern, with yourself situated opposite.
>
> The droplets will fall as before into the vessel beneath without dropping towards the stern, although while the drops are in the air the ship runs many spans.
>
> The fish in the water will swim towards the front of their bowl with no more effort than toward the back, and will go with equal ease to bait placed anywhere around the edges of the bowl.
>
> Finally the butterflies and flies will continue their flights indifferently toward every side, nor will it ever happen that they are concentrated toward the stern, as if tired out from keeping up with the course of the ship, from which they will have been separated during long intervals by keeping themselves in the air....

You can see from this that doing physics was much more fun in those days.

This thought experiment prompts the idea of a family of transformations which we will call "Galilean boosts". A Galilean boost relates two coordinate systems in spacetime, one of which is moving with uniform velocity with respect to the other.

Definition 6.1. Let us denote a point of spacetime by $(\mathbf{r}, t)$. A *Galilean boost* is a transformation of the form

$$B_{\mathbf{v}}(\mathbf{r}, t) = (\mathbf{r} + \mathbf{v}t, t)$$

where $\mathbf{v}$ is some fixed velocity vector.

Note that the Galilean boosts are invertible — in fact,

$$(B_{\mathbf{v}})^{-1} = B_{-\mathbf{v}}.$$

Galilean relativity:

Physics is invariant under Galilean boosts — in fact, it is invariant under the entire Galilean group *which is generated by boosts, symmetries of space and symmetries of time.*

Galilean relativity says that there is no experiment which can distinguish a uniformly moving frame of reference from a stationary frame of reference.

Exercise 6.2. Show that conservation of energy is preserved by Galilean relativity.

Exercise 6.3. The d'Alambert operator

$$\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$$

is *not* invariant under the Galilean transformations.

Exercise 6.3 is bad news for classical physics, because the equation

$$(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2})\varphi = 0$$

comes up in several places in physics — in particular, in Maxwell's theory of electromagnetism. This means that Newtonian electromagnetic theory would change in different Galilean frames of reference. Einstein came up with an ingenious solution to this problem, which we will come to soon enough. But for now, let us examine the Galilean group more thoroughly.

Let E_1 and E_2 be two events in Galilean spacetime. What kind of invariants do we have? Some suggestions might be:

- <u>Distance</u>: But what does this mean exactly? What is the distance between two *events*? Space distance? What is the distance between State College now and Washington DC at ten o'clock last night? Does this make sense in any frame of reference?

If I am stationary relative to State College, then the distance between the two events may be 200 miles. However, if I were in a frame of reference stationary relative to the sun then the distance which I measure between these two non-simultaneous events would be several thousand miles — State College would have moved a considerable distance in the intervening time. I could even carefully choose a Galilean frame of reference such that the space distance between the two events is zero: I could choose my origin to be moving from DC to State College at precisely that speed which puts it in Washington at 10:00pm last night and State College now.

So the space-distance is not *usually* an invariant. It is only an invariant in the case of simultaneous events.

- Time separation between events: Time separation, between events located anywhere, will be measured the same in any Galilean frame of reference. Thus, it is possible to say in classical physics that I ate my lunch an hour ago, even if I ate it somewhere else[4].

One of the philosophical considerations in physics is *causality*. One can ask, if I hurl this piece of chalk into the audience, what possible events in spacetime can it influence? In classical physics, an event can influence any event with a greater time coordinate. We get instantaneous action at a distance.

This is not true of relativity. In Einstein's theory, no information can travel faster than the speed of light: the causal future expands out from us in spacetime as a cone with sides corresponding to the speed of light. Similarly, our past lies in a backward-facing spacetime cone. Thus there are events which lie neither in our past nor in our future.

Picture: Spacetime light cones.

6.2   Surface Geometry

6.2.1   Curves in $\mathbb{R}^2$

Suppose we want to measure the length of my TV cable, which is coiled up on the floor. How do we measure its length? One way is to stretch the cable straight, but let's consider that as cheating. Instead, we divide the cable by chalk marks into pieces which are so small they are approximately straight, and then add up all those minute lengths.

Mathematically, if we write the equation of our curve as

$$\left. \begin{array}{l} x = x(t) \\ y = y(t) \end{array} \right\} \qquad t \in [a, b],$$

---

[4]Strictly speaking, if we are allowing time reflections in our group of Galilean transformations, we cannot speak of a direction in time, only separation. Thus I can only say that I ate my lunch one hour distant from now, and cannot specify past or future.

and then we write the length of the curve as

$$\text{Length} = \int ds, \tag{6.1}$$

where

$$ds^2 = dx^2 + dy^2. \tag{6.2}$$

Equations 6.1 and 6.2 are given in somewhat informal notation. They are short-hand substitutes for writing

$$\text{Length} = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt.$$

This last equation contains no more information than the earlier one, but uses a lot more ink (or pixels), so we will usually abbreviate integrals in the manner of Equations 6.1 and 6.2.

Now, suppose we have two points in $\mathbb{R}^2$, called p and q. We can imagine a whole slew of possible curves from the point p to the point q. Amongst all these curves, one of them is the shortest. Which one? We know the answer to be the straight line. How do we prove this?

Firstly, by rescaling our plane, we can assume that the endpoints are $p = (0,0)$ and $q = (1,0)$. Now let us transform into polar coordinates:

$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta. \end{aligned}$$

So,

$$dx = \frac{\partial x}{\partial r}dr + \frac{\partial y}{\partial r}dr = \cos\theta dr - r\sin\theta d\theta,$$

and similarly,

$$dy = \sin\theta dr + r\cos\theta d\theta.$$

Thus,

$$\begin{aligned} ds^2 &= (\cos\theta dr - r\sin\theta d\theta)^2 + (\sin\theta dr + r\cos\theta d\theta)^2 \\ &= dr^2 + r^2 d\theta^2. \end{aligned} \tag{6.3}$$

Now, looking back at our integral in Equation 6.1, we get

$$\text{Length} = \int \sqrt{dr^2 + r^2 d\theta^2} \geq \int dr = 1.$$

The important feature of this calculation, as far as we are concerned here, is the computation of Equation 6.3. We will come across calculations like this often, so it will be helpful to give a common form for the formulae for ds^2. We write,

$$ds^2 = g_{ij} dx^i dx^j,$$

and call this a *Riemannian metric.*

Consider, for instance, the example of the plane. We have already made two Riemannian metric computations, namely

$$ds^2 = dx^2 + dy^2$$

and

$$ds^2 = dr^2 + r^2 d\theta^2.$$

Both of these Riemannian metrics describe the geometry of the plane.

On the other hand, if we were to define a metric by

$$ds^2 = du^2 + \cos^2\theta dv^2$$

we would get a completely different (non-planar) type of geometry. We can't see yet how different this is, but we will understand once we have studied the features of Riemannian metrics.

Lecture 7

(Wed 17 September 2003)

7.1 Surfaces

We now begin to follow Gauss' 18th century work on the geometry of surfaces, on which Einstein's work was eventually based.

Firstly, we need the definition of a surface. For the time-being, we think of a surface as living in three dimensional space. For instance, think of a sphere in $\mathbb{R}^3$. What makes it a surface? In essence it is the fact that in small regions, the surface looks like a deformed version of the Euclidean plane. This idea suggests the notion of coordinate charts, which we formalize as follows.

Definition 7.1. A surface $\Sigma \subseteq \mathbb{R}^3$ is a subset of $\mathbb{R}^3$ with the following property: For every $\mathbf{p} \in \Sigma$ there is a smooth map

$$\mathbf{r} : U \to \Sigma,$$

where $U \subseteq \mathbb{R}^2$ is an open set containing $\mathbf{0}$, such that

$$\mathbf{r}(\mathbf{0}) = \mathbf{p}$$

and $D\mathbf{r}(\mathbf{0})$ is injective (*ie*, non-singular).

What is this basic idea here? If our surface were like the surface of this page, there would be a good choice of coordinates to describe our position on it — namely, the usual x- and y-coordinates. However, if our surface were something curved, like a surface of the sphere, we would have more difficulty choosing our coordinate system. This need for coordinates is the fulfilled by the function $\mathbf{r}$, which *parameterizes* the surface.

Picture: parametrized section of a surface, with coordinate grid.

Note that there is no preferred choice of coordinate system. We are free to make our own choice. However, we want to ensure that our coordinate system is not completely useless. For instance, we could conceivably choose our x-coordinate and y-coordinate to point in exactly the same direction, but that would not parameterize the surface, it would only parameterize a line in it, and in a rather redundant way at that. So the condition that the derivative be injective is there to ensure we avoid this kind of degenerate situation.

Remark. There are indeed *a lot* of possible choices for local parameterizations. This is just as we know for the plane. We can describe points in the plane via Cartesian coordinates, polar coordinates, oblique coordinates, or any of an infinitude of other possibilities. This makes surface theory somewhat difficult, but it also makes the ideas very flexible to work with.

Example 7.2 (Euclidean plane). The flat plane is a surface in an obvious way.

Example 7.3 (The cylinder). We might parameterize the cylinder by the function

$$\mathbf{r}(u,v) = (\cos u, \sin u, v).$$

<u>Picture</u>: cylinder.

In general, we need more than one coordinate patch to parameterize a surface, but in this case our parameterization works everywhere. To check the non-singularity condition, we compute

$$\begin{aligned} \mathbf{r}_u &= (-\sin u, \cos u, 0) \\ \mathbf{r}_v &= (0, 0, 1). \end{aligned}$$

These are clearly linearly independent for any values of u and v.

Example 7.4 (The sphere). We can describe our position on a sphere by the two coordinates $u = latitude$ and $v = longitude$. After drawing a nice diagram of all the angles involved, we get the following parameterization.

$$\mathbf{r}(u,v) = (\cos u \cos v, \cos u \sin v, \sin u).$$

The partial derivatives are

$$\begin{aligned} \mathbf{r}_u &= (-\sin u \cos v, -\sin u \sin v, \cos u) \\ \mathbf{r}_v &= (-\cos u \sin v, \cos u \cos v, 0). \end{aligned}$$

To see if these are linearly independent, we can take their cross-product:

$$\begin{aligned} \mathbf{r}_u \wedge \mathbf{r}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin u \cos v & -\sin u \sin v & \cos v \\ -\cos u \sin v & \cos u \cos v & 0 \end{vmatrix} \\ &= \cos u (\cos u \cos v, \cos u \sin v, \sin u). \end{aligned} \tag{7.1}$$

For linear independence, this cross-product should be non-zero. The quantity 7.1 is non-zero everywhere, except when $\cos u = 0$. This corresponds geometrically to the north and south poles. Thus our parameterization is non-degenerate everywhere except at the north and south poles.

The fact that our parameterization fails at these two points does not mean, of course, that the sphere fails to be a surface. All it means is that we need a different parameterization to deal with these points. A suitable new parameterization can be produced by turning the sphere around to move the north and south poles to the east and west poles (!?!) and then using the above parameterization for the newly arranged sphere. We again get degeneracies, but at different points of the space, so the sphere is *covered* by good parameterizations.

It is normal to require more than one parameterization to cover our surface, as in this example. With the cylinder, we just got lucky.

Exercise 7.5. Find a parameterization for the torus (the surface of a doughnut).

7.2   Riemannian metrics

The geometry of a surface is determined by the lengths of curves on it.

Definition 7.6. If Σ_1 and Σ_2 are surfaces and

$$T : \Sigma_1 \to \Sigma_2$$

is a map (*ie*, a one-to-one correspondence) which preserves the lengths of curves,

$$\text{Length}_{\Sigma_1}(\gamma) = \text{Length}_{\Sigma_2}(T \circ \gamma),$$

then we call T an *isometry*.

Example 7.7. You can map a patch of a cylinder isometrically to a patch of the plane. This corresponds to the process of unrolling a tube of paper, which does not alter the lengths of curves. However, you cannot unroll a sphere — there is no way to map a piece of a sphere to a piece of the plane without distorting lengths of curves. If you try to flatten a portion of a spherically shaped piece of paper, you would always end up crumpling it. This explains why it is not possible to make a nice map of the world which preserves both angles and areas — something needs to be distorted. We will prove this next lecture.

So, how do we measure the length of a curve on a surface?

Definition 7.8. Define the *first fundamental form*, or *metric tensor*, for a given parameterization $\mathbf{r}(x^1, x^2)$ of a surface, to be

$$g_{ij} = \mathbf{r}_i \cdot \mathbf{r}_j,$$

where we use the notation $\mathbf{r}_i = \frac{\partial \mathbf{r}}{\partial x^i}$.

For a parametrized surface, this gives a collection of four numbers, which can be written in a 2×2 matrix. Classically, one would use the notation,

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix},$$

where

$$
\begin{aligned}
E &= \frac{\partial \mathbf{r}}{\partial x^i} \cdot \frac{\partial \mathbf{r}}{\partial x^i}, \\
F &= \frac{\partial \mathbf{r}}{\partial x^i} \cdot \frac{\partial \mathbf{r}}{\partial x^j}, \\
G &= \frac{\partial \mathbf{r}}{\partial x^j} \cdot \frac{\partial \mathbf{r}}{\partial x^j}.
\end{aligned}
$$

Proposition 7.9. *The length of a curve γ in the surface Σ is given by*

$$\int_\gamma ds,$$

where

$$ds^2 = g_{ij}dx^i dx^j. \tag{7.2}$$

We are using the Einstein summation convention here, so the right-hand side of Equation 7.2 is actually a summation over the indices i and j. We are choosing to use parameters x^1 and x^2 rather than u and v in this definition because it makes the notation easier. However, we reserve the right to switch between the two nomenclatures indiscriminately and unashamedly.

The interpretation of Proposition 7.9 is as follows. The curve is given by the equation

$$t \mapsto x^i(t),$$

in the parameter space. The integral "$\int ds$" means

$$\int \sqrt{g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}}\, \mathrm{d}t.$$

This notation is justified by the fact that $\mathrm{d}t$'s cancel out, formally speaking.

Proof.

$$
\begin{array}{rll}
\left(\frac{ds}{dt}\right)^2 & = \left|\frac{d\mathbf{r}}{dt}\right|^2 & \text{(def'n of length of a curve)} \\[2mm]
& = \left|\mathbf{r}_i\frac{dx^i}{dt}\right|^2 & \text{(chain rule)} \\[2mm]
& = \left(\mathbf{r}_i\frac{dx^i}{dt}\right)\cdot\left(\mathbf{r}_j\frac{dx^j}{dt}\right) & (\text{(length)}^2 = \text{dot product with itself)} \\[2mm]
& = (\mathbf{r}_i\cdot\mathbf{r}_j)\frac{dx^i}{dt}\frac{dx^j}{dt} & \text{(distributivity of the dot product)} \\[2mm]
& = g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} & \text{(defn of } g_{ij})
\end{array}
$$

$\square$

This proposition shows that the lengths of curves on a surface are prescribed entirely by the metric tensor g_{ij}. Since geometry is really about distances, which arise from the lengths of curves, we can see that the geometry of a surface is completely described by the metric tensor.

Example 7.10. Let us compute the metric tensor for the cylinder. Using our calculations above, we see that

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Note that this is exactly the same metric tensor that we would get for the plane, using ordinary x and y coordinates. Observing that these metric tensors are the same is another way (a more *intrinsic* way) of seeing that the cylinder and the plane are locally isometric.

Example 7.11. For the sphere, our earlier calculations yield

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \cos^2 u \end{pmatrix}.$$

Clearly this is different from the metric tensor we got for the cylinder and the plane. Unfortunately, however, this observation is not enough, on its own, to demonstrate that the sphere is *not* locally isometric to the plane or the cylinder. It is conceivable (but false) that we just chose some poor alternative parameterization in which the metric tensor is not so nice. In the ensuing lectures, however, we will extract from this metric tensor some information which is independent of the choice of parameterization. This will allow us to see that the sphere and the cylinder are locally non-isometric.

As a final remark, note that parameterization-independent features of geometry are what Einstein needs for a reasonable description of physics. He wishes to allow us to use all sorts of crazy coordinate systems, and his physical laws should hold on an equal footing in all of them. Thus, the laws of physics should be described in terms of the intrinsic features of the geometry.

Lecture 8

(Fri 19 September 2003)

<u>Handout</u>: Summary sheet of formulae from Riemannian geometry.

Last time, we defined a surface Σ to be a subset of three-dimensional space which is covered by "coordinate patches". A coordinate patch is a vector valued function

$$\mathbf{r} : \mathbb{R}^2 \to \mathbb{R}^3,$$

which satisfies certain non-degeneracy conditions, which we were discussed last time (see Definition 7.1).

<u>Picture</u>: map $\mathbf{r}$ from $\mathbb{R}^2$ to a coordinate patch on a surface.

Example 8.1 (Surface of revolution). We can construct a surface by taking the graph $y = f(x)$ for some function $f(x)$, and rotating it around the x-axis in $\mathbb{R}^3$.

<u>Picture</u>: Surface of revolution.

This *surface of revolution* can be parametrized as follows:

$$\mathbf{r}(u, v) = (u, f(u) \cos v, f(u) \sin v).$$

The partial derivatives of this parameterization are

$$
\begin{aligned}
\mathbf{r}_u &= (1, f'(u) \cos v, f'(u) \sin v), \\
\mathbf{r}_v &= (0, f(u) \sin v, f(u) \cos v).
\end{aligned}
$$

These will be linearly independent so long as f is not zero.

This example encompasses Examples 7.3 and 7.4 from last time. Firstly, we can realize the cylinder as being the surface of revolution of the function $f(x) = 1$. This viewpoint yields the same parameterization as we produced in the last lecture, up to some reordering of the coordinates.

We can also see the sphere as being the surface of revolution of the function $f(x) = \sqrt{1 - x^2}$, whereby we get the parameterization,

$$\mathbf{r}(u, v) = (u, \sqrt{1 - u^2} \cos v, \sqrt{1 - u^2} \sin v).$$

This is a different parameterization to the one we gave last time for the sphere, which underscores the fact that there are *a lot* of different possible parameterizations for any surface.

We defined the metric tensor,

$$g_{ij} = \mathbf{r}_{,i} \cdot \mathbf{r}_{,j} = \frac{\partial \mathbf{r}}{\partial x^i} \cdot \frac{\partial \mathbf{r}}{\partial x^j}.$$

This is comprised of four quantities, which can be written in a 2×2 matrix. Since the dot product is symmetric, the matrix we get is symmetric. Often this matrix is written as

$$M = \begin{pmatrix} E & F \\ F & G \end{pmatrix}.$$

As well as being symmetric, M is *positive definite* — that is,

$$\mathbf{v}^t M \mathbf{v} \geq 0 \quad \text{for all } \mathbf{v} \in \mathbb{R}^2,$$

with equality iff $\mathbf{v} = \mathbf{0}$. To prove this, we calculate:

$$
\begin{aligned}
\mathbf{v}^t M \mathbf{v} &= v^i g_{ij} v^j \\
&= v^i \left(\frac{\partial \mathbf{r}}{\partial x^i} \cdot \frac{\partial \mathbf{r}}{\partial x^j} \right) v^j \\
&= \left(v^i \frac{\partial \mathbf{r}}{\partial x^i} \right) \cdot \left(v^j \frac{\partial \mathbf{r}}{\partial x^j} \right).
\end{aligned}
$$

This last expression is simply the dot product of a new vector $\mathbf{w} = v^i \frac{\partial \mathbf{r}}{\partial x^i}$ with itself, so is positive.

The tensor g_{ij} allows us to measure lengths and areas in our surface Σ. For instance, last time we saw that

$$\text{Length}(\gamma) = \int_\gamma ds,$$

where

$$
\begin{aligned}
ds^2 &= g_{ij} dx^i dx^j \\
&= E du^2 + 2F du dv + G dv^2.
\end{aligned}
$$

Let us apply this to an example. Consider the cylinder, parametrized by

$$\mathbf{r}(u, v) = (\cos u, \sin u, v).$$

On this surface, we take the curve which is given by

$$u(t) = t, \qquad v(t) = t, \qquad\qquad (0 \leq t \leq 2\pi).$$

This describes a helix on the cylinder. Compute the length of this helix.

Picture: Helix on a cylinder.

We can compute by brute force, applying our definition of the Riemannian metric.

$$\text{Length} = \int_\gamma ds = \int_\gamma \sqrt{du^2 + dv^2} = \int_0^{2\pi} \sqrt{dt^2 + dt^2} = \sqrt{2} \int_0^{2\pi} dt = 2\pi\sqrt{2}.$$

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Of course, for this example, we can also compute the answer more simply, by first "unrolling" the cylinder to a flat plane, which we know to be an isometry, and then using Pythagoras' Theorem. But the method above works for all curves on all surfaces.

What about area? It seems sensible that we can compute area as a double integral of some kind:

$$\text{Area} = \iint (\text{something}).$$

To find out what the "something" should be, we consider a tiny little rectangular piece of our coordinate space, with sides given by du and dv.

Picture: Area element being transformed by the parameterization $\mathbf{r}$.

Under the parameterization map $\mathbf{r}$, the sides du and dv are mapped to vectors $\mathbf{r}_u du$ and $\mathbf{r}_v dv$, respectively. It is important to note that these two vectors are not necessarily perpendicular in $\mathbb{R}^3$. The little rectangle we started with is mapped to a parallelogram with area

$$d(\text{Area}) = |\mathbf{r}_u \wedge \mathbf{r}_v| du dv.$$

Remark. We could note the following identity: For any vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{R}^3$,

$$(\mathbf{a} \wedge \mathbf{b}) \cdot (\mathbf{c} \wedge \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}).$$

Applying this in our present situation, we get

$$|\mathbf{r}_u \wedge \mathbf{r}_v|^2 = (\mathbf{r}_u \wedge \mathbf{r}_v) \cdot (\mathbf{r}_u \wedge \mathbf{r}_v) = EG - F^2.$$

In summary, the *element of area*, *ie*, the thing which we need to integrate in order to compute areas, is given by

$$\begin{aligned} dA &= \sqrt{EG - F^2} du dv \\ &= g^{\frac{1}{2}} dx^1 dx^2, \end{aligned}$$

where $g = \det g_{ij}$.

Example 8.2 (Archimedes' tombstone). This discovery of Archimedes was engraved on his tombstone when he died.

Archimedes proved that the *cylindrical projection* of the sphere preserves areas. The cylindrical projection is the map from the unit sphere to the unit cylinder which is obtained by projecting points radially outwards from the z-axis.

Picture: cylindrical projection and representative areas.

Note, however, that the cylindrical projection does not preserve lengths.

To prove the preservation of areas, parameterize the sphere by

$$\mathbf{r}(u, v) = (\cos u \cos v, \cos u \sin v, \sin u),$$

as previously. Computing the Riemannian metric, we get

$$E = 1, F = 0, G = \cos^2 u.$$

If we parameterize the cylinder by letting u and v represent the latitudinal and longitudinal coordinates after cylindrical projection, we get

$$\mathbf{r}(u, v) = (\cos v, \sin v, \sin u).$$

So

$$\mathbf{r}_u = (0, 0, \cos u)$$

and

$$\mathbf{r}_v = (-\sin v, \cos v, 0),$$

and hence,

$$E = \cos^2 u, F = 0, G = 1.$$

Computing the area element in both cases, we get

$$EG - F^2 = \cos^2 u.$$

Thus, although the metric tensors themselves are different for the two surfaces, the area elements are the same, proving Archimedes' Theorem.

8.1 Changes of Coordinates

We want to be able to remove our dependence on the choice of parameterization for a surface — to "intrinsify" the geometry. So firstly, we need to understand effect of changes of coordinates.

Imagine two different (local) coordinate systems,

$$x^1, x^2$$

and

$$\tilde{x}^1, \tilde{x}^2,$$

for a surface Σ. Corresponding to these two coordinate systems, we will have two different metrics, g_{ij} and $\tilde{g}_{ij}$. How do these relate?

By the definition of the metric tensor,

$$\tilde{g}_{rs} = \frac{\partial \mathbf{r}}{\partial \tilde{x}^r} \cdot \frac{\partial \mathbf{r}}{\partial \tilde{x}^s} = g_{ij} \frac{\partial x^i}{\partial \tilde{x}^r} \frac{\partial x^j}{\partial \tilde{x}^s},$$

where the latter equality follows from the chain rule:

$$\frac{\partial \mathbf{r}}{\partial \tilde{x}^r} = \frac{\partial \mathbf{r}}{\partial x^i} \frac{\partial x^i}{\partial \tilde{x}^r},$$

So,

$$\tilde{g}_{rs} = g_{ij} a_r^i a_s^j, \tag{8.1}$$

where
$$a^p_q = \frac{\partial x^p}{\partial \tilde{x}^q}$$
is the matrix of partial derivatives of the transformation.

Equation 8.1 is the reason we call the Riemannian metric a *tensor*. In general, a *tensor* is an object which satisfies exactly this kind of transformation law.

Example 8.3. We have already seen two parameterizations of the plane, which must therefore give the same metric tensor. Namely,

$$dx^2 + dy^2 = dr^2 + r^2 d\theta^2.$$

Of course, these give rise to two different matrices,

$$g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$\tilde{g} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix},$$

but they represent the same tensor via the transformation law of Equation 8.1.

How could we realize that these two matrices correspond to the same metric tensor? If we were ingenious enough to think of the coordinate transformation

$$x = r \cos \theta, \qquad y = r \sin \theta,$$

then we could make the computations, and see directly that the two tensors are related by Equation 8.1.

On the other hand, if we are given coordinate descriptions of two different metric tensors, how are we going to be able to tell that there is *no* change of coordinates which takes one to the other? We need to come up with some kind of invariant to distinguish them.

Lecture 9

(Mon 22 September 2003)

The plan for this lecture is to understand what a straight line is. To make this sound less trivial, we are really interested in the meaning of a straight line on a curved surface. Since surfaces (or in general, *manifolds*) are the natural environments for physics, it is important that we have concepts which are analogous to the well-known features of straight-lines in Euclidean geometry.

9.1 Geodesics

The appropriate analogous concept is a *geodesic*. A geodesic in a surface Σ is basically a path that is no more curved than it absolutely has to be in order to stay on the surface.

How do we find the geodesics? Let's get at this indirectly by asking: supposing we are given a curve in the $\mathbb{R}^2$ (or $\mathbb{R}^3$), How do we measure its *curvature*?

In advanced calculus, we learn that

Curvature = Rate of change of the unit tangent vector with distance.

Let us elucidate on this. We are given a curve

$$\begin{aligned} \gamma : [0,1] &\rightarrow \mathbb{R}^3 \\ t &\mapsto \gamma(t) \end{aligned}$$

For a curve, as for a surface, we can make a different choice of parameterization. In particular, we may choose to parameterize the curve by arc length,

$$s \mapsto \gamma(s).$$

Another way of saying this is that if we consider γ as describing the trajectory of some point moving through $\mathbb{R}^3$, then in the arc-length parameterization, the speed of the particle is always one.

With the arc length parameterization, if we let $\mathbf{t} = \frac{d\gamma}{ds}$ be the tangent vector to the curve, then the length of $\mathbf{t}$ is always one. However, it's direction may be changing. The curvature describes the rate of change of direction of the curve, in other words,

$$\text{Curvature} = \left| \frac{d\mathbf{t}}{ds} \right| = \left| \frac{d^2\gamma}{ds^2} \right|.$$

In $\mathbb{R}^2$ (or $\mathbb{R}^3$, etc), the straight lines are the curves whose unit tangent vectors $\mathbf{t}$ are constant along their path. We will hold on to this fact as the clue for what should be considered a straight line in general surface geometry.

Now suppose we have a surface Σ in $\mathbb{R}^3$.

<u>Picture</u>: path in a surface, with a vector field along the curve.

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Consider a path $\gamma(t)$ which lies in the surface. Suppose, further, that at each point in the path we have a vector $\mathbf{v}(t)$, which is tangent to the surface Σ. What should it mean to say that this family of vectors is constant along the path? Clearly it can't actually be constant, since it is assumed to be always tangent to the surface, which is not flat.

The key trick, which gives us a good definition, is to take $\frac{d\mathbf{v}}{dt}$, which is some vector in $\mathbb{R}^3$ not necessarily tangent to the surface, and resolve it into its components tangent and normal to the surface Σ. Let us see this procedure in an example.

Example 9.1. Consider a field of unit vectors which point along a line of longitude on the surface of a sphere, as shown here.

Picture: tangential field along a circumference of the sphere.

Clearly these vectors are not constant in $\mathbb{R}^3$. They must change their direction in order to remain tangent to the surface. However, these vectors are changing their direction as little as possible in order to remain tangential. The derivative $\frac{d\mathbf{v}}{dt}$ is an inward-pointing normal to the sphere, which corresponds to the fact that $\mathbf{v}$ is being "pushed back" toward the sphere, in order to remain tangent to it. For this reason, we call this field of vectors "parallel" along the curve in the sphere.

Definition 9.2. We say that $\mathbf{v}(t)$ is *parallel* to along the curve $\gamma(t)$ if $\frac{d\mathbf{v}}{dt}$ has no component tangent to Σ — ie, $\frac{d\mathbf{v}}{dt}$ is normal to Σ at each point.

Theorem 9.3. *The notion of a parallel vector field is part of* intrinsic geometry — ie, *it is invariant under isometries.*

To explain this theorem, note that Definition 9.2 above apparently depends on the way in which our surface Σ is embedded in $\mathbb{R}^3$. This would be no good for our description of physics, because we are not given some ambient bizillion-dimensional space in which our four-dimensional spacetime lives. So we need the notion of parallel to be independent of the way a surface is embedded in a Euclidean space. This result (or a close companion of it) is Gauss' *Theorema Egregium.*

Before we tackle the proof, we need to decide how to represent a tangent vector field in the parametric coordinates, rather than with respect to the ambient space $\mathbb{R}^3$. To do so, remember that at any point, the two vectors $\mathbf{r}_{,i}$ and $\mathbf{r}_{,j}$ form a basis of the tangent plane to Σ. Thus, we can always write a vector tangent to the surface as[5]

$$\mathbf{v} = A^i \mathbf{r}_{,i}.$$

We will agree to represent a tangent vector just by these coefficients A^i.

---

[5] Again, using summation notation. Also, recall that the notation $\cdot_{,i}$ means $\frac{\partial \cdot}{\partial x^i}$.

Remark. Next lecture we will see that this representation A^i is meaningful, independent of the choice of coordinate system, as long as we use an appropriate transformation rule to transform to $\tilde{A}^i$ in a different coordinate system, $\tilde{x}^i$.

Proof of Theorem 9.3. We are interested in

$$\frac{d}{dt}(A^i\mathbf{r}_{,i})$$

along the curve

$$\gamma(t) = (x^1(t), x^2(t)).$$

Well,

$$\frac{d}{dt}(A^i\mathbf{r}_{,i}) = A^i_{,j}\frac{dx^j}{dt}\mathbf{r}_{,i} + A^i\mathbf{r}_{,ij}\frac{dx^j}{dt} = (A^i_{,j}\mathbf{r}_{,i} + A^i\mathbf{r}_{,ij})\frac{dx^j}{dt}.$$

For the vector field to be parallel along the curve, we want the above quantity to be perpendicular to the tangent plane. Taking dot products with the tangent vectors $\mathbf{r}_{,k}$, we want

$$(A^i_{,j}\mathbf{r}_{,i} \cdot \mathbf{r}_k + A^i\mathbf{r}_{,ij} \cdot \mathbf{r}_{,k})\frac{dx^j}{dt} = 0. \tag{9.1}$$

Note that $\mathbf{r}_{,i}\mathbf{r}_{,k} = g_{ik}$ is the metric, so the first term in the left-hand side clearly depends only on the metric g_{ik} of the surface. Thus, in order to prove that the whole notion of parallelism is invariant under isometries, all we need to show is that the quantity

$$\Gamma_{ijk} \overset{\text{def}}{=} \mathbf{r}_{,ij} \cdot \mathbf{r}_{,k} \tag{9.2}$$

also depends only on the metric g_{ij}. This is the purpose of the following lemma.

Lemma 9.4.

$$\Gamma_{ijk} = \frac{1}{2}(g_{ik,j} + g_{jk,i} - g_{ij,k}) \tag{9.3}$$

Proof. Computing,

$$g_{ik,j} = \frac{\partial}{\partial x^j}(\mathbf{r}_{,i} \cdot \mathbf{r}_{,k}) = \mathbf{r}_{,ij} \cdot \mathbf{r}_{,k} + \mathbf{r}_{,jk} \cdot \mathbf{r}_i.$$

Similarly,

$$g_{jk,i} = \mathbf{r}_{,ji} \cdot \mathbf{r}_{,k} + \mathbf{r}_{,ki} \cdot \mathbf{r}_j$$

and

$$g_{ij,k} = \mathbf{r}_{,ik} \cdot \mathbf{r}_{,j} + \mathbf{r}_{,jk} \cdot \mathbf{r}_{,i}$$

If we add these, and use the symmetry of second partial derivatives, we get the left-hand side of Equation 9.3, as desired. □

□

Usually this is expressed in a different notation. To explain the new notation, let us first introduce the convention that g^{ij} means the inverse of g_{ij}, so that

$$g_{ij}g^{jk} = \delta_i^k.$$

Then we express Equation 9.1 as

$$A^i_{;j} \frac{dx^j}{dt} = 0,$$

where

$$
\begin{aligned}
A^i_{;j} &= A^i_{,j} + A^l \Gamma_{ljk} g^{ik} \\
&= A^i_{,j} + A^l \Gamma^i_{lj}.
\end{aligned}
\tag{9.4}
$$

Here, we use the notation Γ^i_{lj} as a shorthand for $\Gamma_{ljk} g^{ik}$. This notation fits into the general conventions for raising and lowering indices, which we will discuss in more detail later.

The quantity $A^i_{;j}$ is called the *covariant derivative* of A. We can then say, in light of Equation 9.1, that a vector field along a curve is parallel if and only if its covariant derivative is zero.

Remark. Equation 9.4 shows that the covariant derivative is an ordinary derivative plus a correction term. This correction term is needed in order to make sure that the covariant derivative of a vector field along a curve is again tangent to the surface.

We will notice that the covariant derivative has many properties in common with the ordinary partial derivative. The key difference, however, is that the covariant derivative does not satisfy the symmetry condition of the partial derivative: $\mathbf{r}_{,ij} = \mathbf{r}_{,ji}$.

Lecture 10

(Wed 24 September 2003)

10.1   Some algebra

Last time we talked about g_{ij} and Γ_{ijk}, and pretty soon will be talking about things like R^i_{jkl} and even $R^i_{jkl;m}$. We should spend some time understanding what's going on with these kind of highly decorated quantities. We need to understand what a tensor really is.

As mathematicians, we like to think of coordinate-invariant things. We like to think of some abstract rules for vector spaces, and think of a vector as being just an element in a abstract vector space.

Physicists don't think that way, at least not those who write textbooks on general relativity. They prefer to think of a vector as a list of numbers — the coordinates of the mathematicians' vector. However, physicists are not stupid, and they realize that the choice of a coordinate system is not universally declared. So a vector is not actually a list of coordinates, but rather infinitely many lists of coordinates, one for each possible choice of basis.

As a brief digression, let us note where we will eventually be going with all of this. Consider a surface, such as a sphere, located in $\mathbb{R}^3$. At each point on the surface, the space of tangent vectors is naturally a vector space — identified with a two-dimensional subspace of the ambient space $\mathbb{R}^3$. However, considering the space of *all* tangent vectors to the surface gives us not one vector space, but a whole slew of two-dimensional vector spaces, one at each point in the surface. This is called a *vector bundle*.

Picture: a sphere and a couple of its tangent planes.

A vector field on the surface is then a choice of one vector from each of these vector spaces. For instance, there is a vector field describing the current wind direction over the surface of the earth, which associates to each point a vector tangent to the sphere.

Eventually, our tensors will be built from these vector bundles over surfaces (even manifolds). But for the moment we will forget all about bundles and concentrate on the situation for a single finite-dimensional vector space.

Let V be a vector space (specifically, an n-dimensional vector space over $\mathbb{R}$). Recall that a *basis B* for V is a list $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ of vectors in V such that <u>every</u> $\mathbf{v} \in V$ can be written <u>uniquely</u> as

$$
\begin{aligned}
\mathbf{v} &= \lambda^1 \mathbf{v}_1 + \lambda^2 \mathbf{v}_2 + \cdots + \lambda^n \mathbf{v}_n \\
&= \lambda^i \mathbf{v}_i,
\end{aligned}
$$

where we are using the summation notation as usual.

Note that if we choose a different set of basis vectors $\{\tilde{\mathbf{v}}_1, \dots, \tilde{\mathbf{v}}_n\}$, then we get a different list of coefficients for the vector $\mathbf{v}$. So a vector $\mathbf{v}$ gives ries to a map

$$
\begin{aligned}
\mathcal{B} &\rightarrow \mathbb{R}^n \\
B &\mapsto (\lambda^1, \dots, \lambda^n),
\end{aligned}
$$

where $\mathcal{B}$ is the set of all bases of the vector space. This map corresponds the fact that a choice of basis gives rise to a means of identifying the abstract vector space V with the concrete vector space $\mathbb{R}^n$.

Suppose, now, that I am given some map from the space of all bases $\mathcal{B}$ to $\mathbb{R}^n$. How do I know if this map corresponds to some vector $v \in V$? Is this automatic, or are there some conditions on our map for it to represent a vector?

Certainly there are some constraints. For instance, if the map sends some particular basis to $(0, \dots, 0)$, then it must do so for all bases, since in all bases the zero vector has the same representation. We want to understand which maps $\mathcal{B} \rightarrow \mathbb{R}^n$ actually come from bases.

The answer ensues by considering the following question: if we have two bases $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ and $\tilde{B} = \{\tilde{\mathbf{v}}_1, \dots, \tilde{\mathbf{v}}_n\}$, how can we describe the relationship between the representations of a vector in each of them?

Let's write each $\mathbf{v}_q$ in terms of the $\tilde{\mathbf{v}}_p$ basis vectors. We get

$$
\mathbf{v}_q = [\tilde{B}/B]_q^p \tilde{\mathbf{v}}_p, \tag{10.1}
$$

where $[\tilde{B}/B]$ will be our notation for the change of basis matrix from linear algebra. Now look at the representation of the same vector $\mathbf{v}$ with respect to the two bases:

$$
\begin{aligned}
\mathbf{v} &= \lambda^q \mathbf{v}_q \\
&= \tilde{\lambda}^p \tilde{\mathbf{v}}_p \\
&= \lambda^q [\tilde{B}/B]_q^p \tilde{\mathbf{v}}_p.
\end{aligned}
$$

So

$$
(\tilde{\lambda}^p - \lambda^q [\tilde{B}/B]_q^p) \tilde{\mathbf{v}}_p = \mathbf{0}.
$$

We thus obtain,

$$
\tilde{\lambda}^p = [\tilde{B}/B]_q^p \lambda^q. \tag{10.2}
$$

Equation 10.2 is the answer to our question. A map $\mathcal{B} \rightarrow \mathbb{R}^n$ represents a vector precisely if the numbers λ^q and $\tilde{\lambda}^p$ are related by Equation 10.2. So, to a physicist, a vector is something which is represented in any coordinate system by a list of numbers, such that when we change coordinate systems by Equation 10.1, the coordinate representations change by the transformation law, Equation 10.2.

Note that the location of the $\tilde{\ }$ is reversed in moving from the definition of the change of basis matrix of Equation 10.1 to the transformation law of Equation 10.2. For this reason, the transformation law 10.2 is called the *contragredient transformation law*, and something that satisfies it is called a *contravariant vector*.

Remark. A technical comment: consider two spaces X and Y and a group G which acts on both of them. A map $f : X \to Y$ is called *equivariant* if

$$f(g(x)) = g(f(x)) \quad \text{for all } x \in X, g \in G.$$

So the discussion above can be rephrased by saying that a vector is a map from the space $\mathcal{B}$ of bases to the space $\mathbb{R}^n$, which is equivariant with respect to the actions of the group of invertible matrices on both which are described by Equations 10.1 and 10.2.

Other actions will lead to other types of objects, as we will see.

10.1.1   The dual space

The dual space of V is defined by

$$V^* = \text{Hom}(V; \mathbb{R}),$$

ie, it is the space of all linear maps from V to $\mathbb{R}$. It is a vector space. A basis $\{v_1, \dots, v_n\}$ of V gives rise to a *dual basis* $\{\phi^1, \dots, \phi^n\}$ of V^*, which is determined by

$$\phi^i(v_j) = \delta^i_j.$$

We want to know how to describe an element ϕ of the dual space V^* in this new physics language.

We can write $\phi = \mu_i \phi^i$. If we change the originally given basis $B = \{v_1, \dots, v_n\}$, then the dual basis $\{\phi^1, \dots, \phi^n\}$ will also change, and thus the coefficients μ_i will have to change in some way. Our mission is to work out how this transformation depends on the change of basis matrix $[\tilde{B}/B]$.

We look at

$$\phi(\mathbf{v}) = \mu_i \lambda^j \phi^i(\mathbf{v}_j) = \mu_i \lambda^j \delta^i_j = \mu_i \lambda^i,$$

where $\mathbf{v} = \lambda^j \mathbf{v}_j$ is an arbitrary vector in V. Then we get the equation

$$\tilde{\mu}_i \tilde{\lambda}^i = \mu_i \lambda^i.$$

But from earlier,

$$\tilde{\lambda}^i = [\tilde{B}/B]^i_j \lambda^j.$$

So,

$$(\tilde{\mu}_i [\tilde{B}/B]^i_j - \mu_j) \lambda^j = 0,$$

for all choices of λ^j, and hence

$$\mu_j = [\tilde{B}/B]^i_j \tilde{\mu}_i. \tag{10.3}$$

Equation 10.3 looks similar to Equation 10.2, but the location of the $\tilde{\ }$ is reversed. We call Equation 10.3 the *cogredient transformation*, and an object which satisfies it is called a *covariant vector*.

Remember that we expect the laws of physics to be formulated in a way which is independent of the choice of a coordinate system. Einstein described such a situation as a *generally covariant equation*. He noted one such possibility in [Ein3]:

If, therefore, a law of nature is expressed by equating all the components of a tensor to zero, it is generally covariant. By examining the laws of the formation of tensors, we acquire the means of formulating generally covariant laws.

That was a guiding thought which inspired him to formulate the laws of general relativity.

Lecture 11

(Fri 26 September 2003)

We are currently considering a finite-dimensional vector space V and its dual space V^*, which is the space of linear maps from V to $\mathbb{R}$. We are interested in the way we represent elements of these spaces with respect to different bases of V.

A vector in V is represented by $\lambda^i \mathbf{v}_i$, where the $\mathbf{v}_i$ are basis vectors of V. A covector in V^* can be represented by $\mu_j \phi^j$, where $\{\phi^j\}$ is the dual basis to $\{\mathbf{v}_i\}$. Both can be represented by a list of n coefficients, but they differ in the way these lists of coefficients transform when we change our choice of basis for V.

Specifically, the transformation laws are

$$\lambda^q = [B/\tilde{B}]^q_p \tilde{\lambda}^p$$

for vectors and

$$\mu_q = [\tilde{B}/B]^p_q \tilde{\mu}_q$$

for covectors, as shown last time.

Note that many mathematical constructions most naturally yield covectors rather than vectors.

Example 11.1 (The gradient of a function). The gradient of a real-valued function on $\mathbb{R}^n$ is most naturally thought of as a *covector field*, rather than a vector field as it is often described in basic calculus courses. To understand this reasoning, we should think of how the gradient of a function is used.

Given a function $f(x, y)$ on $\mathbb{R}^2$, the gradient of f is used for computing directional derivatives of the function f. To compute a directional derivative, calculus students are often taught that one should take the dot product of the gradient vector with the appropriate direction vector. However, this process involves the "unnatural" step of introducing the dot product. A more natural way is to think of the gradient as a covector with components

$$f_{,i} = \frac{\partial f}{\partial x^i}.$$

Then the directional derivative of f in direction of the vector λ^j is simply given by

$$\lambda^i f_{,i}.$$

11.1   Raising and lowering indices

It is clear from Example 11.1 that there is some connection between the relationship of vectors and covectors and the introduction of a dot product.

Definition 11.2. Recall that an *inner product* on a vector space V is a map

$$V \times V \to \mathbb{R}$$

(which we will denote by $\cdot$, as for the usual dot product), such that

(i) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$,

(ii) $\mathbf{a} \cdot \mathbf{a} \geq 0$, with equality iff $\mathbf{a} = \mathbf{0}$, and

(iii) $(\lambda_1 \mathbf{a}_1 + \lambda_2 \mathbf{a}_2) \cdot \mathbf{b} = \lambda_1 \mathbf{a}_1 \cdot \mathbf{b} + \lambda_2 \mathbf{a}_2 \cdot \mathbf{b}$.

A vector space V equipped with an inner product is called an *inner product space.*

If we have an inner product space V, we can define a linear map

$$\Phi : V \to V^*$$

by

$$\mathbf{v} \mapsto (\mathbf{w} \mapsto \mathbf{v} \cdot \mathbf{w}).$$

To rephrase this, we can write

$$\Phi(\mathbf{v}) = \phi_{\mathbf{v}} \in V^*,$$

where $\phi_{\mathbf{v}}$ is the linear map defined by

$$\phi_{\mathbf{v}}(\mathbf{w}) = \mathbf{v} \cdot \mathbf{w}.$$

Theorem 11.3. *The map $\Phi : V \to V^*$ is an isomorphism (if V is finite dimensional).*

Proof. Since $\dim V = \dim V^*$, we need only show that $\ker(\Phi) = (0)$. But

$$\ker(\Phi) = \{\mathbf{v} | \phi_{\mathbf{v}} = \mathbf{0}\}.$$

If $\phi_{\mathbf{v}} = 0$ then, in particular, $\phi_{\mathbf{v}}(\mathbf{v}) = \mathbf{v} \cdot \mathbf{v} = 0$. So $\mathbf{v} = \mathbf{0}$ by (ii) above. $\square$

Theorem 11.3 is a piece of mathematics, but of course it can be also interpreted by the physicists. In that case, the theorem yields a flurry of indices, which we will now reproduce.

Firstly, an inner product is described by its coefficients

$$g_{ij} = \mathbf{v}_i \cdot \mathbf{v}_j,$$

where the $\mathbf{v}_i$ are basis vectors for our vector space V. But of course, knowing what we know about this physics notation, we must describe not just the entries of the object g_{ij}, but also the way in which this object transforms under a change of basis for V. As an exercise, the reader can confirm that the transformation law for an inner product is

$$g_{ij} = [\tilde{B}/B]_i^p [\tilde{B}/B]_j^q \tilde{g}_{pq}.$$

The axioms for an inner product, (i)–(iii) of Definition 11.2, become the requirements that g_{ij} is symmetric (*ie*, $g_{ij} = g_{ji}$) as well as a notion of *positive definiteness* for a tensor.

So now, with this interpretation, what becomes of Theorem 11.3? In particular, which covector should correspond to a given vector $\lambda^i \mathbf{v}_i$?

The answer is $\mu_j \phi^j$, where

$$\mu_j = g_{ij} \lambda^i. \tag{11.1}$$

To explain why Equation 11.1 is indeed the correct answer, we note that in order to identify a covector, we need to see what it does to all the basis vectors v_k. So, we check:

$$\phi_{\mathbf{v}}(\mathbf{v}_k) = \mathbf{v} \cdot \mathbf{v}_k = \lambda^i \mathbf{v}_i \cdot \mathbf{v}_k = \lambda^i g_{ij}. \tag{11.2}$$

On the other hand, for an arbitrary covector $\mu_j \phi^j$, evaluation on the basis vector $\mathbf{v}_k$ gives

$$(\mu_j \phi^j)(\mathbf{v}_k) = \mu_j \phi^j(\mathbf{v}_k) = \mu_j \delta_k^j = \mu_k. \tag{11.3}$$

Comparing Equations 11.2 and 11.3, we see that the required values of μ_j are as given in Equation 11.1..

This procedure of turning a vector λ^i into its corresponding covector μ_j, via a chosen inner product g_{ij} is aptly called "lowering an index". In the economy of physics notation, we will often represent the "lowered" covector coming from a vector A^i as A_j, so that then

$$A_j = g_{ij} A^i.$$

We can also "raise an index", a procedure which, given a choice of inner product, takes a covector and produces a vector. Let us define g^{ij} to be the matrix inverse to g_{ij}. We raise indices by multiplying by g^{ij} — that is,

$$A^j = g^{ij} A_i.$$

You can check that if you take some vector and lower an index, and then raise that index again, you will end up with exactly the same vector that you first thought of. The same goes for these operations in reverse.

11.2 Tensors

Definition 11.4. A *tensor* is a "thing" which is represented in any coordinate system by a list of quantities indexed by superscripts and subscripts, such that the quantities transform contragrediently for the upper indices and cogrediently for the lower ones.

So, for instance, if we have a tensor described by quantities R^i_{jkl}, then under a change of basis from B to $\tilde{B}$, the new representation $\tilde{R}^i_{jkl}$ of the tensor is related to the original by

$$R^i_{jkl} = [B/\tilde{B}]^i_{\tilde{i}} [\tilde{B}/B]^{\tilde{j}}_j [\tilde{B}/B]^{\tilde{k}}_k [\tilde{B}/B]^{\tilde{l}}_l \tilde{R}^{\tilde{i}}_{\tilde{j}\tilde{k}\tilde{l}}.$$

Note that if a tensor has all its components zero with respect to some particular choice of basis, then its components will be zero with respect to *every* choice of basis.

As a caution, we should point out at this stage that there are various quantities in physics which are decorated with indices, but which are *not* tensors — ie, they do not transform in the manner just described. Let us see some examples.

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- The coordinate transformation matrix $[B/\tilde{B}]_q^p$ is not a tensor. This is for the rather stupid reason that a change of basis matrix between two fixed bases B and $\tilde{B}$ will not change no matter what other choice of basis $\hat{B}$ we are currently using.

- More interestingly, the quantity Γ_{ijk} which we defined in Lecture 9 is not a tensor. This is because it was given by the quantities $\mathbf{r}_{,ij} \cdot \mathbf{r}_k$, which are not tensors.

  Why not? Consider a surface in $\mathbb{R}^3$, and a point p on it. Without loss of generality (by relocating the surface in $\mathbb{R}^3$), we can assume that the surface is tangent to the (x, y)-plane at the point p. Now if we choose our local parameterization for the surface to be given in terms of the x and y coordinates of $\mathbb{R}^3$ (*ie*, parameterizing the surface locally as a graph of a function $f(x, y)$), then we will see that the quantities $\mathbf{r}_{,ij} \cdot \mathbf{r}_k$ all vanish at the point p.

  It follows that $\mathbf{r}_{,ij} \cdot \mathbf{r}_k$ cannot be a tensor: certainly it cannot be zero in *all* choices of coordinate systems, or we would have proven that every surface is flat.

Lecture 12

(Mon 29 September 2003)

12.1 Curvature

Consider a surface Σ in $\mathbb{R}^3$. A fundamental feature of the surface, at an intuitive level, is its curvature. How are we to analyze its curvature mathematically?

Picture: Surface with a unit normal at a point.

Since the surface is embedded in $\mathbb{R}^3$, at each point p of the surface we can find a outward-pointing unit normal vector. Let us call this $\mathbf{n}(p)$. Specifically, $\mathbf{n}(p)$ is the unit vector in the direction of $\mathbf{r}_{,1} \wedge \mathbf{r}_{,2}$ (or the negative of this if we wrote our coordinates in the wrong order). So $\mathbf{n}$ defines a map

$$\mathbf{n} : \Sigma \to S^2,$$

where the image space S^2 is the space of unit vectors in $\mathbb{R}^3$, also known as the unit sphere. The map $\mathbf{n}$ is called the *Gauss map*.

As we move over a curved surface, the unit normal vector $\mathbf{n}(p)$ will change. We can measure the curvature of the surface by the rate of change of the Gauss map $\mathbf{n}$. To make the quantitative, we use the following observation.

Lemma 12.1. *The vector $\mathbf{n}_{,1} \wedge \mathbf{n}_{,2}$ is parallel[6] to $\mathbf{n}$.*

Proof. The key observation is that the derivative of a vector-valued function with constant length (such as $\mathbf{n}$) is orthogonal to the function itself. To see this, note that the product rule applied to $\mathbf{n} \cdot \mathbf{n}$ implies

$$\mathbf{n}_{,1} \cdot \mathbf{n} = \frac{1}{2}(\mathbf{n} \cdot \mathbf{n})_{,1}.$$

But since $\mathbf{n}$ is always length one, the right-hand side vanishes. The same holds for differentiation with respect to x^2. We therefore see that $\mathbf{n}_1$ and $\mathbf{n}_2$ are both perpendicular to $\mathbf{n}$. The result follows. $\qquad\qquad\square$

Thus $\mathbf{n}_1 \wedge \mathbf{n}_2$ is a scalar multiple of $\mathbf{r}_1 \wedge \mathbf{r}_2$. We can now make the following definition.

Definition 12.2. The *Gauss curvature K* is defined by

$$\mathbf{n}_1 \wedge \mathbf{n}_2 = K(\mathbf{r}_1 \wedge \mathbf{r}_2).$$

---

[6]The definition of two vectors being *parallel* is that one is a scalar multiple of the other, which includes the possibility that $\mathbf{n}_{,1} \wedge \mathbf{n}_{,2}$ is zero.

It is possible to give a geometric interpretation of the curvature as follows. The area element of the surface Σ is given by

$$(\mathbf{r}_{,1} \wedge \mathbf{r}_{,2})dx^1 dx^2.$$

The map $\mathbf{n}$ gives a map from Σ to the unit sphere S^2, where the area element will be

$$(\mathbf{n}_{,1} \wedge \mathbf{n}_{,2})dx^1 dx^2.$$

Thus, the Gauss curvature at a point p on the surface is the ratio of the area traced out on the unit sphere by the Gauss map over a small neighbourhood of $p \in \Sigma$ to the area of that small neighbourhood on Σ itself. (Of course, there is an issue of sign here, but for the moment we will sweep this under the rug.)

Picture: small area on the surface Σ and the corresponding area traced out on S^2 by the Gauss map.

Example 12.3. The Gauss curvature of a sphere of radius R is constant at $1/R^2$. This follows from the geometric interpretation just given, since the Gauss map simply maps a point on the sphere to the vector pointing in the same direction on the unit sphere.

Example 12.4. The Gauss curvature of the plane is zero, as one would expect. The Gauss map sends the entire plane to a single point on the unit sphere.

According to Definition 12.2 above, the Gauss curvature appears to be an extrinsic quantity — it ostensibly depends upon the way in which the surface is embedded in $\mathbb{R}^3$. This is what makes the following theorem remarkable.

Theorem 12.5 (Gauss' *Theorema Egregium*). *The curvature K is intrinsic, ie, it depends only on the metric g_{ij} (and its derivatives).*

Proof. Step A:
We will need to consider the second derivatives of the parameterization function $\mathbf{r}$. Let us write these second derivatives in terms of the basis $\{\mathbf{r}_{,1}, \mathbf{r}_{,2}, \mathbf{n}\}$. Doing so is is precisely the rôle of the Γ^i_{jk} defined in Lecture 9. Recall the definition of the Christoffel symbol (Equation 9.2):

$$\Gamma^i_{jk} = g^{il}\Gamma_{jkl} = g^{il}\mathbf{r}_{,jk} \cdot \mathbf{r}_{,l}.$$

Using this, the reader can check that

$$\mathbf{r}_{,jk} = \Gamma^i_{jk}\mathbf{r}_{,i} + b_{jk}\mathbf{n}, \tag{12.1}$$

where the b_{jk} are some real numbers.

Note that the Γ^i_{jk} are themselves defined only in terms of the metric g_{ij}, as we saw in Lemma 9.3. Furthermore, observe that they are symmetric in j and k (ie, $\Gamma^i_{jk} = \Gamma^i_{kj}$) as the formula of Lemma 9.3 also shows.

Step B:

We can, in fact, calculate Gauss curvature from the b_{jk}'s of in Equation 12.1. In fact,

$$K = b/g, \tag{12.2}$$

where $b = \det(b_{ij})$ and $g = \det(g_{ij})$.

Why? Note that

$$b_{jk} = \mathbf{r}_{,jk} \cdot \mathbf{n}.$$

This can be rewritten as

$$b_{jk} = -\mathbf{r}_{,j} \cdot \mathbf{n}_{,k}, \tag{12.3}$$

by applying the product rule in differentiating the constant (zero) quantity $\mathbf{r}_{,j} \cdot \mathbf{n}$.

Thus,

$$b = b_{11}b_{22} - b_{12}b_{21} = (\mathbf{r}_{,1} \cdot \mathbf{n}_{,1})(\mathbf{r}_{,2} \cdot \mathbf{n}_{,2}) - (\mathbf{r}_{,1} \cdot \mathbf{n}_{,2})(\mathbf{r}_{,2} \cdot \mathbf{n}_{,1}),$$

by applying Equation 12.3 four times. Now we employ the identity,

$$(\mathbf{a} \wedge \mathbf{b}) \cdot (\mathbf{c} \wedge \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}), \tag{12.4}$$

and get

$$b = (\mathbf{r}_{,1} \wedge \mathbf{r}_{,2}) \cdot (\mathbf{n}_{,1} \wedge \mathbf{n}_{,2}).$$

By applying the definition of Gauss curvature, and the identity 12.4 again, we get

$$b = Kg.$$

Step C:

Differentiate Equation 12.1 again:

$$\mathbf{r}_{,jkl} = \Gamma^i_{jk,l}\mathbf{r}_{,i} + \Gamma^i_{jk}\mathbf{r}_{,il} + b_{jk,l}\mathbf{n} + b_{jk}\mathbf{n}_{,l}.$$

Now take the dot product with $\mathbf{r}_{,m}$, to get

$$\mathbf{r}_{,jkl} \cdot \mathbf{r}_{,m} = \Gamma^i_{jk,l}g_{im} + \Gamma^i_{jk}\Gamma_{ilm} - b_{jk}b_{lm},$$

where the last term is a result of Equation 12.3. If we raise an index in the second term (and perform some relabelling), we get a common factor of g_{im}:

$$\mathbf{r}_{,jkl} \cdot \mathbf{r}_{,m} = g_{im}(\Gamma^i_{jk,l} + \Gamma^p_{jk}\Gamma^i_{pl}) - b_{jk}b_{lm}. \tag{12.5}$$

Notice that the left-hand side of Equation 12.5 is symmetric in k and l. This is not immediately obvious for the right-hand side, so the observation must be telling us something. We therefore write the corresponding formula for $\mathbf{r}_{,jlk} \cdot \mathbf{r}_m$, set them equal, and subtract. When we have done this, we will have,

$$b_{jk}b_{lm} - b_{jl}b_{km} = g_{im}(\Gamma^i_{jk,l} - \Gamma^i_{jl,k} + \Gamma^p_{jk}\Gamma^i_{pl} - \Gamma^q_{jl}\Gamma^i_{qk}). \tag{12.6}$$

The quantity $\Gamma^i_{jk,l} - \Gamma^i_{jl,k} + \Gamma^p_{jk}\Gamma^i_{pl} - \Gamma^q_{jl}\Gamma^i_{qk}$ arises frequently, and so is given a name: the *Riemann curvature tensor*. It is denoted by R^i_{jkl}. Of course, we have

not yet proven that it is actually a tensor (*ie*, that it transforms appropriately), but it is.

Step D:
    We now have

$$b_{jk}b_{lm} - b_{jl}b_{km} = R_{mjkl}.$$

This equation represents sixteen different equations, most of which are extremely boring. For instance, putting $(m, j, k, l) = (1, 1, 1, 1)$, we get a left-hand side which is clearly zero and a right-hand side which is almost as clearly zero. But amongst all those uninteresting equations lie a couple of interesting ones.
    Put $(m, j, k, l) = (2, 1, 1, 2)$. We get

$$b = b_{22}b_{11} - b_{12}b_{21} = R_{2112}.$$

But R_{mjkl} has been constructed entirely from g_{ij} and its derivatives, so the theorem is proven.

$\square$

Corollary 12.6. *You can't make a decent map of the world — something has to give.*

Proof. To make an undistorted map of the world, we would want to isometrically map the sphere to the plane. Well, actually, an isometry would lead to a fairly impractical map, since the scale would be one to one, but an isometry followed by some rescaling would be desirable.
    The *Theorema Egregium* says that curvature is an invariant of isometry. But the curvature of the plane is zero, while that of the sphere is nonzero. Some distortion of distances must occur.

$\square$

Remark. In all the example considered so far, the Gauss curvature has been constant over the surface. This is far from the case in general. The Gauss curvature is a real-valued function on the surface.

    The issue we have not discussed is, what is the meaning of the sign of the Gauss curvature? The answer is as follows: the sign of of $K(p)$ indicates whether the surface is curving in the same direction no matter which way we move from the point p (positive curvature), or whether it in different directions, like a saddle (negative curvature).

Picture: surfaces with positive and negative Gauss curvature.

Example 12.7. Imagine a torus — the surface of a donut — in $\mathbb{R}^3$. On the outer part of the donut, where you take your first bite, the surface curves away in the same direction, somewhat like a sphere. The Gauss curvature is positive. On the top, where the frosting goes, the surface is like a cylinder, which is

locally isometric to the plane, and hence the Gauss curvature there is zero. On the inside, near the hole, the surface curves like a saddle, and the Gauss curvature is negative.

An interesting question is, how much curvature is there total? On a slow night, with a good pencil sharpener and a steady supply of freshly-brewed coffee, one could do the computation, and you would find,

$$\iint_{\text{Donut}} K \, dA = 0.$$

This is an amazing fact. What is more amazing is that it doesn't matter whether you get a nice symmetric donut or a hideously deformed reject from the Dunkin Donuts factory, the answer is always zero. This result is a consequence of the Gauss-Bonet Theorem, which we may discuss later if time permits.

Lecture 13

(Mon 6 October 2003)

Last time we introduced the Riemann curvature tensor,

$$R^i_{jkl} = \frac{\partial \Gamma^i_{jl}}{\partial x^k} - \frac{\partial \Gamma^i_{jk}}{\partial x^l} + \Gamma^p_{jl}\Gamma^i_{pk} - \Gamma^p_{jk}\Gamma^i_{pl}.$$

At the time, the production of this tensor may have seemed somewhat mysterious. In fact, this object is a very natural one to consider. To explain why is our objective for this lecture and the next.

But before doing so, let us make one observation to begin with — that the Riemann curvature tensor is a non-linear operator. The presence of the $\Gamma^p_{jl}\Gamma^i_{pk}$ and $\Gamma^p_{jk}\Gamma^i_{pl}$ introduces a non-linear factor. Since Einstein's theory of gravity is based upon the curvature of space-time, this means that the equations of gravity will be a non-linear differential equations. This is a significant difference from Newtonian gravity, where the fundamental equation is linear. It makes the gravitational equations extremely difficult to solve.

Remark. In Einstein's papers, he used somewhat different notation for the Christoffel symbols than we have used here. To aid the reader in studying those papers, we mention Einstein's own notation here:

$$\{ij, k\} = \Gamma_{ijk}$$
$$\begin{bmatrix} k \\ ij \end{bmatrix} = \Gamma^k_{ij}$$

13.1 Tensors and covariant derivatives

Another matter which we shall attend to in the course of this lecture is to demonstrate that R^i_{jkl} is indeed a tensor. So far, we have simply named it a tensor, but we have not yet observed that it actually is a tensor, *ie*, that it transforms under changes of variables in the appropriate way.

Let us first remind ourselves of the defining property of a tensor. Tensors are required to obey certain transformation laws, as described in Lectures 10 and 11. At that time, we described the transformation laws in terms of the change of basis matrix of a vector space. In our present situation, however, we are dealing not with some arbitrary vector space, but specifically with the space of tangent vectors to a surface at a point. A basis for this tangent vector space is given by the partial derivatives of the chosen parameterization,

$$\{\mathbf{r}_{,1}, \mathbf{r}_{,2}\}.$$

A change of basis will be induced by a change of parameterization — *ie*, a change of local coordinates for the surface — and the corresponding change of basis matrix will be the Jacobian matrix of the coordinate change,

$$[B/\tilde{B}]^i_j = \left(\frac{\partial x^i}{\partial \tilde{x}^j}\right).$$

This Jacobian matrix will always be invertible, from the definition of a coordinate system (Definition 7.1).

With this in mind, recall the transformation laws. We use the term *vector field* (*covector field, tensor field, etc*) to describe a tangent vector (covector, tensor, *etc*) at each point of the surface (or some region within the surface). However, having said that, we will often suppress mention of the word *field* in practice.

- A contravariant vector field A^i, refers to the vector field $A^i \mathbf{r}_{,i}$. It transforms according to the law

$$\tilde{A}^j = A^i \frac{\partial \tilde{x}^j}{\partial x^i}. \tag{13.1}$$

Remark. Equation 13.1 may be considered the defining property of a contravariant vector field, but note that we can deduce this law — and indeed have done so — as a consequence of the chain rule.

- A covariant vector field B_i transforms according to

$$\tilde{B}_j = B_i \frac{\partial x^i}{\partial \tilde{x}^j}.$$

Remark. Note that the partial derivatives used in this transformation law represent the inverse of the matrix of partial derivatives from the contravariant transformation law of Equation 13.1. One can observe this directly: the product of the matrix is

$$\frac{\partial x^i}{\partial \tilde{x}^j} \frac{\partial \tilde{x}^j}{\partial x^k} = \frac{\partial x^i}{\partial x^k} = \delta^i_k.$$

It follows that

$$A^i B_i = \tilde{A}^i \tilde{B}_i.$$

The reader could take it as a simple exercise to verify the details of this claim.

- A general tensor field is denoted by a quantity with a combination of upper indices and lower indices, and the transformation law will arise from a contravariant transformation for each upper index and a covariant transformation for each lower index. For instance,

$$\tilde{R}^i_{jkl} = R^m_{npq} \frac{\partial \tilde{x}^i}{\partial x^m} \frac{\partial x^n}{\partial \tilde{x}^j} \frac{\partial x^p}{\partial \tilde{x}^k} \frac{\partial x^q}{\partial \tilde{x}^l}.$$

Now let us consider the relationship between these tensor transformation laws and derivatives. First, following our previous notation for the partial derivatives of a vector-valued function, we set up notation for the partial derivative of anything:

Definition 13.1. Let (thing) be a thing. Then

$$(\text{thing})_{,i} = \frac{\partial \text{thing}}{\partial x^i}.$$

Our first observation is that the derivative of a scalar-valued function is a tensor — in particular a covariant vector.

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Proposition 13.2. *Let Φ be a scalar field (ie, a function on the surface). Then $\Phi_{,i}$ is a covariant vector field.*

Remark. Note that we are thinking of Φ as being a function on the surface Σ, but it can also be described as a function from the various local coordinate spaces $\{x^1, x^2\}$. In particular, when we want to do calculus with functions on a surface, the computations will employ its derivatives with respect to the coordinate variables.

Clearly, different choices of parameterization will lead to different partial derivatives. Proposition 13.2 says that these differences are described by the transformation law of a covariant vector field.

Proof. By the chain rule,

$$
\begin{aligned}
\tilde{\Phi}_{,i} \quad &\stackrel{\text{def}}{=} \quad \frac{\partial \Phi}{\partial \tilde{x}^i} \\
&= \quad \frac{\partial \Phi}{\partial x^j} \frac{\partial x^j}{\partial \tilde{x}^i} \\
&= \quad \Phi_{,j} \frac{\partial x^j}{\partial \tilde{x}^i}.
\end{aligned}
$$

$\square$

On the other hand, the derivatives of a vector field *don't* form a tensor field. To show this, let A^i be a contravariant vector field. We need to determine the transformation law for $A^i_{,j}$. Well,

$$
\begin{aligned}
\tilde{A}^i_{,j} \quad &= \quad \frac{\partial}{\partial \tilde{x}^j} \tilde{A}^i \\
&= \quad \frac{\partial}{\partial \tilde{x}^j} \left(A^k \frac{\partial \tilde{x}^i}{\partial x^k} \right) \\
&= \quad \frac{\partial}{\partial x^l} \left(A^k \frac{\partial \tilde{x}^i}{\partial x^k} \right) \frac{\partial x^l}{\partial \tilde{x}^j}.
\end{aligned}
$$

But now we need to deploy the product rule, and as a result we get an extra term:

$$
\frac{\partial}{\partial x^l} \left(A^k \frac{\partial \tilde{x}^i}{\partial x^k} \right) \frac{\partial x^l}{\partial \tilde{x}^j} = A^k_{,l} \frac{\partial \tilde{x}^i}{\partial x^k} \frac{\partial x^l}{\partial \tilde{x}^j} + A^k \frac{\partial^2 \tilde{x}^i}{\partial x^l \partial x^k} \frac{\partial x^l}{\partial \tilde{x}^j}.
$$

This would be a tensor but for the second term, which we will refer to as the *tensor correction term*.

We should try to understand why this term appears. When one differentiates, one is considering the quantity,

$$
\frac{f(x+h) - f(x)}{h}.
$$

This is fine if we are dealing with scalar fields, because we can always subtract scalars. However, if the function f represents a vector field, then the vector $f(x+h)$ and the vector $f(x)$ will lie in different vector spaces, namely, the tangent spaces at two different points on Σ. There is no way in general we can subtract two vectors which lie in different vector spaces.

The only way one can subtract vectors from different vector spaces is to employ some method of identifying the two vector spaces: subtracting two vectors from the same vector space is no problem. What we did, effectively, in the above computation, was to use our coordinate system to identify the tangent spaces at two different points of our surface. However, if we chose a different coordinate system, we would obtain a different identification of the two tangent spaces. The tensor correction term that appears above shows the dependence of our notion of derivative upon the choice of coordinates.

So, how do we get around this problem? We have a notion of parallel transport. This provides a means for moving vectors form the $f(x+h)$-space to the $f(x)$-space, where we can then subtract the two vectors. Invoking this will give us a notion of differentiation of a vector field which does not depend upon the choice of coordinates.

In fact, we do not even need to use the full power of parallel transport, since we only need to transport vectors over infinitesimal distances. Instead, we simply differentiate our vector in the ambient space $\mathbb{R}^3$, and then take the component tangential to the surface Σ. This process will yield a tensor.

Let us do the computations. We have

$$\frac{\partial}{\partial x^i}(A^i \mathbf{r}_{,i}) = A^i_{,j}\mathbf{r}_{,i} + A^i \mathbf{r}_{,ij}.$$

But recall that

$$\mathbf{r}_{,ij} = \Gamma^k_{ij}\mathbf{r}_k + b_{ij}\mathbf{n}.$$

So

$$\frac{\partial}{\partial x^j}(A^i \mathbf{r}_{,i}) = (A^i\Gamma^k_{ij} + A^k_{,j})\mathbf{r}_{,k} + (\text{some multiple of } \mathbf{n}).$$

So we define

$$A^i_{;j} = A^i_{,j} + \Gamma^i_{kj}A^k. \tag{13.2}$$

This is now a tensor. It is called the *covariant derivative* of A^i.

Geometrically, the covariant derivative is computed by identifying the tangent spaces along a curve in the desired direction of differentiation via parallel transport, and then applying the usual limit definition of a derivative. This realization is enough to demonstrate that it is a tensor, because it is defined purely geometrically on the surface.

But one can also compute directly that it is a tensor. We have already noted that the first term on the right hand side of Equation 13.2 is not a tensor. The second term is also not a tensor — we have observed previously that the Christoffel symbols are non-tensorial. One can check that their respective "non-tensorialnesses" precisely compensate.

Clearly, if physics is to be independent of choices of coordinate systems, this kind of coordinate-independent differentiation must be crucial to doing physics in curved space.

Covariant derivatives enjoy many of the same properties as ordinary derivatives. But one key property that they do not enjoy is that of the symmetry of mixed derivatives:

$$A^i_{;jk} \neq A^i_{;kj}.$$

The difference of the two mixed derivatives is measured by the Riemann curvature tensor.

This messes up the whole of physics! Every time a classical physical law involves second partial derivatives, one must make a choice about whether to use the $(;ij)$-derivative or the $(;ji)$-derivative. Thus, every classical theory, once brought into the world of Einsteinian gravity, will have the Riemann curvature tensor introduced.

Because of this, we will want to take covariant derivatives of more complicated objects than simply vector fields. We will summarize the definitions of the covariant derivatives of general tensors without proof. Casually speaking, there will appear one tensorial correction term for each upper and each lower index of the tensor being differentiated, with each correction term being of a form similar to that for either contravariant or covariant vector fields, depending on whether the index is upper or lower.

Proposition 13.3. *Rules for covariant derivatives of more complicated things:*

- $B_{i;j} = B_{i,j} - B_k \Gamma^k_{ij}$.

- $X_{ij;k} = X_{ij,k} - X_{lj}\Gamma^l_{ik} - X_{il}\Gamma^l_{jk}$.

Exercise 13.4. Prove that

$$g_{ij;k} = 0.$$

This fact is a result of the way that the covariant derivative is defined: since the covariant derivative is defined in terms of the metric itself, it is reasonable to expect that the metric tensor should be constant with respect to itself on the surface.

Lecture 14

(Wed 8 October, 2003)

14.1 Covariant differentiation

Last time we were discussing the notion of covariant differentiation. This is a directional derivative for vector fields. In particular, $A^i_{;j}$ is a derivative in the "j-direction". The covariant derivative uses the notion of parallel transport along a curve to identify the different tangent spaces along a curve.

We would also like to use the notion of parallel transport to attempt to extend a vector $\mathbf{v}$ at a point $p \in \Sigma$ to a vector field over the entire surface in such a way that it is everywhere parallel. Recall that a vector field A^i is parallel along the curve $x^i(t)$ if

$$A^i_{;j} \dot{x}^i(t) = 0,$$

where $\dot{x}^i(t)$ is the derivative (ie, tangent vector) to the curve $x(t)$.

We might try to produce this everywhere parallel vector field as follows. First, we take a curve through p in the x^1-direction (ie, $x^2 = constant$). We can parallel transport the vector $\mathbf{v}$ along this curve. Now we take the family of curves in the x^2-direction, and parallel transport the new vectors along each of these curves. Alternatively, we could first transport in the x^2-direction, and then along curves in the x^1-direction.

Picture: parallel transport along x-curve then y-curves, or y-curve and then x-curves.

Both these procedures work in the plane, producing an everywhere parallel field. Unfortunately, they do not necessarily work for a general surface. In fact, for a general surface, the two notions of parallel transport might not match up. In general, we can't extend a vector at a point to be parallel over a two-dimensional (or larger) region.

Why not? Suppose we had an everywhere parallel vector field A^i. That means it's covariant derivative in every direction would be zero. That is,

$$A^i_{;j} = 0,$$

for every j. So

$$A^i_{;jk} = A^i_{;kj}.$$

But this is not true in general.

For an illustration of this, think of the sphere. Starting at the north pole, the southward-pointing tangent vectors are parallel along any chosen line of longitude. If we transport the vector down this line of longitude, and then along the equator, it remains southward-pointing. Finally, if we transport this

back up another line of longitude, we obtain a vector at the north pole which points in a different direction from that originally chosen.

<center>Picture: some parallel vector fields along curves on the sphere.</center>

All of this indicates that we should be interested in looking at these second derivatives $A^j_{;kl}$. Noting that the first covariant derivative $A^j_{;k}$ is a tensor, we can use the laws of covariant derivatives of tensors that were mentioned at the end of last lecture (Proposition 13.3) to compute:

$$
\begin{aligned}
A^j_{;kl} &= (A^j_{,k} + \Gamma^j_{pk} A^p)_{;l} \\
&= A^j_{,kl} + A^p_{,k}\Gamma^j_{pl} - A^j_{,q}\Gamma^q_{kl} + \Gamma^j_{pk,l}a^p + \Gamma^j_{pk}A^p_{,l} + \Gamma^q_{pk}\Gamma^j_{ql}A^p - \Gamma^j_{pq}\Gamma^q_{kl}A^p
\end{aligned}
$$

Comparing this with $A^j_{;lk}$ by taking their difference, we get a lot of cancellation from symmetric terms, finally yielding

$$
\begin{aligned}
A^j_{;kl} - A^j_{;lk} &= \Gamma^j_{pk,l} - \Gamma^j_{pl,k} + \Gamma^q_{pk}\Gamma^j_{ql} - \Gamma^q_{pl}\Gamma^j_{qk} A^p \\
&= R^j_{plk} A^p.
\end{aligned}
$$

To summarize, the difference between mixed covariant derivatives taken in different orders is measured by the Riemann curvature tensor.

Proposition 14.1. *If A^j is a vector field,*

$$
A^j_{;kl} - A^j_{;lk} = R^j_{plk} A^p. \tag{14.1}
$$

Since the left-hand side of Equation 14.1 is clearly a tensor, we get as a corollary the justification of the terminology "Riemann curvature *tensor*".

Corollary 14.2. $R^j_{plk} A^p$ *is a tensor.*

To illustrate one of the many implications of the Riemann curvature tensor, consider the following fact.

Theorem 14.3. *The Riemann curvature tensor of a space Σ is zero (ie, all its components are zero) if and only if Σ is locally isometric to Euclidean space.*

Sketch of proof. Basically, we do everything we did at the start of this lecture, but backwards. Pick a point p in the space Σ. Pick a covariant vector at the point p. We now parallel transport this vector over all of Σ along curves. The fact that the Riemann curvature tensor vanishes is enough show that parallel transport along any two curves between the same start and end points will give the same answer. So we end up with a covector field A_i on Σ which is everywhere parallel:

$$
A_{i;j} = 0,
$$

for every j.

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Now we ask, can we find some scalar function Φ, such that A_i is actually the gradient covector field $\Phi_{,i}$ of Φ? The equation

$$A_i = \Phi_{,i}$$

is a system of partial differential equations for Φ. As one learns in a calculus course, it will be solvable as long as we have a certain integrability condition, namely that

$$A_{i,j} = A_{j,i}.$$

But in our case,

$$A_{i,j} = \Gamma_{ij}^k A_k,$$

which is symmetric in i and j. Therefore, there exists such a Φ.

Finally, suppose we started not with a single covector at p, but a family of orthonormal covectors at p. When we parallel transport these all over the surface, we end up with a family of covariant vector fields, which are everywhere orthonormal. The corresponding family of functions Φ, will give a set of coordinate functions. The fact that the gradients of these coordinate functions are everywhere orthonormal is enough to show that they are (locally) the coordinate functions for Euclidean space.

□

Lecture 15

(Monday 20 October 2003)

In this lecture, we'll start with something new. There is still a little geometry that we need to understand, but for now, let's do some physics, and we'll deal with the extra geometry as it comes along. We will now turn gravity off for a while, and talk about special relativity.

15.1 Special relativity

What is Einsteinian relativity?

Recall that Einstein did not invent the idea of relativity. The idea of relativity is that there is some group acting on our space — a group of coordinate transformations — under which the laws of physics are invariant. Galileo noted that physics, at least the "classical" physics that he observed, was invariant under the group of *Galilean transformations*, which we discussed in Lecture 6.

The Galilean transformation group was a group of transformations of space-time, thought of as a four-dimensional vector space. This makes drawing pictures difficult. However, there is no significant loss in supposing, for the moment, that we are inhabiting a universe of one spatial dimension, so that our space-time is a two-dimensional space.

The key transformation which was noted by Galileo was the *Galilean boost*

$$(x, t) \mapsto (x + vt, t).$$

It was because of the invariance of physics under this transformation that we noted previously that no meaningful statement can be made about distance between two events occurring at different times.

In fact, there were two key components to Galileo's physics. The fact just mentioned — that there is no observable phenomenon which allows us to detect the difference between two frames of reference moving with uniform speed with respect to one another — is the first. The second is better known to us as Newton's first law of motion — that a body moving with uniform velocity tends to remain in that state unless acted upon by some external body.

However, Galilean relativity is incompatible with one of the great triumphs of 19th century physics, Maxwell's theory of electromagnetism. Maxwell's theory can be summarized by a set of differential equations which describe the behaviour of electric and magnetic fields. In particular, they describe the propagation of electromagnetic waves, *ie* light, through space. In solving these equations, one can deduce the speed of light in free space of roughly $3 \times 10^8 ms^{-1}$.

This theory is clearly incompatible with Galilean physics. Under the assumption of relativity, the physical laws must be the same in any uniformly moving frame of reference. In particular, Maxwell's equations must be invariant, and hence their solution as well: the speed of light must be constant in all frames of reference. But if I am observing a photon travelling away from me at the speed of light, and you pass me in a sports-car at 200 miles per hour, then in the framework of Galilean physics you must observe the photon receding from you at the speed of light minus 200 miles per hour.

So how do we resolve this incompatibility? One possible solution was to suppose the existence of the "ether". The ether was a supposed medium, pervading the universe, through which light travelled. The speed of light was then a constant relative to the ether. In effect, this singles out one particular frame of reference as a distinguished coordinate system for Maxwell's electromagnetic theory.

From the point of view of modern physics, this idea seems anathema to us — as distasteful as the suggestion that the universe revolves about the earth. But towards the end of the nineteenth century, this was the predominant theory for resolving the incompatibilities. The verification of the ether, and the determination of the earth's velocity through it, were the purpose of an experiment by Michelson and Morley in 1887.

15.2   The Michelson-Morley Experiment

The hypothesis of the ether yields certain physical consequences, which Michelson and Morley attempted to exploit in their famous experiment. We can explain the theoretical phenomenon most easily with the following analogy.

Imagine a river, say 100 metres wide, moving with a uniform flow velocity v. Now imagine two swimmers, both equally strong, who take off at the same time from a point on one of the banks with speed c relative to the water. The first swimmer swims 100 metres upstream, then turns and swims back to their starting point. The second swims 100m across the stream and back, in such a way that his path is perpendicular to the flow of the river. Which swimmer gets back first?

<u>Picture</u>: swimmers in the river.

With the aid of Pythagoras' theorem and some algebra, the time taken by each swimmer can be calculated. Let's simplify by supposing each swimmer swims 1 unit of distance away and back. The swimmer going up- and downstream returns at time

$$\frac{1}{c-v} + \frac{1}{c+v},$$

while the swimmer going across the stream returns at

$$\frac{2}{\sqrt{c^2 - v^2}}.$$

The important fact to note is that these times are not the same (although, one can check specifically that it is the cross-stream swimmer who comes back first).

Michelson and Morley applied this observation by measuring variations in the speed of light in perpendicular directions, just as in the swimming race. The experiment was repeated with different orientations and at different times of the year, in order to remove the possibility that the earth was temporarily stationary in the ether at the time the experiment was performed. Despite a

very high sensitivity of the experimenatal design, under no circumstances was any variation in the speed of light detected. So, either Michelson and Morley's lab coincidentally happened to be at the centre of the universe, with all other bodies moving around it, or there is no observable ether.

15.3 Einstein's solution

It is not clear whether or not Einstein knew of Michelson and Morley's experiment at the time he produced his own theory to resolve the incompatibility of Galilean relativity and Maxwell's electrodynamics. Either way, In his 1905 paper, "On the Electrodynamics of Moving Bodies" [Ein1], he noted that the principle of relativity could be reinstated for Maxwell's electrodynamic theory, as long as the group of acceptable space-time transformations was adjusted appropriately. In short, he replaced the Galilean transformations by the group of Lorentz transformations.

What are the quantities in space-time which the group of Galilean transformation preserves? We saw in Lecture 6 that it preserves the time between two events[7], and the distance between two *simultaneous* events. Moreover, if we declare that we want to consider all transformations which preserve these two quantities, we will precisely recover the Galilean group.

The best *mathematical* way to come up with Einstein's Special relativity is to decide which quantities should be preserved by the group of transformations, and determine the group of transformations from there. Firstly, we will assume that the coordinate transformations are linear[8]. Secondly, we will declare that all coordinate transformations preserve the *Lorentz distance* from an event (x, t) to the origin, which is defined as the quantity

$$x^2 - c^2 t^2.$$

One can show that the preservation of the Lorentz interval is a consequence of the requirement that the speed of light be the same in all frames of reference. This is done by means of certain thought experiments, involving relatively moving observers shining light at mirrors and so on. We may go into this later. The key idea that Einstein had, however, is that when you set up a coordinate system for describing events in space-time you have to be very careful about what you mean.

Our naive idea about setting up a coordinate system is based upon the assumption that speed of light is much faster than any of the things I wish to measure. So, if I want to set up a coordinate system to describe objects in my room, I can just look around the room to determine the location of various events. But if my room were extraordinarily large, so that light took a long time to get form one corner of the room than the other, then after observing a distant event, I would need to do some calculations in order to work out at what time and place, in my chosen coordinate system, the event actually took place. If the speed of light is the same for relatively moving frames of reference, it is not surprising that such calculations might differ.

---

[7]Recall that an *event* is a point in space-time, marked by both a position and a time

[8]Actually, we will allow our transformations to be affine, *ie*, we will also allow also translations in both the space and time directions. But for now, let us consider only those transformations which fix the origin $(0, 0)$ of space-time.

We now identify the Lorentz group, *ie*, determine which are the transformations which preserve the Lorentz distance to the origin. To make the mathematics simpler, let us choose units so that the speed of light is $c = 1$.

The correct mathematical way of writing the Lorentz distance is

$$x^2 - t^2 = \begin{pmatrix} x & t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}.$$

To put this in a suggestive form, we write this as

$$\mathbf{v}^T g \mathbf{v},$$

(or in Einstein notation, $g_{ij} v^i v^j$), where

$$g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

So if L is the matrix of a Lorentz transformation, then

$$(L\mathbf{v})^T g (L\mathbf{v}) = \mathbf{v}^t g \mathbf{v},$$

for all $\mathbf{v}$.

It follows that,

$$L^T g L = g.$$

Putting $L = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, we get

$$\begin{aligned} L^T g L &= \begin{pmatrix} p & r \\ q & s \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} \\ &= \begin{pmatrix} p & -r \\ q & -s \end{pmatrix} \begin{pmatrix} p & r \\ q & s \end{pmatrix} \\ &= \begin{pmatrix} p^2 - r^2 & pq - rs \\ pq - rs & q^2 - s^2 \end{pmatrix}, \end{aligned}$$

and we want this to equal

$$g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

How do we identify such matrices? The key is to take an analogy with the group $O(2)$ of orthogonal 2×2-matrices, which preserve the bilinear form

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In that case, we know that the relevant matrices are the rotations (ignoring the reflections momentarily), which are given by

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

In our case, by considering the (1,1)-entry of the expression above, and under the assumption that $p > 0$, we see that we can find some ϕ such that

$$p = \cosh\phi, \; r = \sinh\phi,$$

Similarly, from the (2,2)-entry, with $s > 0$, we can find ψ so that

$$q = \sinh\psi, \ s = \cosh\psi.$$

But then we also need

$$0 = pq - rs = \cosh\phi\sinh\psi - \sinh\phi\cosh\psi = \sinh(\phi - \psi).$$

Hence $\psi = \phi$. Our transformation is therefore

$$L_\phi = \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix}.$$

Explicitly, the corresponding transformation of space-time is described by

$$\begin{aligned} x &= \cosh\phi x' + \sinh\phi t', \\ t &= \sinh\phi t' + \cosh\phi x'. \end{aligned}$$

The picture representing such a coordinate transformation is as follows.

<u>Picture</u>: Coordinate grids for two Lorentzian coordinate systems.

Looking at this picture, one notes that the line $t' = 0$ represents the world line of a particle at the spatial origin of the (x', t')-coordinate system, as seen in (x, t)-coordinates. It is moving with uniform velocity through the (x, t)-frame of reference. In other words, this transformation represents the coordinate transformation between two frames of reference moving with uniform velocity with respect to one another.

Furthermore, we can compute what the relative velocity v is, and hence write the transformation in a more physically intuitive form. Computing the slope of the path, we get

$$v = \tanh\phi.$$

Thus,

$$\cosh\phi = \frac{1}{\sqrt{1 - \tanh^2\phi}} = \frac{1}{1 - v^2},$$

and

$$\sinh\phi = \frac{\tanh\phi}{\sqrt{1 - \tanh^2\phi}} = \frac{v}{1 - v^2}.$$

So,

$$L_\phi = \frac{1}{\sqrt{1 - v^2}} \begin{pmatrix} 1 & v \\ v & 1 \end{pmatrix}.$$

If we use units in which the speed of light is not 1, this becomes

$$L_\phi = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \begin{pmatrix} 1 & v \\ v/c^2 & 1 \end{pmatrix}. \tag{15.1}$$

A notable feature of this transformation group is that the time coordinate, as well as the space coordinate, is transformed under a Lorentz transformation. This is in distinct contrast with the Galilean transformations, where the time coordinate remains unchanged under a Galilean boost.

A striking physical consequence of this is that, under Einstein's relativity, different observers may disagree on the notion of simultaneity of events. Suppose I were to meet you on the street and tell you that, at this very instant, an intergalactic battle-fleet was taking off from the Andromeda galaxy on its way to destroy the earth. If you were walking past me with any velocity, you might not agree with me. Of course, you might not agree with me under any circumstances, but even if it were true, and we were both watching the military operation through our high-power telescopes, you would not observe the event to be simultaneous with our meeting.

This fact that the Lorentz group is the correct one for physics has further consequences. Consider the following question. We have three observers, called A, B and C. Observer B sees A pass her at half the speed of light. Meanwhile, observer C sees B pass him in the same direction at half the speed of light. At what speed does A move, relative to A? In Galilean physics, we would know simply to add the two relative velocities. However, this does not work in special relativity, as we will now see.

We know that B's and C's frames of reference are related by the transformation matrix

$$\frac{2}{\sqrt{3}} \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}.$$

Similarly, A's and B's coordinates are related by

$$\frac{2}{\sqrt{3}} \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}.$$

So to work out the relationship between A's and C's, we need to compose the two transformations, to get

$$\frac{4}{3} \begin{pmatrix} \frac{5}{4} & 1 \\ 1 & \frac{5}{4} \end{pmatrix}.$$

In order to compute the relative speed which gives rise to this Lorentz transformation, we need to rewrite it as

$$\frac{5}{3} \begin{pmatrix} 1 & \frac{4}{5} \\ \frac{4}{5} & 1 \end{pmatrix},$$

and compare this to Equation 15.1. We find that C sees A move past at four-fifths the speed of light. In the screwy world of special relativity, $\frac{1}{2} + \frac{1}{2} = \frac{4}{5}$.

Lecture 16

(Wednesday 22 October 2003)

In this lecture, we're going to discuss the paradoxes of special relativity, or some of them at least.

In the last lecture we discussed Einstein's method for making the relativity of physics compatible with Maxwell's theory of electromagnetism. Passing to three spatial dimensions, Einstein's idea was to declare that physics operates so as to preserve the *Lorentz inner product*

$$g(\mathbf{v}, \mathbf{w}) = g_{ij} v^i w^j,$$

where

$$g = \begin{pmatrix} c^2 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}.$$

Note that, from this lecture on, we will write the time coordinate first, and we are reversing our sign convention from the previous lecture for reasons that will become clear later.

The *Lorentz length* of a four-dimensional space-time vector $\mathbf{v} = (t, x, y, z)$ is

$$g(\mathbf{v}, \mathbf{v}) = c^2 t^2 - x^2 - y^2 - z^2.$$

From the assumption of the invariance of this quantity, we derived the *Lorentz transformations*, which in one spacial dimension were

$$\begin{pmatrix} x \\ t \end{pmatrix} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \begin{pmatrix} 1 & v \\ \frac{v}{c^2} & 1 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{t} \end{pmatrix}.$$

It is worth noting here that this situation is exactly like that which we discussed in Lecture 10 for changes of coordinates between coordinate systems $\mathbf{v}$ and $\tilde{\mathbf{v}}$. In Einsteinian physics, the particular group of coordinate changes under which physics is invariant is the Lorentz group. What I wish to do now is discuss some of the peculiarities which arise as consequences of this.

16.1   Simultaneity in relativity

Imagine you flick on the radio in State College, and it is playing the BBC world service. You hear the BBC announcer, in his plummy BBC English accent, saying, "This is the BBC World Service. It is 6:00am GMT."

At the instant at which you hear him say that, the radio signal has already had to traverse some distance across the Atlantic. In order for you to deduce when he actually said it, you must perform some calculations, involving the distance between you and him, and the speed of light.

What we conclude from these thoughts is that the notion of simultaneity has to be *defined*. Einstein discussed this at length in his original paper [Ein1], and provided a suitable definition, now known as the "radar definition" of simultaneity, which works as follows.

We use the observation that the speed of light is constant in all frames of reference. So if I want to find out what is going on in London, what I can do is send a beam of light over to London, where the BBC man will bounce it back immediately, say by reflecting from a mirror. We denote the event of the light striking the mirror by E. By deducing the time at which the event E occurs, we are deducing the time at which an event simultaneous with E occurs in the BBC studio.

Picture: Space-time diagram for the beam of light.

Suppose the beam of light leaves my position at time t_0 and returns at time t_1, according to my watch in State College. I declare that, in my set of coordinates, the time at which event E occurred is $\frac{1}{2}(t_0 + t_1)$. This seems a very reasonable definition. In fact, it seems so reasonable that it is hard to see just how revolutionary it is.

But now consider observations of an observer moving relative to me. We add the moving observer's space-time coordinates to the above space-time diagram. Lines of equal time coordinate t' in this observer's frame of reference appear as lines parallel to the x' axis. Thus, the time t' at which the moving observer calculates the event E happened will be given by the time t' of the event E' marked on the diagram. However, it is clear that E and E' will not appear simultaneous to me.

Picture: Space-time diagram for the beam of light in the above frame of reference, and the coordinate grid for a moving frame of reference. Event E is the reflection of the light beam. Event E' is the event on the moving observer's world line which that observer notes as simultaneous..

The point is that the moving observer thinks that the beam of light took exactly the same time to go out to E as it did to come back to her, because, of course, to her the speed of the light was constant for both the outward and return journeys. To me, that does not seem to be the case — it seems as if the light came back to her much more quickly than it took to get to E, because of the constancy of the speed of light my frame of reference. So to her the events E and E' seem simultaneous, and not so for us.

Remark. Einstein originally did the reverse procedure from that just undertaken by us: he used this thought experiment to deduce what the Lorentz transformation must be for passing to the frame of reference of the moving observer. This is the traditional method for deriving special relativity, while we are taking a somewhat dogmatic mathematical approach.

16.2 Time dilation

I am holding a watch. At 12:00 noon, a white rabbit rushes past me at half the speed of light. It is also holding a watch, and the watch reads the same time as mine as he passes. An hour later, by my watch, I want to check what time the rabbit's watch is registering.

I don't mean, of course, that I simply look up in the sky and read the watch. The rabbit has been receding from me at great speed, so the light that I am currently receiving will have left the rabbit some time ago. Instead I must do some calculations to compute what time the rabbit's watch is presently reading. You can see that life becomes much more complicated when physical systems are at a scales of around the speed of light.

What we will see is that, by my best calculations, my watch and the rabbit's watch no longer coincide.

<u>Picture</u>: space-time diagram.

More formally, we are considering two observers O_1 and O_2, where O_2 is moving relative to O_1 with speed v. They synchronize their clocks as they pass. Our question is, what time t does O_1 assign to the event that O_2's clock registers time t'?

<u>Solution 1</u>: Use the Lorentz transformation.

We know that
$$\begin{pmatrix} x \\ t \end{pmatrix} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \begin{pmatrix} 1 & v \\ \frac{v}{c^2} & 1 \end{pmatrix} \begin{pmatrix} 0 \\ t' \end{pmatrix}.$$

So
$$t = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} t'.$$

Let us verify this calculation in another way.

<u>Solution 2</u>: Use the invariance of the Lorentz interval.

Since the position of the observer O_2 in his own frame of reference is always $x' = 0$, we have

$$\begin{aligned} c^2 t'^2 &=& c^2 t^2 - x^2 \\ &=& c^2 t^2 - v^2 t^2 \end{aligned}$$

Solving for t, we get
$$t^2 = \frac{1}{1 - \frac{v^2}{c^2}} t'^2.$$

We could summarize this calculation by saying, "moving clocks run slow". Note, however, that we have to be careful with such simple statements — it is also true that O_2 sees O_1's clock running slow in his coordinate system.

Next, let us take this idea even further. Let us suppose that at some time in the future, somebody gives the white rabbit an almighty kick, so that the

rabbit starts moving back towards me again at half the speed of light. What happens when the rabbit eventually arrives back by me? What times do our two watches read?

Assuming the watch is an ideal watch that is capable of surviving such an intense shock, and the rabbit is similarly an ideal rabbit, the above calculation shows that when he returns to me, the rabbit's watch must register an earlier time than mine. The watch is measuring time in the rabbit's frame of reference, so in fact, the rabbit has aged less on his journey than I have staying home.

Suitably rephrased, this is the "twin paradox" of special relativity. Why is it that the rabbit's journey has allowed him to stay young, while I have aged much more? The important thing to notice is that the situation is not symmetric. My motion was with constant velocity in any inertial frame of reference, while the rabbit's journey involved an enormous kick up the rear, *ie* an acceleration, at some point.

It is possible to make from this potential "lemon", some very interesting "lemonade". It turns out that, by staying at constant velocity, I have aged as much as possible for any path through space-time to the same event. This observation is better phrased as.

The Principle of Action:

A particle, in its natural state, moves so as to take as long as possible, according to its own clock, to get where it is going.

16.3   Fitzgerald contraction

Next let us consider the situation where the object which is moving relatively to me is not a point-like object, like a rabbit, but an object of some significant length, like an extremely long rabbit. The question I am interested in is how long is this object in my frame of reference. In order to answer this question, I need to determine the position of the two ends of the object at the same time t.

Suppose the object is a rod with length l in its own frame of reference. The space-time diagram below shows the world-lines of the two ends of the rod. The event P marks the position of the front end of the rod at the same time as the rear end is passing the origin, according to its own frame of reference. Thus the coordinates of this point are $(x', t') = (l, 0)$. This transforms to my frame of reference as

$$\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}(l, l\frac{v}{c}).$$

Picture: space-time diagram with world-lines of two ends of the rod.

In order to compute the length in my frame of reference, I am interested in the x-coordinate of the event Q, which we now compute. The slope of the

world-lines for the moving object in (x, t)-coordinates is $\frac{v}{c}$, from which we obtain

$$x = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}(l - l\frac{v}{c}\frac{v}{c}) = l\sqrt{1 - \frac{v^2}{c^2}}.$$

Thus the rod appears to have contracted by a factor of $\sqrt{1 - \frac{v^2}{c^2}}$.

Remark. This contraction was originally used by Fitzgerald as an explanation of the failure of the Michelson-Morley experiment to detect the ether. It was supposed that there was a drag on the experimental apparatus as it moved through the ether, and that this drag caused a physical contraction of the experimental apparatus which exactly compensated for the change in the speed of light, yielding it undetectable. This sort of conspiracy of the universe is now considered too tortuous an explanation. The Fitzgerald contraction has been instead realized as a feature of Einstein's relativistic theory.

Lecture 17

(Friday 24 October 2003)

17.1 Minkowski Space

Our plan for this lecture is to consider special relativity from the point of view of four-dimensional geometry. Let us begin by discussing inner products.

Let V be a finite-dimensional vector space over $\mathbb{R}$. We have defined the notion of an inner product on V: it is a symmetric bilinear form, satisfying the condition of positivity (*cf* Lecture 11). We now want to relax the requirement of positivity, since we will want to consider quantities like $ct^2 - x^2 - y^2 - z^2$, which are not positive.

Definition 17.1. We will say that an *inner product* on V is a symmetric bilinear form $g(\cdot, \cdot)$ which is *non-degenerate*, in the sense that if $g(\mathbf{v}, \mathbf{w}) = 0$ for all $\mathbf{w} \in V$, then $\mathbf{v} = 0$.

Remark. Strictly speaking, such an object should not be called an inner product, but a non-degenerate pseudo-inner product. The effect of sticking to this strict terminology would only be to waste ink, so we will not bother ourselves about this indiscretion.

We define the *length* of a vector to be $g(\mathbf{v}, \mathbf{v})$ (now blurring the distinction between length and length-squared). Note that with our generalized notion of an inner product, one can have nonzero vectors $\mathbf{v}$ in V with length $g(\mathbf{v}, \mathbf{v}) = 0$.

Fortunately, though, much of what is true for genuine inner products is also true for these pseudo-inner products. In particular, we can still perform a variant of Gram-Schmidt orthonormalization procedure: we can find a (pseudo)-orthonormal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ such that

- $g(\mathbf{v}_i, \mathbf{v}_j) = 0$ for $i \neq j$, and

- $g(\mathbf{v}_i, \mathbf{v}_i) = \pm 1$.

We define the *signature* of the inner product g to be (m, n), where m is the number of $+1$'s and n the number of -1's appearing as $g(\mathbf{v}_i, \mathbf{v}_i)$ in the above pseudo-orthonormal basis. It is a fact of linear algebra that the signature is an invariant of the inner product — *ie*, all such pseudo-orthonormal bases will give the same signature.

Basic postulate of special relativity:
   The displacement vectors between events in space-time form a vector space which is equipped with an inner product of signature $(1, 3)$.

Remark. In fact, as we will eventually see when we move to discussing general relativity, the appropriate vector space will not be the space of displacement vectors, but the tangent spaces to our four-dimensional space-time. What we are doing here is the simplified situation of special relativity, wherein we assume that our space-time is itself a a four-dimension vector (or affine) space.

A four-dimensional vector space with a $(1,3)$-inner product is called *Minkowski space*. The inner product on Minkowski space is defined by

$$g(\mathbf{v}, \mathbf{v}) = t^2 - x^2 - y^2 - z^2,$$

where $\mathbf{v} = (t, x, y, z)$. Elements of Minkowski space will be referred to as *4-vectors*.

Let us analyze the geometry of Minkowski-space. The set of *null vectors* — vectors with zero length — form a (double) cone in Minkowski space, whose constant time cross-sections are three-dimensional spheres. This set is called the *light cone*, as it represents the directions in space-time along which light travels. Inside the light cone are the vectors for which $g(\mathbf{v}, \mathbf{v}) > 0$. Such vectors are called *time-like*. They correspond to the possible directions of motion of ordinary physical objects through space-time. The space-like vectors are separated into two regions, those which lie in the past $(t < 0)$, and those that lie in the future $(t > 0)$. Outside the light cone lie the vectors with negative length, which are called *space-like*.

Note that all of this structure follows purely from the existence of the Minkowski inner-product.

Picture: picture of 3-dimensional Minkowski-space, with light cone, time-like regions (past and future), and space-like region.

Assumption:

Any two observers agree on which is the future and which is the past, corresponding to the $t > 0$ and $t < 0$ time-like regions.

This is a separate postulate, independent of the other postulates of special relativity.

Proposition 17.2.

(i)   *If the separation of two events is time-like, one can find an inertial frame with respect to which they occur at the same place.*

(ii)   *If the separation of two events is space-like, one can find an inertial frame with respect to which they occur at the same time.*

Idea of proof. If the displacement vector between two events is time-like, that means that they both lie on a straight line with slope v, which is strictly less than c. Then, we simply consider the frame of reference which moves with velocity v relative to our given frame.

The second result can be dealt with similarly.

□

It is because of the second part of Proposition 17.2 that people say that we cannot affect events outside our future light-cone. For, given any event outside the light-cone, we can find a frame of reference in which it has $t = 0$, and in fact

we can go further to find a frame in which it has $t < 0$. If we were to influence this event, it would appear to some other observer that we had influenced an event in our past, which seems ridiculous. We conclude:

Corollary 17.3. *Space-like separated events are causally isolated — neither can influence the other.*

Exercise 17.4 (Book report). Read "Absolutely elsewhere", by Dorothy Sayers, for an amusing application of causality in a detective novel.

We finish off today's lecture by looking at some particularly interesting 4-vectors. We'll find that we can do some of the things that were done last lecutre with the Lorentz transformations, by simply considering certain 4-vectors.

17.1.1 Proper time

Suppose that a particle moves through space-time along a path γ. Our first question is, how are we to parameterize this path? Certainly, we can not parameterize by time, since time is dependent upon the observer. We instead use the following natural parameter.

Definition 17.5. We define the *proper time* of the particle to be the time measured by an ideal clock traveling with it. We denote it by τ.

For instance, recall the twin paradox from last lecture. The proper time of each of the two twins is the time shown on each of their wrist-watches.

Let us now compute the formula for the proper time of a path. We will see that it looks awfully familiar.

For a curved path through space-time, we consider the infinitesimal lengths of an infinitesimal portion of the curve.

<u>Picture</u>: infinitesimal motion of the particle through space-time.

The length of this infinitesimal 4-vector, according to the given frame of reference, is
$$dt^2 - dx^2 - dy^2 - dz^2.$$

However, in the frame of reference of the particle itself, its spatial coordinates remain at **0**, so that the infinitesimal Minkowskian length is just $d\tau^2$. Since the Minkowskian length is an invariant under the Lorentz transformations, we see that the infinitesimal proper time is given by

$$d\tau^2 = dt^2 - dx^2 - dy^2 - dz^2.$$

Integrating this, the proper time of the particle is given by

$$\tau = \int_\gamma \sqrt{dt^2 - dx^2 - dy^2 - dz^2}.$$

This is just like the arc-length formula for a path through a (standard) inner product space.

Note that there is no such natural parameterization for photons. The length of a light-like vector is zero, which says that the proper time for a photon is always zero. Time does not progress for something moving at the speed of light.

Next, let us consider the "velocity" of a space-time curve.

Definition 17.6. The *4-velocity* of a particle is

$$\mathbf{V} = (\frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau})$$

Example 17.7 (Steady motion). Suppose the 4-position of a particle is $(t, \mathbf{v}t)$. Then

$$\tau = \sqrt{1 - v^2}t,$$

and thus the 4-velocity is

$$\mathbf{V} = \frac{1}{\sqrt{1 - v^2}}(1, \mathbf{v}).$$

Note that $g(\mathbf{V}, \mathbf{V}) = 1$. In other words, every particle moves through space-time at the speed of light — there is simply a question of how much of that motion is motion through space and how much is motion through time.

Example 17.8 (Conservation of 4-momentum). Imagine two identical sports cars. One is initially at rest, while the other is moving towards it at 30mph. Suppose that the two collide and stick to each other. What is their final velocity?

From Newtonian physics, we know that their final velocity will be 15mph. How do we know this? By conservation of momentum.

This argument is good in Newtonian physics, but it isn't compatible with relativity. To see this, you could transform the above problem under a Lorentz transformation, and you will find that you do not arrive at the same answer, violating the postulate that physics looks the same in all frames of reference. So how do we fix this? This was the subject of Einstein's 1905 paper, "Does the inertia of a body depend upon its energy-content?" [Ein2].

To solve the problem, we postulate that every particle has a "rest mass" m. We then define the *4-momentum* of a particle by

$$\mathbf{P} = m\mathbf{V}.$$

It turns out that, in relativistic collisions, the 4-momentum is conserved.

Let us think about what this means if the particles are not moving too fast. Suppose $\mathbf{v}$ is small compared to the speed of light. Then

$$\mathbf{P} = m\mathbf{V} = \frac{m}{\sqrt{1 - v^2}}(1, \mathbf{v}).$$

Taking a Taylor expansion, and ignoring terms of order 3 and higher, this is approximately

$$\mathbf{P} = (m(1 + \frac{1}{2}|\mathbf{v}|^2), m\mathbf{v}).$$

Note that the first component of the 4-momentum includes a term that we know to represent the Newtonian kinetic energy of the particle. Thus, the conservation of 4-momentum includes both the conservation of energy and the conservation of momentum, in the Newtonian approximation.

Einstein took the bold step of defining the *total energy* of an object to be the first component of its 4-momentum. Considering a body at rest, this yields the now famous equation $E = mc^2$. It is interesting to note that the way in which the equation is commonly interpreted these days is that it represents an amount of energy which can be extracted from matter, by means of nuclear reactions and so on. But the way in which Einstein originally conceived it was in fact a limitation on the amount of energy which can be stored in a body of a given mass. As the energy-content of a body increases, so too must its mass.

Lecture 18

We have already talked about the Fitzgerald contraction, in Lecture 16. However, we will begin this lecture by looking at this again, this time as an application of the Minkowskian geometry of Lecture 17.

Recall the situation: an object, say a rod, of length l_{rod} (as measured when stationary) is moving at uniform velocity with respect to ourframe of reference. We are interested in the length of the rod as we observe it, which we will call l_{us}.

Picture: Space-time diagram, with events A=rear of rod at origin, B=front of rod simultaneously in t' coords, C=front of rod simultaneously in t coords, l_{us}=length of rod in our frame, l_{rod}=length of rod in its frame.

Suppose the rod is moving with velocity $v\mathbf{i}$ in our frame of reference, where $\mathbf{i}$ is the unit vector in the first spatial dimension. We choose our coordinates so that both frames of reference agree on the origin $(0,0)$ of space time. We can further assume that the rear end of the rod passes through the space-time origin, and we mark this event by A. Events B and C mark the positions of the front end of the rod at time $t = 0$ in our frame of reference, and at time $t' = 0$ in the frame of reference of the rod, respectively.

The rod's 4-velocity is

$$\mathbf{V} = \frac{1}{\sqrt{1 - v^2}}(1, v\mathbf{i}),$$

as computed last lecture. The space-time displacement vector $\overrightarrow{BC}$ is parallel to this, so

$$\overrightarrow{BC} = \lambda \mathbf{V},$$

for some $\lambda \in \mathbb{R}$.

We now make two observations:

- $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$, and

- $g(\overrightarrow{AC}, \overrightarrow{BC}) = 0$.

The second says that the two vectors $\overrightarrow{AC}$ and $\overrightarrow{BC}$ are "orthogonal". Of course, they do not appear to be so in the above space-time diagram because of the screwy nature of the Minkowski inner product. Nevertheless Pythagoras' Theorem still applies, in the sense that

$$g(\overrightarrow{AB}, \overrightarrow{AB}) = g(\overrightarrow{AC}, \overrightarrow{AC}) + g(\overrightarrow{BC}, \overrightarrow{BC}), \tag{18.1}$$

as the reader may check.

Now we look at the inner product $g(\overrightarrow{AB}, \mathbf{V})$. We get firstly that

$$g(\overrightarrow{AB}, \mathbf{V}) = g(\overrightarrow{AC} - \lambda \mathbf{V}, \mathbf{V}) = -\lambda,$$

but also

$$g(\overrightarrow{AB}, \mathbf{V}) = g((0, l_{us}\mathbf{i}), \frac{1}{\sqrt{1-v^2}}(1, v\mathbf{i})) = -\frac{1}{\sqrt{1-v^2}}vl_{us}.$$

Hence, we get an expression for λ in terms of l_{us}. Substituting this into Equation 18.1, we get

$$
\begin{aligned}
-l_{rod}^2 &= -l_{us}^2 - \left(\frac{l_{us}v}{\sqrt{1-v^2}}\right)^2 \\
&= -l_{us}^2\left(1 + \frac{v^2}{1-v^2}\right) \\
&= \frac{-l_{us}^2}{1-v^2}.
\end{aligned}
$$

So

$$l_{us} = l_{rod}\sqrt{1-v^2},$$

as computed last time.

As we move towards general relativity, we will want to take this idea of using the Minkowskian inner product to solve physical problems further, and we will obtain general Minkowskian geometry. But before we do that, let us digress briefly to use what we now have to talk about hyperbolic space.

18.1   Digression: Hyperbolic geometry

Let us consider the space $M = \mathbb{R}^{2,1}$, which is three-dimensional Minkowski space, *ie*, a three dimensional vector space equipped with an inner product of signature (2,1). Inside M, let H be the subspace of future-pointing unit vectors:

$$H = \{\mathbf{v} \; : \; g(\mathbf{v}, \mathbf{v}) = 1\}.$$

More specifically,

$$H = \{(t, x, y) \; : \; t^2 - x^2 - y^2 = 1, \; t > 0\}.$$

<u>Picture</u>: Picture of the subspace H in M.

Proposition 18.1. *All tangent vectors to H are space-like.*

Proof. The tangent space to H at $\mathbf{v}$ is spanned by those vectors $\mathbf{w}$ such that

$$g(\mathbf{v}, \mathbf{w}) = 0. \qquad (18.2)$$

This can be observed from the following geometric argument. Firstly, it is easily checked that all tangent vectors at the point $(1, 0, 0)$ satisfy the above. But now, as in the case of the unit sphere in Euclidean space, there is an isometry of $\mathbb{R}^{2,1}$ which moves any point of H to $(1, 0, 0)$. This is the essence of Proposition 17.2. Since isometries preserve the inner product g, the result follows. Of course, it is also possible to check the veracity of the Equation 18.2 directly — the reader may wish to do so as an exercise.

Picture: Tangent vector $\mathbf{w}$ to the space H at $\mathbf{v}$.

Now suppose that such a vector $\mathbf{w}$ were not space-like, *ie*,

$$g(\mathbf{w}, \mathbf{w}) \geq 0.$$

Remember that we also have

$$g(\mathbf{v}, \mathbf{v}) = 1,$$

and

$$g(\mathbf{v}, \mathbf{w}) = 0.$$

Thus, $\mathbf{v}$ and $\mathbf{w}$ would span a vector subspace of M with signature (0,2), which is impossible in $\mathbb{R}^{2,1}$. $\qquad\qquad\square$

It follows that the form

$$ds^2 = -dt^2 + dx^2 + dy^2$$

restricts to a positive definite form on each tangent space to H. In other words, it gives us a metric tensor on H. To evaluate this explicitly, we note that there are natural coordinates for the subspace H of M, which are analogous to the usual spherical polar coordinates for the unit sphere in the Euclidean space $\mathbb{R}^3$. Specifically, the parameterization is

$$(r, \theta) \quad \mapsto \quad (\cosh r, \sinh r \cos \theta, \sinh r \sin \theta)$$
$$(= (t, x, y)).$$

One can check that this maps $\mathbb{R}^2$ to H by using the hyperbolic trigonometric identities.

We then have

$$
\begin{aligned}
ds^2 &= -dt^2 + dx^2 + dy^2 \\
&= -\sinh^2 r\, dr^2 + (\cosh r \cos \theta\, dr - \sinh r \sin \theta\, d\theta)^2 \\
&\qquad\qquad + (\cosh r \sin \theta\, dr + \sinh r \cos \theta\, d\theta)^2 \\
&= dr^2 + \sinh^2 r\, d\theta^2. \qquad\qquad (18.3)
\end{aligned}
$$

Again, one can compare this to the metric $ds^2 = dr^2 + \sin^2 r d\theta^2$ on the unit sphere.

The space H is called the *hyperbolic plane*. It has much in common with the usual sphere and also, clearly, some significant differences. For a start, it continues out forever (it is not *closed*). But just like the sphere, there is a group of isometries — here, the Lorentz group — which allows one to move any point on H to any other point, and also includes a subgroup of rotations by any angle about a given point.

We might now compute its Gauss curvature. We employ the following computational proposition.

Proposition 18.2. *If a metric $g(r, \theta)$ on a surface is given by*

$$g_{11} = 1, g_{12} = 0, g_{22} = f(r),$$

then its Gauss curvature is

$$K = \frac{-1}{2\sqrt{f}} \frac{\partial}{\partial r} \left(\frac{1}{\sqrt{f}} \frac{\partial f}{\partial r} \right).$$

Let us check this with the example of the sphere. In that case, with the usual paramterization, we have $f(r) = \sin^2 r$. Then

$$\frac{\partial f}{\partial r} = 2 \sin r \cos r,$$

and we get that the curvature is 1 everywhere, as expected.

On the other hand, in the case of the hyperbolic plane H we get $K = -1$ everywhere. It is the classic example of a space of constant negative curvature.

Proof of proposition 18.2. One just needs to slog through the computation. One finds that the expression

$$K = \frac{-1}{2f} \frac{\partial^2 f}{\partial r^2} + \frac{1}{4f^2} \left(\frac{\partial f}{\partial r} \right)^2,$$

comes from the expressions for the Christoffel symbols of the surface — the first term corresponding to $\Gamma^2_{12,1}$ and the second coming from $\Gamma^1_{21}\Gamma^1_{21}$. $\qquad\square$

One final remark about the geometry of hyperbolic space. In the hyperbolic plane, the circumference of a circle of radius r is $2\pi \sinh r$. This can be deduced directly from the metric of Equation 18.3. The consequence of this fact is that hyperbolic space is incredibly roomy — in Euclidian space, increasing the radius of a circle by one unit causes a proportional increase in its circumference, while in hyperbolic space the circumference increases exponentially. As a wise mathematician once said, if you lose your keys in Euclidean space, it's not good, but if you lose them in hyperbolic space, you're sunk.

There are numerous consequences of this exponential growth phenomenon of hyperbolic space. One is that there is some absolute fixed constant δ [..should compute this and write it in..] such that, for any geodesic triangle, each point on a side of the triangle lies within a distance δ of some point on one of the other

two sides. This is in stark contrast with the scalability of triangles in Euclidean space.

Perhaps the most famous consequence, however, is that Euclid's fifth postulate does not hold in hyperbolic geometry — given a line (*ie*, a geodesic) and a point not on it, there are infinitely many lines through the point which do not intersect the given line. It was a breakthrough of 19th century mathematics to realize that such geometries could exist.

Lecture 19

(Wednesday 29 October 2003)

19.1   Kinematical assumptions for general relativity

We now make our move towards general relativity. Today, we must discuss three basic assumptions of general relativity. The first two are concerned with the geometry of space-time. The third is the replacement for Newton's first law of motion.

Assumption 1:

   Space-time is a 4-dimensional smooth manifold.

   This means that, whatever "space-time" actually is, it is covered by the domains of observations of local observers. More specifically,

(i)   near any given event, a local observer can parameterize space-time by four coordinates x^0, x^1, x^2, x^3, and

(ii)   the observations of two observers O and Õ on the overlap of their domains are related by smooth functions:

$$\tilde{x^0} \;=\; \tilde{x^0}(x^0, x^1, x^2, x^3)$$
$$\vdots$$
$$\tilde{x^3} \;=\; \tilde{x^3}(x^0, x^1, x^2, x^3),$$

where the Jacobian matrix $\left(\frac{\partial \tilde{x^i}}{\partial x^j} \right)$ is non-singular.

Picture: picture of overlapping coordinate charts on a surface.

   This mathematical set-up is entirely analogous to our description of surface geometry in Lecture 7, and it is worthwhile employing this analogy to understand what is going on.

   One remark is in order, however. While the surfaces of Lecture 7 were embedded in an ambient three-dimensional Euclidean space, we will *not* assume that our four-dimensional space-time is embedded in an ambient linear space of higher dimension. (So, we are assuming that one cannot leave the four-dimensional universe we inhabit, or, at least, that if we were to leave, we could not come back to report what it was that we saw.) The ambient space was an unnecessary crutch which we will now kick out.

   Of course, what we lose in the kicking out of our ambient crutch, is that we do not have an induced metric tensor to give us a notion of lengths of curves, and so on. Under Assumption 1 alone, space-time is a very flabby, pliable object, without any *geometry*. The geometry comes from Assumption 2.

Assumption 2:
   Each tangent space (to space-time) carries a Lorentz metric.

Remark. Strictly speaking, there is an additional technical requirement, which is that these metrics vary smoothly over space-time.

Assumption 2 can be rephrased physically as "special relativity is valid to first order". Special relativity says that the whole of space-time carries a Lorenz metric. General relativity says that each infinitesimal piece of space-time carries a Lorentz metric, but these metric may vary across the space. The way in which these metrics vary is what is responsible for gravity in Einstein's theory.

Mathematically, Assumption 2 says that each observer can determine a matrix-valued function g_{ij} on space-time such that the inner product of two 4-vector fields V^i and W^j is $g_{ij}V^iW^j$, and that this inner product has signature (1,3). The smoothness condition of the remark is a requirement from the mathematical lawyers that says that the matrices g_{ij} are smooth functions of the coordinates.

Note that this metric g is a covariant symmetric 2-tensor, just as for surfaces.

Remark. You might well ask what we mean by a tangent vector to space-time, now that we have discraded the notion of an ambient Euclidean space in which space-time is embedded. The most straight-forward, but perhaps least satisfactory definition is to say that the tangent space is simply the collection of all four-component objects V^i which transform in the way of a contravariant vector under changes of coordinates:

$$\tilde{V}^i = \frac{\partial x^i}{\partial \tilde{x}^j} V^j.$$

Of course, there is also a more intrinsic mathematical definition[9]. Fix a point p in space-time. Consider a curve through p, given by a map from $\mathbb{R}$ to a local parameter space,

$$c : t \mapsto (x^0(t), x^1(t), x^2(t), x^3(t)),$$

such that $c(0) = p$. Now consider all the curves through p which agree with c up to first order in t. Because of the way in which our coordinate transformations were defined, the fact that two curves agree up to first order is independent of the choice of local coordinates, as the reader can check. A tangent vector at p can then be defined to be a family of such curves. In order to recreate from this our working definition of a tangent vector, one simply takes the derivative

$$c'(0) = \left(\frac{dx^0}{dt}(0), \frac{dx^1}{dt}(0), \frac{dx^2}{dt}(0), \frac{dx^3}{dt}(0) \right).$$

of any one of the family of curves.

<u>Picture</u>: tangent curves through a point.

---

[9]In fact, there are several intrinsic mathematical definitions, of which this is the most conceptual.

From the point of view of physics, it is usually most convenient to think of a tangent vector as being simply a displacement between to extremely close points.

Assumption 3 is the one we will spend most of today's lecture discussing. Recall that Newton's first law of motion describes the motion of a particle which is not subject to the action of any external forces. Assumption 3 tells us about such a particle in the theory of General Relativity. These particles will be called *freely falling.*

Assumption 3:
   The path of a freely falling particle (or photon) is a geodesic of the metric g.

Recall that a *geodesic* in M is a curve whose tangent vector is "covariant constant" along itself — *ie*, its covariant derivative along itself is zero. Equivalently, the tangent vector field of the curve is parallel along the curve, or more heuristically, the trajectory is as straight as is possible for a path through this "curved" space-time.

We want to deduce a mathematical formulation of the geodesic condition. To this end, suppose we have a curve in space-time, which is given by a parameterization

$$(x^0(s), x^1(s), x^2(s), x^3(s)).$$

Note that for any quantity Q defined along the curve,

$$\frac{dQ}{ds} = Q_{;i} \frac{dx^i}{ds}.$$

We use this to write the condition that the tangent vector $A^i = \frac{dx^i}{ds}$ to the curve has covariant derivative zero along the curve, and get

$$A^i_{;j} A^j = 0. \tag{19.1}$$

We can rewrite the left-hand side somewhat to get

$$
\begin{aligned}
A^i_{;j} A^j &= A^i_{,j} A^j + \Gamma^i_{jk} A^j A^k \\
&= \frac{d^2 x^i}{ds^2} + \Gamma^i_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds}.
\end{aligned}
\tag{19.2}
$$

Thus, we see that Equation 19.1 is just a system of second-order differential equations. The solution to this system will be unique, given appropriate initial conditions: an initial position and an initial velocity (tangent vector). In general relativity, Equation 19.1 is the equation of motion.

Example 19.1. In the case of special relativity, our space-time is $\mathbb{R}^{1,3}$, for which we have a particularly nice set of coordinates. In those standard coordinates, the Christoffel symbols are all zero, which is to say that space-time is not curved in special relativity. One can see from Equation 19.2 that the equation of motion will reduce to Newton's law. Of course, one could also choose stupid coordinates to make the differential equations appear much more complicated, but the physical result would be the same.

Theorem 19.2. *The "Lorentz norm" of the tangent vector is constant along a geodesic.*

Remark. It follows that geodesics come in three types — time-like, like-light, and space-like — according to whether their initial tangent vector (and hence every tangent vector) has Lorentz norm positive, zero or negative, respectively. A light-like geodesic corresponds to the track of a photon[10] through space-time. A space-light geodesic would correspond to a particle travelling with speed faster than the speed of light, were that possible — such an object is called a *tachyon*. Tachyons are generally thought not to exist for physical reasons. A material particle moves on a time-like geodesic, and we may parameterize its geodesic path by the particle's proper time τ, as defined in the previous lecture.

Proof. We want to prove that the expression $g_{ij}\frac{dx^i}{ds}\frac{dx^i}{ds}$ is constant along the curve. So we compute its derivative:

$$\frac{d}{ds}\left(g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}\right) = g_{ij,k}\frac{dx^k}{ds}\frac{dx^i}{ds}\frac{dx^j}{ds} + 2g_{ij}\frac{d^2x^i}{ds^2}\frac{dx^j}{ds}.$$

Now, using the geodesic equation, this equals

$$g_{ij,k}\frac{dx^k}{ds}\frac{dx^i}{ds}\frac{dx^j}{ds} - 2g_{ij}\Gamma^i_{lk}\frac{dx^j}{ds}\frac{dx^k}{ds}\frac{dx^l}{ds}$$
$$= (g_{ij,k} - 2\Gamma_{ikj})\frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds}.$$

But, if we recall that, $\Gamma_{ikj} = \frac{1}{2}(g_{ij,k} + g_{kj,i} - g_{ik,j})$, we see that the above quantity is zero. □

In principle, we now have enough mathematics to be able to solve motion problems in Einsteinian gravity. If someone tells us the appropriate Lorentzian metric for a system with gravity, such as the Schwarzschild solution for a spherical body which will discuss next lecture, we can write down and try to solve the geodesic equation in order to compute the motion of a freely falling body.

---

[10]or any massless particle, such as a neutrino.

Lecture 20

(Friday 31 October 2003)

As noted at the end of last lecture, we now have enough mathematical machinery to determine the motion of freely-falling bodies in Einsteinian gravity. Our next major goal is to do just that — determine the orbit of a freely-falling particle about a spherically symmetric mass, by simply stating the Schwarzchild metric for the system without proof. However, before we get to that, we will make a couple of observations about the mathematical properties of geodesics which are physically interesting.

20.1 Extremal property of geodesics

Recall from last lecture that (time-like) geodesics in space-time are the trajectories of freely-falling bodies.

Proposition 20.1. *A geodesic is a path which (locally[11]) extremizes the Lorentz interval between its endpoints.*

For a time-like curve, the Lorentz interval (*ie*, the length) of the path is the proper time of a particle moving along it. In this case, one may show that geodesics in fact maximize the proper-time between their endpoints. Compare this with the discussion of the twin-paradox and the Principle of Action of Section 16.2.

In the proof that follows, we will prove only that geodesics are critical for the Lorentz interval — *ie*, that the first variation is stationary. We will not continue with the second-order analysis which is necessary to show that they are in fact extremal.

Proof. Let's imagine a particle moving along a trajectory, $x^i(s)$, for $0 \leq s \leq 1$. Since the length of a curve is independent of parameterization, we can assume that the parameter s is the proper time along the curve.

We compare this trajectory to a nearby trajectory, which is given by $x^i(s) + \delta x^i(s)$, where $\delta x^i(0) = \delta x^i(1) = 0$ so that the nearby trajectory has the same endpoints.

In particular, we look at the proper time lapse along the nearby trajectory, which differs from that for the original curve by:

$$\int_{s=0}^{1} d\tau,$$

where $d\tau^2 = g_{ij}dx^i dx^j$. Since we are trying to find the extrema for this quantity, we are interested in its first-order variation,

$$\delta \int d\tau = \int \delta d\tau.$$

(This kind of informal argument may bristle with mathematical purists, but the effect of making the notation rigorous would only be to add obfuscation, not content.)

---

[11]*ie*, amongst all paths which are close to the given path.

Differentiating the expression for $d\tau^2$, we get

$$
\begin{aligned}
2d\tau\delta(d\tau) &= \delta(g_{ij})dx^i dx^j + 2g_{ij}dx^i\delta(dx^j) \\
&= g_{ij,k}\delta x^k dx^i dx^j + 2g_{ij}dx^i\delta(dx^j) \\
&= (g_{ij,k}\delta x^k dx^j + 2g_{ij}\delta(dx^j))dx^i.
\end{aligned}
$$

So,

$$
\delta(d\tau) = (\frac{1}{2}g_{ij,k}\delta x^k dx^j + g_{ij}\delta(dx^j))\frac{dx^i}{d\tau}.
$$

Therefore, we have

$$
\delta\int_0^1 d\tau = \int_0^1 \left(\frac{1}{2}g_{ij,k}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}\delta x^k + g_{ij}\frac{d(\delta x^j)}{d\tau}\frac{dx^i}{d\tau}\right) d\tau.
$$

At this point, observe that

$$
\frac{d}{d\tau}\left(g_{ij}\frac{dx^i}{d\tau}\delta x^j\right) = g_{ij}\frac{d(\delta x^j)}{d\tau}\frac{dx^i}{d\tau} + \frac{d}{d\tau}\left(g_{ik}\frac{dx^i}{d\tau}\right),
$$

and furthermore, since $\delta x^0(0) = \delta x^1(0) = 0$, we have

$$
\int_0^1 \frac{d}{d\tau}\left(g_{ij}\frac{dx^i}{d\tau}\delta x^j\right) d\tau = 0.
$$

Thus, since we parametrized the original curve by $s = \tau$, we have

$$
\delta\int_0^1 d\tau = \int_0^1 \left(\frac{1}{2}g_{ij,k}\frac{dx^i}{ds}\frac{dx^j}{ds} - \frac{d}{ds}\left(g_{ik}\frac{dx^i}{ds}\right)\right)\delta x^k ds.
$$

If the original curve $x^i(s)$ was extreme, we must therefore have

$$
\frac{d}{ds}\left(g_{ik}\frac{dx^i}{ds}\right) - \frac{1}{2}g_{ij,k}\frac{dx^i}{ds}\frac{dx^j}{ds} = 0.
$$

But

$$
\frac{d}{ds}\left(g_{ik}\frac{dx^j}{ds}\right) = g_{ik}\frac{d^2x^i}{ds^2} + g_{ik,l}\frac{dx^i}{ds}\frac{dx^l}{ds},
$$

and thus, we get

$$
g_{ik}\frac{d^2x^i}{ds^2} + \frac{1}{2}(g_{ik,l} + g_{lk,i} - g_{il,k})\frac{dx^i}{ds}\frac{dx^l}{ds} = g_{ik}\frac{d^2x^i}{ds^2} + \Gamma_{ilk}\frac{dx^i}{ds}\frac{dx^l}{ds} = 0,
$$

which is precisely the geodesic equation.

$\qquad\square$

We will not make heavy use of this fact in the course of these lectures, but it is important physically, since it realizes geodesics, and hence the trajectories of particles, in terms of the minimization of some quantity — a "Lagrangian".

20.2 Symmetries and conservation laws

Often in physics we come across conserved quantities — momentum, angular momentum, *etc.* These conserved quantities come about as a result of symmetries of the physical system. For instance, the invariance of a system under the group of rotations yields the conservation of angular momentum. The purpose of this section is to prove and elaborate on this idea. As well as yielding physical understanding, this will be an aid to solving the equations of motion for systems with additional symmetry.

Proposition 20.2. *Suppose that all of the components of the metric tensor g_{ij} do not depend on a certain coordinate, say x^0 (such a coordinate is called a cyclic coordinate). Then the quantity*

$$g_{0i}\frac{dx^i}{ds}$$

is constant along any geodesic.

Remark. The conserved quantity is called the *conjugate momentum* of x^0, and x^0 is called a *cyclic coordinate.*

Proof. We look at the derivative of the conjugate momentum along a geodesic:

$$
\begin{aligned}
\frac{d}{ds}\left(g_{0i}\frac{dx^i}{ds}\right) &= \frac{d^2x^i}{ds^2}g_{0i} + g_{0i,j}\frac{dx^j}{ds}\frac{dx^i}{ds} \\
&= -g_{0i}\Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} + g_{0j,k}\frac{dx^j}{ds}\frac{dx^k}{ds} \\
&= (-\Gamma_{jk0} + g_{0j,k})\frac{dx^j}{ds}\frac{dx^k}{ds} \\
&= (-\frac{1}{2}g_{k0,j} + \frac{1}{2}g_{j0,k} + \frac{1}{2}g_{jk,0})\frac{dx^j}{ds}\frac{dx^k}{ds} \\
&= 0,
\end{aligned}
$$

by the assumption that $g_{jk,0} = 0$. ☐

Of course, if we do not choose our coordinate system (x^0, x^1, x^2, x^3) appropriately, the independence of g on a certain coordinate may not be realized. Worse, we may have two (or more) conserved quantities which occur as a result of the independence of the metric in different coordinate systems. For these reasons, we want to introduce a fancier language in which to phrase Proposition 20.2.

Definition 20.3. A *Killing field* is a vector field which is tangent to a 1-paramater group of isometries.

This definition requires some explanation. A 1-parameter group of isometries is is a family of isometries $\{\Phi_t\}$ (of space-time, say), parameterized by a real number t, which form a group. Specifically, if we denote the isometries by Φ_t, then

- Φ_0 is the identity map, and

- $\Phi_{t_1} \circ \Phi_{t_2} = \Phi_{t_1+t_2}$.

In the case of a metric which independent of the coordinate x^0, the corresponding 1-parameter family of isometries is given by translating in the x^0-coordinate. The Killing field, is then the vector field with componenets $(1, 0, 0, 0)$.

In this language, Proposition 20.2 says that, associated to a 1-parameter group of isometries (or Killing field), there is a conserved quantity. Of course, if we have two or three parametrized families of isometries, we obtain two or three conserved quantities.

20.3   Orbital motion for the Schwarzschild metric

Now, as promised, we will pull the Schwarzschild metric out of a hat, and do some kinematics with it.

Schwarzschild metric:

$$\left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 d\theta^2$$

($M \in \mathbb{R}^+$ is some constant).

You can imagine that the constant M is, more or less, the mass of some body — a star or a black hole — at the centre of our system. The chosen coordinate names, r, θ and t, are clearly suggestive, corresponding basically to something like the polar coordinates for the plane (r, θ) and time t. Of course, such notions should be taken with a grain of salt in the world of Einsteinian relativity.

Note, also, that the metric we have written down is a metric for only three-dimensional space-time. This is justified by the fact that, for the moment, we will only be interested in planar motion around the body, so we can ignore the third spatial dimension.

The first thing to observe is that the Schwarzchild metric does not depend on the coordinates θ or t. We therefore have two conserved quantities, which are, respectively,

$$r^2 \frac{d\theta}{d\tau},$$

and

$$\left(1 - \frac{2M}{r}\right) \frac{dt}{d\tau}.$$

Physically, the first corresponds to the conservation of angular momentum. The second is more complicated — it involves $\frac{dt}{d\tau}$, the first component of 4-momentum, which Einstein recognized to be the energy of the particle. However, the conserved quantity is not energy itself, but some function of r times the energy. This leads to the phenomenon of "gravitational red-shift". For consider a photon originating from the vicinity of some gravitationally massive body. The conserved quantity shows that as the distance of the photon from the body increases, its energy must decrease. But the energy of a photon is linked with its frequency (*ie*, colour) by Planck's Law, $E = h\nu$. Thus, an escaping photon experiences a shift down in frequency.

It is this phenomenon which is responsible for the influence of general relativity on the Global Positioning System, which we plan to discuss later in the course.

Lecture 21

(Monday, 3 November 2003)

The Schwarzschild metric which we wrote down last time corresponds physically to the geometry of space-time near a non-rotating point-mass. We think of this physically as representing the gravitational effect of a large gravitating body such as a planet, the sun, or a black hole on a much smaller orbiting body. IN this lecture we will analyze the orbits of particles for this metric. But first, let us revisit the predictions of Newtonian gravity.

21.1   Newtonian orbit theory

<u>Picture</u>: System: Stationary (huge) mass M and small particle with polar coords (r, θ).

In Newtonian theory, the conserved quantities for a freely falling body are (in polar coordinates)

- angular momentum: $L = r^2 \frac{d\theta}{dt}$

- total energy E, which is the sum of

   - kinetic energy: $\frac{1}{2} \left(\frac{dr}{dt} \right)^2 + \frac{1}{2} r^2 \left(\frac{d\theta}{dt} \right)^2$, and
   - potential energy: $-M/r$.

For convenience, we rewrite the total energy as

$$E = \frac{1}{2} \left(\frac{dr}{dt} \right)^2 + \frac{1}{2} \frac{L^2}{r^2} - \frac{M}{r}.$$

The fact that these quantities are conserved is enough to deduce how the radius r and angular position θ of a freely falling body change with time.

To this end, let us rewrite the last equation to get

$$
\begin{aligned}
\frac{1}{2} \left(\frac{dr}{dt} \right)^2 &= E - \frac{1}{2} \frac{L^2}{r^2} + \frac{M}{r} \\
&= E - V(r),
\end{aligned}
\tag{21.1}
$$

where the quantity

$$V(r) = \frac{1}{2} \frac{L^2}{r^2} - \frac{M}{r} \tag{21.2}$$

is called the *effective potential*. The advantage of writing the energy in this form is that we obtain an equation for r only, which we can solve to determine the radial position of the body.

Heuristically, the effective potential is a potential which takes into account not just the gravitational potential, but also a "centrifugal potential". As r decreases, the conservation of angular momentum shows that the angular velocity of the body increases, and so does the apparent "centrifugal force", pushing it outward from the gravitating body. Of course, the centrifugal force is not an actual force, only an artifact of ignoring the angular component in our dynamical system.

Picture: graph of the effective potential $V(r)$ against r, with line of constant energy E.

Equation 21.1 describes the rate of change of radial position of the orbiting particle as a function of its total energy and the effective potential $V(r)$. We can think of $V(r)$ as representing a potential well on the line.

For $E < 0$, we imagine a particle oscillating back and forth within this potential well, with some period. From the conservation of angular momentum we will obtain another equation describing the angular motion of the particle. It is a miracle of Newtonian theory that the period of the angular motion and the period of the radial motion precisely agree. Thus the orbits of freely falling bodies around a point mass close up — in fact giving elliptical orbits. This does not happen in general relativity, as we will see.

For $E > 0$ the radial position of the freely falling particle will decrease to a point, stop, and increase again to infinity. This corresponds a hyperbolic orbit.

Picture: elliptical and hyperbolic orbits.

21.2 Relativistic orbit theory

Now we undertake the same kind of qualitative analysis for the orbital motion about a gravitational body in the theory of general relativity. Note that the kinematic principle of general relativity states that all freely falling particles travel along geodesics of space-time, so this analysis will apply equally well to photons as to regular material particles.

Recall the Schwarzschild metric from last lecture:

$$d\tau^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 d\theta^2.$$

The cyclic coordinates t and θ gave rise to two conserved quantities:

$$L = r^2 \frac{d\theta}{d\tau}, \tag{21.3}$$

and

$$E = \left(1 - \frac{2M}{r}\right) \frac{dt}{d\tau}. \tag{21.4}$$

The conserved quantity E does not look much like energy, as written, but there is good reason to think of it as such.

A priori, it seems we don't have enough to solve our kinematic problem, because we now have three coordinates (t,r and θ), and so far only two conserved quantities. However, in general relativity there is one quantity which is always conserved, namely the length of a tangent vector to the space-time geodesic, which is always one:

$$\left(1 - \frac{2M}{r}\right) \left(\frac{dt}{d\tau}\right)^2 - \left(1 - \frac{2M}{r}\right)^{-1} \left(\frac{dr}{d\tau}\right)^2 - r^2 \left(\frac{d\theta}{d\tau}\right)^2 = 1. \tag{21.5}$$

What this says is simply that particles travel through space-time at constant speed — we are parameterizing the geodesic by proper time.

If we substitute Equation 21.3 and Equation 21.4 into Equation 21.5, we get

$$-\frac{L^2}{r^2} + \left(1 - \frac{2M}{r}\right)^{-1} \left(E^2 - \left(\frac{d\theta}{d\tau}\right)^2\right) = 1.$$

Reorganizing,

$$\begin{aligned}
\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 &= \frac{E^2 - 1}{2} + \frac{M}{r} - \frac{1}{2}\frac{L^2}{r^2} + \frac{ML^2}{r^3}. \\
&= \frac{E^2 - 1}{2} - V(r), \tag{21.6}
\end{aligned}$$

where

$$V(r) = -\frac{M}{r} + \frac{1}{2}\frac{L^2}{r^2} - \frac{ML^2}{r^3}. \tag{21.7}$$

Compare Equation 21.6 with its Newtonian analogue, Equation 21.1. The first term has changed somewhat — we will discuss why shortly. The Einsteinian effective potential $V(r)$ includes an additional term of order $1/r^3$. For large r this term is negligible, and we see that the Newtonian theory holds very well at large distances from the gravitating body. However, at small distances, the Newtonian potential is completely dominated by the new relativistic term, which introduces a potential well of infinite depth at the point mass itself. Clearly, this leads to radically different behaviour.

So why is the first term of Equation 21.6 as it is? To answer this, we must look at what happens at infinity. Here, the metric is approximately that of special relativity (given in polar coordinates), *ie*, special relativity is valid. The kinetic energy of a body in special relativity is the first component of its 4-momentum, which is

$$\frac{1}{\sqrt{1 - v^2}}.$$

For small speeds v, we get

$$\frac{E^2 - 1}{2} \approx \frac{v^2}{2}.$$

In other words, the general relativistic and Newtonian theories yield approximately equal solutions in the region for which we expect them to do so – at small velocities out near infinity.

We can rewrite $V(r)$ as

$$V(r) = \frac{1}{2}\left(1 - \frac{2M}{r}\right)\left(1 + \frac{L^2}{r^2}\right) - \frac{1}{2},$$

and again imagine this as a potential on the line. Drawing the graph of this, we see the qualitative behaviour of various freely falling particles.

<u>Picture</u>: Effective potential $V(r)$ for a value of L which gives a potential "hump", with various different constant energy lines..

For a reasonably large value of the angular momentum L we get a potential as in the above diagramme. In this case, for "energy" $(E^2 - 1)/2 < 0$, we get bounded orbits, as in the Newtonian case, and for positive values of $(E^2 - 1)/2$ below some critical value we get orbits which originate from and return to infinity. However, once we pass the critical value, the trajectory of the particle will continue inwards forever. We have a "black hole". This is a feature of general relativity which is unseen in Newtonian theory. We will return to discuss this in greater detail later in the course.

<u>Picture</u>: Effective potential $V(r)$ for various values of L..

For smaller values of L, the potential hump will be smaller in magnitude, or may not appear at all, as shown in Figure **??**. To examine this behaviour, we introduce more convenient variables, namely,

$$\rho = \frac{r}{M}, \qquad \lambda = \frac{L}{M}.$$

We now ask about the critical points of the effective potential

$$V(\rho) = \left(1 - \frac{2}{\rho}\right)\left(1 + \frac{\lambda^2}{\rho^2}\right).$$

This is a first year calculus problem. We find that $V'(\rho) = 0$ if and only if

$$2\rho^2 - 2\lambda^2\rho + 6\lambda^2 = 0,$$

which gives

$$\rho = \frac{\lambda^2}{2}(1 \pm \sqrt{1 - 12/\lambda^2}).$$

Therefore, we get no critical points for the effective potential if $\lambda \leq \sqrt{12}$ — all particles with such small angular momentum will drop into the black hole. On the other hand, for $\lambda > \sqrt{12}$ there are stable orbits.

Is this solution physically reasonable? Do we see the phenomena just derived occurring for actual physical bodies?

We should compare general relativity with the Newtonian theory for some specific examples. This will be the purpose of the next two lectures, to discuss the first two great confirmations of Einstein's new theory of gravity: firstly the deflection of light as it passes the sun, and secondly the precession of the orbit of Mercury.

Lecture 22

(Wednesday, 5 November 2003)

In this lecture, we will discuss the deflection of rays of light around the sun, which is a classic confirmation of general relativity. We begin in the era before general relativity existed.

Newton's theory of gravity provides no insight in to the affect of gravity on the propagation of electromagnetic radiation. However, the process of setting up his theory of special relativity led Einstein to consider the question. In 1911, Einstein was working towards the ideas that would eventually become general relativity, and he wrote a paper, "On the influence of gravitation on the propagation of light"[Ein1911], in which he put forward the following argument which predicts, quantitatively, the deflection of light by a large gravitational mass.

He was working to extend his principle of special relativity. His key idea is that there is no way to differentiate between a frame of reference at rest within a uniform gravitational field and an accelerating frame of reference. For, consider the following thought experiment. I fall from the ceiling of my room while watching a model train travelling along a table-top. In my frame of reference, it appears that the train moves upward along a parabolic trajectory, as if it were being pulled up by a gravitational force equal to my downward acceleration. This led Einstein to believe that a light ray should be bent downwards, according to the following calculation.

Imagine a beam of light which is passing near to the sun. We assume its angular deflection is small and do some infinitesimal analysis. We look at the angular deflection over a time interval dt, given by a downward acceleration of a.

Picture: light beam passing the sun, perpendicular distance to center of the sun is r_0.

Picture: small parabolic deflection of an initially horizontal path by a downward acceleration a.

The small angular deflection over this time period is

$$d\alpha = \frac{a}{c}dt = \frac{a}{c^2}ds.$$

Now we compute the total deflection by integrating. We assume, first that the speed of light is constant. We also need to compute the perpendicular

component of the acceleration of the sun's gravitational field at the location of the light particle, which is

$$a = \frac{GM}{r^2} \cos \psi,$$

where r is its distance from the sun and ψ its angle from the perpendicular to the path. We get

$$
\begin{aligned}
\Delta \alpha &= \int d\alpha \\
&= \frac{1}{c^2} \int \frac{GM}{r^2} \cos \psi ds \\
&= \frac{1}{c^2} \int \frac{GM}{r_0^2} \cos^3 \psi \sec^2 \psi d\psi \\
&= \frac{GM}{r_0 c^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \psi d\psi \\
&= \frac{2GM}{r_0 c^2}.
\end{aligned}
$$

This was Einstein's 1911 solution. It's wrong.

The prediction is verifiable. One needs a light source somewhere behind the sun. There is a plentiful supply of these given all the stars in the universe. The biggest problem is in seeing the light from the distant star against of the huge amount of light given off by the sun itself. The solution is to wait for an eclipse.

There was a solar eclipse in 1915. Unfortunately, the European nations were involved in one of their periodic squabbles at the time, so no expedition could be launched to perform the experiment. But by the time the next eclipse occured on May 27, 1919, Einstein had had enough time to produce his general theory of relativity and correct his eroneous prediction. His corrected prediction, of about one second of arc deflcetion for a beam of light just grazing the surface of the sun, was indeed confirmed.

22.1   Solution from general relativity

With hindsight, the error in Einstein's original prediction is not surprising. The physical assumptions made in his argument are a comprise between special relativity and Newtonian theory: the light is being accelerated by the sun's gravity in the perpendicular direction, but not in the tangential direction to its path. To obtain the correct prediction, we use the full power of general relativity, by again appealing to the Schwarzchild metric.

Recall that we have two conserved quantities E and L. We cannot parameterize a light-like geodesic by proper time, so we will use a parameter λ to parameterize the photon's tranjectory. From last lecture,

$$\frac{d\theta}{d\lambda} = \frac{L}{r^2},$$

and also

$$\left(\frac{dr}{d\lambda} \right)^2 = E^2 - \left(1 - \frac{2M}{r} \right) \frac{L^2}{r^2}.$$

For the geodesic trajectory of a light beam, the tangent vectors have length zero. This fact, combined with the above conserved quantities, yields

$$\frac{d\theta}{dr} = \frac{L}{r^2\sqrt{E^2 - \left(1 - \frac{2M}{r}\right)\frac{L^2}{r^2}}}.$$

We now make a change of coordinates, $u = 1/r$, to get

$$\begin{aligned}
\frac{du}{d\theta} &= -\frac{1}{r^2}\frac{dr}{d\theta} \\
&= -\frac{1}{r^2}\left(\frac{d\theta}{dr}\right)^{-1} \\
&= -\frac{1}{L}\sqrt{E^2 - (1 - 2Mu)L^2u^2}.
\end{aligned}$$

Thus,

$$\left(\frac{du}{d\theta}\right)^2 = \frac{E^2}{L^2} - (1 - 2Mu)u^2.$$

For convenience, we put $a = E/L$. The total variation of θ over the path is then

$$2\int_0^{u_0} d\theta = 2\int_0^{u_0} \frac{du}{\sqrt{a^2 - (1 - 2Mu)u^2}}, \tag{22.1}$$

where u_0 is the maximum attainable value of the parameter u, namely the root of $a^2 - u_0^2 - 2Mu_0^3 = 0$. This is a tough integral, which would in principle need a heady amount of elliptic function theory to compute. But even without that level of theory, we can still make some useful physical observations.

Firstly, it is comforting to note that we get the expected result if the sun were to have mass $M = 0$. In that case, the total change in the angular parameter θ is computed to be

$$2\int_0^a \frac{du}{\sqrt{a^2 - u^2}} = \pi,$$

as expected for a straight line.

But we can also note that the effect of the sun's gravity on a passing beam of light is small. We can therefore compute the deflection very satisfactorily by expanding the integral to first order in M. Effectively, this means taking $\frac{d}{dM}$ of Equation 22.1 and evaluating at $M = 0$.

We first make some more variable changes:

$$v = u/u_0$$

and

$$\mu = u_0 M,$$

so that

$$a^2 = u_0^2(1 - 2\mu).$$

Then,

$$\begin{aligned}
\theta &= 2\int_0^1 \frac{u_0 dv}{\sqrt{u_0^2(1 - 2\mu) - u^2(1 - 2\mu/u_0)}} \\
&= 2\int_0^1 \frac{dv}{\sqrt{1 - 2\mu - v^2 + 2\mu v^3}}.
\end{aligned}$$

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We now want to differentiate this integral with respect to μ. In the spirit of physics, we ignore the theorems that tell us when it is legal to differentiate under the integral, although they do apply, and just go ahead and do it:

$$
\begin{aligned}
\left.\frac{d\theta}{d\mu}\right|_{\mu=0} &= 2\int_0^1 \frac{\partial}{\partial \mu}\left(\frac{1}{\sqrt{1 - 2\mu - v^2 + 2\mu v^3}}\right)\Bigg|_{\mu=0} dv \\
&= 2\int_0^1 \frac{1 - v^3}{(1 - v^2)^{3/2}}dv \\
&= 4,
\end{aligned}
$$

where the last integral is evaluated by the substitution $v = \sin\gamma$. Thus, to first order, the total deflection is $4\mu = 4M/r_0$. This is the answer in coordinates for which both c and G are one, so after reintroducing these constants we get

$$
\text{Angluar deflection} = \frac{4MG}{c^2 r_0},
$$

precisely twice Einstein's original prediction.

Lecture 23

(Friday, 7 November 2003)

23.1 Perihelion precession

According to Lecture 21, the orbits of planets about the sun as predicted by general relativity should differ from the Newtonian predictions. This fact provided one of the most beautiful and startling confirmations of Einstein's theory of gravitation — the prediction of the precession of the orbit of Mercury.

In order to understand the corrections to the Newtonian theory that general relativity provides, we need to first understand what the Newtonian theory tells us.

23.1.1 Newtonian orbit theory revisited

We need a more precise analysis than that of Section 21.1. The dynamics of the Newtonian orbit are given by the equations

$$\frac{d\theta}{dt} = \frac{L}{r^2}, \tag{23.1}$$

and

$$\left(\frac{dr}{dt}\right)^2 = 2E + \frac{2M}{r} - \frac{L^2}{r^2}. \tag{23.2}$$

Equation 23.1 can be recognized as Kepler's third law of planetary motion — the area of sectors swept out in a given time by the motion of a planet about the sun is constant — since the area of a small sector is $r^2 d\theta$.

Combining Equations 23.1 and 23.2, we get

$$
\begin{aligned}
\left(\frac{dr}{d\theta}\right)^2 &= \frac{r^4}{L^2}\left(\frac{dr}{dt}\right)^2 \\
&= r^4\left(\frac{2E}{L^2} + \frac{2M}{L^2 r} - \frac{1}{r^2}\right).
\end{aligned}
$$

It is convenient to use the variable $u = 1/r$, for which

$$\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta},$$

and thus

$$
\begin{aligned}
\left(\frac{du}{d\theta}\right)^2 &= \frac{2E}{L^2} + \frac{2M}{L^2}u - u^2 \\
&= \frac{2E}{L^2} + \frac{M^2}{L^4} - \left(u - \frac{M}{L^2}\right)^2. \\
&= b^2 - \left(u - \frac{M}{L^2}\right)^2,
\end{aligned}
$$

where

$$b^2 = \frac{2E}{L^2} + \frac{M^2}{L^4}.$$

This is a separable differential equation, which we deal with in the usual way:

$$\theta = \int \frac{du}{\sqrt{b^2 - \left(u - \frac{M}{L^2}\right)^2}} \tag{23.3}$$

$$= \cos^{-1}\left(\frac{u - \frac{M}{L^2}}{b}\right) + C,$$

Different values of the constant of integration here amounts to a different choice of axes for the system, so with an appropriate choice we get

$$\frac{1}{r} = u = \frac{M}{L^2} + b\cos\theta. \tag{23.4}$$

We can recognize Equation 23.4 as the equation of an ellipse, particularly if we recall the following definition of an ellipse as Apollonius would have it:

Definition 23.1. An ellipse is the locus of a body that moves so that its distance from a fixes point F (the *focus*) bears a ratio e (the *eccentricity*, with $0 < e < 1$) to its distance from a fixed line (the *directrix*).

Picture: vertical directrix, focus at distance d from the directrix, point R on the locus at distance r from F and distance $d = r\cos\theta$ from the line, where θ is the angle between RF and the altitude from F to the directrix..

The equation comes from Apollonius' definition by considering the above picture, wherein we require

$$r = e(d - r\cos\theta),$$

that is,

$$r(1 + e\cos\theta) = ed,$$

or

$$\frac{1}{r} = \frac{1}{ed} + \frac{1}{d}\cos\theta,$$

which is essentially Equation 23.4 above.

Remark. More conventionally, one describes an ellipse in terms of its eccentricity e and its *semi-major axis*, which is defined to be

$$a = \frac{1}{2}(r_{\max} + r_{\min})$$

$$= \frac{1}{2}ed\left(\frac{1}{1-e} + \frac{1}{1+e}\right)$$

$$= \frac{ed}{1-e^2}.$$

We can use this framework to describe the geometry of the orbit in a more direct way.

23.1.2 The relativistic perturbation

In the relativistic world we have the equations

$$\frac{d\theta}{d\tau} = \frac{L}{r^2}$$

and

$$\left(\frac{dr}{d\tau}\right)^2 = E^2 - \left(1 - \frac{2M}{r}\right)\left(1 + \frac{L^2}{r^2}\right).$$

In this case, calculations along the lines of that just undertaken in the Newtonian case eventually yield

$$\theta = \int \frac{du}{\sqrt{\frac{E^2-1}{L^2} + \frac{2M}{L^2}u - u^2 + 2Mu^3}}. \tag{23.5}$$

This integral is obviously similar to its Newtonian analogue, Equation 23.3, with the key difference being the inclusion of the term of order u^3 under the square root in the denominator. As we have seen before, this factor is small if r remains large, giving a good approximation to the Newtonian theory for orbits far from the gravitating body. We are now interested in how this cubic term influences the orbital behaviour.

Recall the graph of the "effective potential" that we produced in Lecture 21.

<u>Picture</u>: Graph of $\frac{2}{L^2}V(r) = -2\frac{M}{L^2}u + u^2 - 2Mu^3$, with line of constant negative "energy" marked, and roots of the corresponding cubic marked at $r_{\max}$, $r_{\min}$ (the bounds of the stable orbit) and r_0 (the extra root due from GR).

A body in a bounded orbit, such as a planet orbiting the sun, has an "energy" which puts in inside the potential well of this graph, oscillating between $r_{\max}$ ($= u_{\min}^{-1}$) and $r_{\min}$ ($= u_{\max}^{-1}$). The cubic term means that the "effective potential" has a third value at this energy level, which we will designate by r_0 ($= u_0^{-1}$). This corresponds to the new unstable regime of the relativistic dynamics which will not be visited by our stable physical system. Nevertheless, its existence has a mathematical effect on the dynamics.

We are interested in the relationship between the periods of the angular motion and the radial motion of the orbiting body. If we compute the integral of Equation 23.5 from $u_{\min}$ to $u_{\max}$, we will find the angular variation of the body as moves through half a radial cycle. We consider this as a perturbation of the Newtonian integral, taking an expansion to first order in the mass M of the gravitating body.

Note that as M varies, not only will the integrand change, but also the limits. Specifically, in the Newtonian limit, we get that

$$u_{\text{max}} \quad \rightarrow \quad \frac{1}{a(1-e)},$$

$$u_{\text{min}} \quad \rightarrow \quad \frac{1}{a(1+e)},$$

and

$$u_0 \rightarrow \infty.$$

Since u_{max}, u_{min} and u_0 are the roots of the cubic

$$2M(u - u_{\text{max}})(u - u_{\text{min}})(u - u_0) = \frac{E^2 - 1}{L^2} + \frac{2M}{L^2}u - u^2 + 2Mu^3,$$

we also have that

$$u_0 + u_{\text{max}} + u_{\text{min}} = \frac{1}{2M}.$$

We write the integral of Equation 23.5 as

$$2 \int_{u_{\text{min}}}^{u_{\text{max}}} \frac{du}{\sqrt{2M(u - u_{\text{max}})(u - u_{\text{min}})(u - u_0)}}$$

$$= \quad 2 \int_{u_{\text{min}}}^{u_{\text{max}}} \frac{du}{\sqrt{(1 - 2M(u + u_{\text{max}} + u_{\text{min}}))(u_{\text{max}} - u)(u - u_{\text{min}})}}.$$

Let us now compute the term of first order with respect to M, by taking the derivative of θ at $M = 0$:

$$\left.\frac{\partial \theta}{\partial M}\right|_{M=0} \quad = \quad 2 \int_{u_{\text{min}}}^{u_{\text{max}}} \frac{u + u_{\text{max}} + u_{\text{min}}}{\sqrt{(u - u_{\text{max}})(u_{\text{min}} - u)}} du$$

$$= \quad 3\pi(u_{\text{max}} + u_{\text{min}})$$

$$= \quad \frac{6\pi}{a(1 - e^2)}.$$

Thus, to first order, and reintroducing the constants c and G, the predicted precession of the orbit is

$$\frac{6\pi MG}{a(1 - e^2)c^2} \text{ per revolution.}$$

That's the math taken care of. Now we need to look for the physical confirmation. Our solar system is comprised of a bunch of orbiting bodies, and amongst these, it is clear that we should look to Mercury. Mercury makes for an excellent subject, not just because its proximity to the sun makes it most affected by the relativistic factors, but also because its orbit is particularly eccentric, making accurate determination of the perihelion (point of closest approach to the sun) feasible.

The numbers relevant to Mercury's orbit are as follows: the eccentricity is

$$e \approx 0.206$$

and the semi-axis,

$$a \approx 5.79 \times 10^{10} m.$$

Invoking the computations we just made, general relativity predicts a precession of

$$5 \times 10^{-7} \text{radians per revolution.}$$

Mercury makes 415 revolutions about the sun per century, so this amounts to an advancement of the perihelion by 43 seconds of arc per century.

Einstein made this calculation early in the history of general relativity, and he has been recorded variously as remarking that, "it was as if something was broken in my head," or more reservedly, and poetically, "it was as if the universe had whispered in my ear". Whatever the truth, his reaction was certainly one of amazement, for there was indeed a previously inexplicable precession of 43 seconds of arc per century which had been a confounding puzzle in 19th century astronomy.

In fact, the actual advancement of the perihelion of Mercury is several hundred seconds of arc per century. However, all but 43 seconds had been accounted for in the Newtonian theory, by the gravitational influence of the other planets in the solar system. This kind of Newtonian analysis of the mechanics of the solar system had been studied heavily. For instance, the two outermost planets, Neptune and Pluto[12], were discovered as a result of their effects upon the orbits of the other planets well before they were seen directly. In fact, the missing 43 seconds of Mercury's precession were, for a time, speculated to be evidence of an undiscovered planet even closer in to the sun. This theory gained enough credibility that at one stage the hypothetical planet was even given a name, Vulcan. The planet was of course never found, and Einstein's beautiful new theory of gravity refuted Vulcan's existence, freeing up the name for the makers of Star Trek.

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[12]Pluto's status as a planet is presently falling out of fashion.

Lecture 24

(Monday, 10 November 2003)

Having studied the Schwarzschild metric without actually justifying its physical relevance, we now want to go back and actually write down Einstein's theory of gravity, from which the Schwarzschild metric can be deduced as a solution. In order to do this, we will have to get back to doing abstract differential geometry. We'll catch up enough of it in the next few lectures to write down Einstein's field equations and give some justification for them.

The first order of business is to look at the symmetries of the Riemann curvature tensor R^i_{jkl}. We could do this all in the index notation that we have been using in the course so far. However, this is a good time to introduce the standard coordinate-free notation that mathematicians like to use. This introduction will take the form of a "dictionary" for translating the geometric notions we've encountered so far into the their coordinate-free terminology.

24.1 Coordinate-free notation

24.1.1 Vector fields

We first introduce an abstract notion of the tangent space of which mathematicians are particularly fond. In our original set-up of differential geometry — the case of surfaces in $\mathbb{R}^3$ — we noted that the space of tangent vectors at a point was spanned by basis vectors $\frac{\partial \mathbf{r}}{\partial x^1}, \frac{\partial \mathbf{r}}{\partial x^2}$. Now, associated with each tangent vector at a point is the notion of differentiating a function in that direction. For instance, differentiating a function f in the direction of $\frac{\partial \mathbf{r}}{\partial x^1}$ amounts to applying the differential operator $\frac{\partial}{\partial x^i}$ to f at that point. We now choose to identify the basis vector with its corresponding differential operator. Under this identification, vector fields now have the form

$$V = V^i \tfrac{\partial}{\partial x^i}$$

with respect to some coordinate system $x^0, \dots, x^n$.

This abstract definition has several compelling advantages. Perhaps most importantly, this definition still makes good sense even though we have long since stopped to think of our manifold as embedded in some ambient Euclidean space. Additionally, the contravariant transformation law for tangent vectors (Equation 13.1) is now realized as simply being the chain rule for a change of coordinates. Furthermore, we can now write the directional derivative of a function f in the direction of a vector field V as just $V.f$, which in coordinates amounts to

$$V.f = V^i \frac{\partial f}{\partial x^i}.$$

Throughout the following, the letters V, W, X, Y, Z will denote vector fields.

If our tangent spaces are equipped with a metric tensor, we write the inner product of two vector fields V and W as

$$\langle V, W \rangle,$$

replacing the coordinate-dependent notation

$$g_{ij} V^i W^j.$$

This operation results in a real-valued function on the manifold.

We can also form the *commutator* (or *Lie bracket*) of two vector fields, which is the vector field $[X, Y]$ defined by the equation

$$[X, Y]f = X.Y.f - Y.X.f.$$

Note that it is not immediately clear that this is a vector field. In our new viewpoint, vector fields are first-order linear differential operators, while the commutator appears to be some kind of second order differential operator. However, the symmetry of mixed second order partial derivatives comes to the rescue, causing all second order terms to cancel:

$$
\begin{aligned}
X.Y.f &= X.(Y^i \frac{\partial f}{\partial x^i}) \\
&= X^j \frac{\partial}{\partial x^j}(Y^i \frac{\partial f}{\partial x^i}) \\
&= X^j Y^i_{,j} \frac{\partial f}{\partial x^i} + X^j Y^i \frac{\partial^2 f}{\partial x^i \partial x^j},
\end{aligned}
$$

and similarly,

$$Y.X.f = Y^j X^i_{,j} \frac{\partial f}{\partial x^i} + X^i Y^j \frac{\partial^2 f}{\partial x^i \partial x^j},$$

and thus their difference is

$$X.Y.f - Y.X.f = (X^j Y^i_{,j} - Y^j X^i_{,j}) \frac{\partial f}{\partial x^i}.$$

So, in coordinates,

$$[X, Y] = X^j Y^i_{,j} - Y^j X^i_{,j}.$$

Note that if X and Y are coordinate vector fields, $\frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial x^j}$, for some local coordinate system $x^1, \ldots, x^n$, then their commutator is zero. This is just the fact that ordinary mixed partials commute. So, in some sense, the commutator of two vector fields measures the extent to which they cannot be coordinate vector fields for some coordinate system.

A useful property of the commutator, which can easily be checked by the reader, is that

$$[fX, Y] = f[X, Y] - (Y.f)X.$$

24.1.2 Covariant derivative

We use the notation

$$\nabla_V W$$

to denote the directional covariant derivative of W in the V-direction. In our earlier notation,

$$
\begin{aligned}
(\nabla_V W)^i &= W^i_{;j} V^j \\
&= W^i_{,j} V^j + \Gamma^i_{jk} V^j W^k.
\end{aligned}
\tag{24.1}
$$

Several properties of ∇ can be observed from Equation 24.1. Firstly, the symmetry of the Christoffel symbols,

$$\Gamma^i_{jk} = \Gamma^i_{kj},$$

proves the identity

$$\nabla_V W - \nabla_W V = [V, W]. \tag{24.2}$$

Secondly, the reader can check the results of scaling the vector fields by a function:

- $\nabla_V(fW) = f\nabla_V W + (V.f)W$,

- $\nabla_{fV} W = f\nabla_V W$.

Finally, recall the result of Exercise 13.4 which says that the covariant derivatives of the metric tensor g_{ij} are zero. The coordinate-free analogue of this observation is the identity

$$\langle \nabla_X V, W \rangle + \langle Y, \nabla_X Z \rangle = X.\langle Y, Z \rangle. \tag{24.3}$$

24.1.3   Curvature

Definition 24.1. Let X, Y, Z be vector fields. Then we define

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

This is the coordinate-free definition of the Riemann curvature tensor R^i_{jkl} (see Lecture 13). In the case where the vector fields X, Y and Z are coordinate vector fields $\frac{\partial}{\partial x^j}$, $\frac{\partial}{\partial x^k}$ and $\frac{\partial}{\partial x^l}$, respectively, the commutator term vanishes and we find that

$$\left(R\left(\tfrac{\partial}{\partial x^j}, \tfrac{\partial}{\partial x^k} \right) \tfrac{\partial}{\partial x^l} \right)^i = R^i_{jkl},$$

which matches up with our old definition.

We now come to an amazing feature of the Riemann curvature.

Proposition 24.2. *The value of $R(X, Y)Z$ at a point p depends only on the values of the fields X, Y and Z at p.*

To understand how amazing this property is, note that it is not the case for almost any other differential operator. For instance, the derivative of a function f at p certainly depends upon more information than just the value $f(p)$.

Proof. We will show that, for any fields X, Y and Z and any function f,

$$R(fX, Y)Z = fR(X, Y)Z. \tag{24.4}$$

There are also two similar identities, namely,

$$R(X, fY)Z = fR(X, Y)Z, \tag{24.5}$$

and

$$R(X, Y)fZ = fR(X, Y)Z, \tag{24.6}$$

which we leave to the reader.

To prove (24.4), first note that

$$\nabla_{fX} \nabla_Y Z = f\nabla_X \nabla_Y Z. \tag{24.7}$$

Secondly, we have

$$\begin{aligned}
\nabla_Y \nabla_{fX} Z &= \nabla_Y(f\nabla_X Z) \\
&= (Y.f)\nabla_X Z + f\nabla_Y \nabla_X Z, \tag{24.8}
\end{aligned}$$

and thirdly,

$$\begin{aligned}
\nabla_{[fX,Y]} Z &= \nabla_{f[X,Y]} Z - \nabla_{(Y.f)X} Z \\
&= f\nabla_{[X,Y]} Z - (Y.f)\nabla_X Z. \tag{24.9}
\end{aligned}$$

Adding Equations 24.7, 24.8 and 24.9, we get Equation 24.4.

Equations 24.4, 24.5 and 24.6 are enough to prove the Proposition. For consider the case when the field X vanishes at the point p. We can use the basis of coordinate vector fields $X_i = \frac{\partial}{\partial x^i}$ to write the vector field X as

$$X = \sum_{i=1}^{n} f_i X_i,$$

and note that the coefficient functions f_i must all vanish at p. Then we see that

$$\begin{aligned}
R(X,Y)Z &= \sum_{i=1}^{n} R(f_i X_i, Y)Z \\
&= \sum_{i=1}^{n} f_i R(X_i, Y)Z \\
&= 0 \text{ at } p.
\end{aligned}$$

But now, if two vector fields X and X' agree at the point p, then their difference $X - X'$ vanishes at p, so that we get

$$0 = R(X - X', Y, Z) = R(X, Y, Z) - R(X', Y, Z).$$

Similar arguments hold for Y and Z. $\qquad\square$

We can now completely understand the relationship between our previous notion of the Riemann curvature tensor, and our new coordinate-free definition. Specifically, for *any* fields X, Y and Z, $R(X,Y,Z)$ is a vector field with coordinates

$$(R(X,Y)Z)^i = R^i_{jkl} Z^j X^k Y^l. \tag{24.10}$$

This entry in our coordinate-free dictionary is rooted in the result of Proposition 24.2. We observed above that Equation 24.10 holds for the special case of coordinate vector fields. But any vector at a point p can be expressed as a linear combination of the coordinate vector fields at p, and thus Proposition 24.2 extends the equality to arbitrary vector fields.

The proof of Proposition 24.2 justifies the inclusion of the commutator term in Definition 24.1: it is introduced precisely so as to ensure the linearity properties of Equations (24.4)–(24.6), which imply the point-wise definition property of $R(X,Y)Z$.

Lecture 25
(Wednesday, 12 November 2003)

Let's take another look at the formula defining the Riemann curvature tensor, Definition 24.1.

Recall how we first came upon the Riemann curvature tensor. In observations from Lecture 13 we saw that the Riemann curvature tensor measures the degree to which second order mixed covariant derivatives are not symmetric, namely:

$$V^i_{;kl} - V^i_{;lk} = R^i_{jkl} V^j.$$

This formula describes the relationship between mixed derivatives of the coordinate vector fields $\frac{\partial}{\partial x^k}$ and $\frac{\partial}{\partial x^l}$, but what is the difference of mixed derivatives in the direction of general vector fields? That is, in our new coordinate-free notation, what is

$$\nabla_X \nabla_Y - \nabla_Y \nabla_X?$$

Recall that the expression for $\nabla_X V$ in coordinates is

$$(\nabla_X V)^i = V^i_{;k} X^k.$$

So let us consider the fields

$$
\begin{aligned}
\nabla_Y (\nabla_X V)^i &= (V^i_{;k})_{;l} Y^l \\
&= V^i_{;kl} X^k Y^l + V^i_{;k} X^k_{;l} Y^l,
\end{aligned}
$$

and

$$\nabla_X (\nabla_Y V)^i = V^i_{;lk} X^k Y^l + V^i_{;k} Y^k_{;l} X^l.$$

When we take the difference of these two fields, we note that the difference of the first terms,

$$V^i_{;kl} X^k Y^l - V^i_{;kl} X^k Y^l,$$

depends only on the values of the fields X and Y at each point — there is no differentiation of X and Y in these terms. It is this quantity which we want to denote by $R(X, Y)V$.

The difference of the two remaining terms,

$$V^i_{;k} Y^k_{;l} X^l - V^i_{;k} X^k_{;l} Y^l,$$

is equal to

$$V^i_{;k} [X, Y]^k,$$

as the reader can check. This explains the presence of the term

$$\nabla_{[X,Y]} V$$

in Definition 24.1.

25.1 Symmetries of the Riemann curvature tensor

As a tensor quantity with four indices on four-dimensional space-time, the Riemann curvature tensor has, *a priori*, $4^4 = 256$ components. However, it is a tensor with a great deal of symmetry, and as a result actually has only 24 independent entries.

Proposition 25.1. *For any vector fields X, Y, Z and W,*

(i) $R(X,Y)Z + R(Y,X)Z = 0$ *(ie, $R^i_{jkl} + R^i_{jlk} = 0$).*

(ii) $R(X,Y)Z + R(Y,Z)X + R(Z,Y)X = 0$ *(ie, $R^i_{jkl} + R^i_{klj} + R^i_{ljk}$).*

(iii) $\langle R(X,Y)Z, W \rangle + \langle R(X,Y)W, Z \rangle = 0$ *(ie, $R_{ijkl} + R_{jikl} = 0$).*

(iv) $\langle R(X,Y)Z, W \rangle + \langle R(Z,W)X, Y \rangle = 0$ *(ie, $R_{ijkl} = R_{klij}$).*

That the Riemann curvature has so much symmetry should be very surprising. In particular, the reasons for the three-fold nature of symmetry (ii) and the involvement of the metric tensor in (iii) and (iv) are not immediately obvious.

Proof. (i):   Obvious from the definition.
(ii):   Since we have already observed that the value of $R(X,Y)Z$ depends only on the values of X, Y and Z at a point, and since every vector field can be written as a linear combination of the coordinate vector fields, we can, without loss of generality, assume that X, Y and Z are coordinate vector fields, and hence commute[13]. Then,

$$
\begin{aligned}
R(X,Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z \\
R(Y,Z)X &= \nabla_Y \nabla_Z X - \nabla_Z \nabla_Y X \\
R(Z,X)Y &= \nabla_Z \nabla_X Y - \nabla_X \nabla_Z Y
\end{aligned}
$$

Adding these up, we get

$$ \nabla_X(\nabla_Y Z - \nabla_Z Y) + \nabla_Y(\nabla_Z X - \nabla_X Z) + \nabla_Z(\nabla_X Y - \nabla_Y X). $$

But, by Equation 24.2,

$$ \nabla_Y Z - \nabla_Z Y = [Y, Z] = 0, $$

and the other two terms are zero similarly.
     This particular symmetry is referred to as the *First Bianchi Identity.*
(iii):   It will suffice to prove that

$$ \langle R(X,Y)Z, Z \rangle = 0, $$

for any vector field Z, as the reader can check by expanding out the expression

$$ \langle R(X,Y)(W + Z), (W + Z) \rangle. $$

---

[13]Vector fields A and B *commute* if $[A, B] = 0$. In particular, the vector fields corresponding to the coordinate directions of some local coordinate system commute, by the symmetry of ordinary mixed partial derivatives.

Once again, we can assume that $[X, Y] = 0$. Equation 24.3 shows that

$$X.\langle Z, Z \rangle = 2\langle \nabla_X Z, Z \rangle,$$

and hence that,

$$Y.X.\langle Z, Z \rangle = 2\langle \nabla_Y \nabla_X Z, Z \rangle + 2\langle \nabla_X Z, \nabla_Y Z \rangle.$$

Consequently,

$$
\begin{aligned}
2\langle R(X, Y)Z, Z \rangle &= X.Y.\langle Z, Z \rangle - Y.X\langle Z, Z \rangle \\
&= [X, Y].\langle Z, Z \rangle = 0.
\end{aligned}
$$

(iv): The final symmetry is a non-trivial consequence of the first three.

Picture: Octahedral picture of the final identity in terms of the first Bianchi identity and friends.

$\square$

Definition 25.2. The *Ricci Tensor* R_{jl} is the *contraction* of the Riemann curvature tensor R^i_{jkl}, defined by

$$R_{jl} = R^i_{jil}.$$

Corollary 25.3. *The Ricci Tensor is symmetric in j and l.*

Proof. This is obvious, since

$$R^i_{jil} = g^{pq} R_{pjql}.$$

$\square$

What is this Ricci tensor? Note that a tensor A^i_k with one upper and one lower index is a matrix, and that the quantity A^i_i represents the trace of this matrix. Thus, the Ricci curvature is essentially a trace of the Riemann curvature tensor.

Lecture 26

(Friday, 14 November 2003)

Today, at last, we will derive Einstein's equation for gravity. Of course, one cannot mathematically prove a physical law, but we can make it seem plausible by showing that it looks like the law of Newtonian gravity that we already know and love. The way to do this is to look at the tidal forces, which we discussed in Lecture 2.

26.1 Newtonian tidal forces revisited

Imagine again a cluster of particles, freely falling under the influence of the earth's gravity.

<u>Picture</u>: Gravitational forces on five particles and the tidal effect.

If we look at this picture, not from the point of view of the some external observer, stationary with respect to the earth, but from the point of view of one of the falling particles, there will be observed forces acting on the nearby particles. These apparent *tidal forces* will push particles apart along the axis of the gravitational attraction, and pull them inward in the perpendicular directions.

<u>Picture</u>: Same picture in the reference frame of freely falling observer.

In Lecture 2 we noted that the equations which govern Newton's theory of gravity can be expressed by saying that these tidal forces average to zero. For the tidal forces are given by the matrix of second partials

$$\frac{\partial^2 \Phi}{\partial x^i \partial x^j}$$

of the gravitational potential Φ, and the Laplace Equation formulation of Newton's Law of Gravity (Equation 2.2) says that the trace of this matrix is zero:

$$\text{Trace} \frac{\partial^2 \Phi}{\partial x^i \partial x^j} = \nabla^2 \Phi = 0.$$

Let us reformulate this into a slightly different language. Consider a line of particles falling under (Newtonian) gravity. Each particle has some trajectory $(x^k(t))$, and if we parameterize the family of particles by a real variable q, then

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the family of trajectories becomes a function of two variables $x^k(t, q)$. We will look at the time dependence of the vector

$$V^k = \frac{\partial x^k}{\partial q}.$$

This vector represents the infinitesimal displacement between two nearby particles at the same time, and hence our analysis will describe the *relative* motion of the particles as they fall towards the centre of the earth.

The equation of motion for each fixed q is

$$\frac{\partial^2 x^k}{\partial t^2} = -\frac{\partial \Phi}{\partial x^k}, \qquad (26.1)$$

where Φ is the Newtonian potential.

Remark. From the point of view of our Einstein notation conventions, this equation seems odd since the index k appears as an upper index on the left, and a lower index on the right. To reconcile this, there should be a metric tensor involved to facilitate an index lowering, which does not appear explicitly in the Newtonian theory. Its presence will be felt when we get to Einstein's theory.

Applying $\frac{\partial}{\partial q}$ to Equation 26.1, we get

$$\begin{aligned}
\frac{\partial^2}{\partial t^2} V^k &= \frac{\partial^3 x^k}{\partial t^2 \partial q} \\
&= -\frac{\partial^2 \Phi}{\partial x^k \partial x^l} \frac{\partial x^l}{\partial q} \\
&= -M.V,
\end{aligned}$$

where M is the matrix whose entries are

$$\frac{\partial^2 \Phi}{\partial x^k \partial x^l}.$$

Thus the particles' relative motion is described by a particularly simple family of differential equations, involving multiplication by the matrix M which is derived from the gravitational potential.

Of course, we have not introduced any theory of gravity yet — that is done by stating some condition upon the potential Φ. For this, we have the Laplace Equation formulation which, as we noted above, says that M must satisfy

$$\text{Trace} M = 0.$$

26.2 General relativistic counterpart

Again, consider a family of freely falling particles, whose trajectories are again described by a function of two variables:

$$\gamma : \mathbb{R}^2 \to \text{space-time},$$

where $t \mapsto \gamma(t, q)$ is the track of the qth particle.

As in the Newtonian case, we will shortly need to take derivatives of up to third order, but now they will become covariant derivatives. It is worth taking the time to make some mathematical observations about this situation.

The function γ has *two* tangent vectors — one in the direction of t and one in the direction of q, given by $\frac{\partial\gamma}{\partial t}$ and $\frac{\partial\gamma}{\partial q}$, respectively. We will want to take covariant derivatives in the direction of these tangent vectors. For notational convenience, we will write

$$\nabla_t = \nabla_{\frac{\partial\gamma}{\partial t}}$$

and

$$\nabla_q = \nabla_{\frac{\partial\gamma}{\partial q}}.$$

According to the physical assumptions of General Relativity, particles travel along geodesics through space-time. The geodesic equation says

$$\nabla_t\left(\frac{\partial\gamma}{\partial t}\right) = 0.$$

Now, for any vector field V, we have that

$$(\nabla_t V)^i = \frac{\partial V^i}{\partial t} + \Gamma^i_{jk}V^j\frac{\partial x^k}{\partial t},$$

where

$$\gamma = (x^i(t,q))$$

is the coordinate description of γ.

Lemma 26.1.

$$\nabla_t\left(\frac{\partial\gamma}{\partial q}\right) = \nabla_q\left(\frac{\partial\gamma}{\partial t}\right).$$

Proof. The left-hand side, in coordinates, comes to

$$\frac{\partial^2 x^i}{\partial t\,\partial q} + \Gamma^i_{jk}\frac{\partial x^j}{\partial t}\frac{\partial x^k}{\partial q}.$$

This is symmetric in j and k, and interchanging them yields the right-hand side. $\qquad\square$

Remark. The mathematical basis of Lemma 26.1 is that the two tangent vector fields to γ act as coordinate vector fields, and hence their Lie bracket is zero.

As before, we will be interested in the relative motion of the particles, as described by the vector field

$$V = \frac{\partial\gamma}{\partial q},$$

We differentiate the geodesic equation with respect to q, to get

$$\begin{aligned}
0 &= \nabla_q\nabla_t\frac{\partial\gamma}{\partial t} \\
&= R\left(\frac{\partial\gamma}{\partial t},\frac{\partial\gamma}{\partial q}\right)\frac{\partial\gamma}{\partial t} + \nabla_t\nabla_q\frac{\partial\gamma}{\partial t} \\
&= R\left(\frac{\partial\gamma}{\partial t},V\right)\frac{\partial\gamma}{\partial t} + \nabla_t^2 V.
\end{aligned}$$

Again, we have a second order differential equation which describes the relative motion of freely-falling particles, and again, this equation has the form

$$\nabla_t^2 V = -M.V \ .$$

However, in this case the matrix M depends on the velocity $\frac{\partial \gamma}{\partial t}$ of the particle — specifically, M is the matrix

$$M_l^i = R_{jlk}^i \frac{\partial x^j}{\partial t} \frac{\partial x^k}{\partial t}.$$

Now comes Einstein's key manoeuvre. We demand, analogously with the Laplace Equation for Newtonian gravity, that the trace of the matrix M is zero, whatever the velocity vector $\frac{\partial \gamma}{\partial t}$ may be. Writing this down,

$$
\begin{aligned}
\text{Trace} M \quad &= \quad M_i^i \\
&= \quad R_{jk} \frac{\partial x^j}{\partial t} \frac{\partial x^k}{\partial t} \\
&= \quad R_{jk} v^j v^k.
\end{aligned}
$$

This will vanish for all v if and only if every entry of the Ricci curvature tensor is zero. We have thus arrived at Einstein's Law of Gravity.

Einstein's Law of Gravity:
For a region of space-time not containing any matter,

$$R_{jk} = 0. \tag{26.2}$$

Equation 26.2 seems a far cry from Newton's Law of Gravity. However, it is worth remembering that this equation is just a family of second order differential equations (in the components of the metric tensor), which should be thought of as being analogous to the second order equations governing the Newtonian potential. It is an interesting exercise, which we will undertake next lecture, to recover Newton's inverse square law from a low-order approximation of Einstein's Law.

To conclude this lecture, let us make some remarks on solving Einstein's equations. Firstly, the equations have ten unknowns — the ten independent components of the symmetric tensor g_{ij}. They are governed by ten differential equations — the components of the Ricci curvature tensor. This is good news. If there were more equations than unknowns we might have problems with the existence of the universe, and if there were fewer it might not be uniquely determined.

Unfortunately, this is not the whole story. The metric tensor is itself subject to certain constraints, introduced by the necessary compatibility of changes of coordinates (?) and bestowed with the name *gauge transformations*. These use up four of our ten degrees of freedom in choosing the entries g_{ij}. However, it turns out that the ten components of the Ricci curvature tensor are themselves subject to constraints, namely a set of four differential equations relating the entries, called the *Bianchi identities*. In fact, when these four differential equations are applied to a general relativistic system involving mass, we obtain four conservation laws — the conservation of energy and the conservation of the three components of momentum. Einstein's theory of gravity thus has the mechanical conservation laws built into it.

Lecture 27
(Monday, 17 November 2003)

We now have Einstein's equation for gravity:

$$R_{ik} = 0$$

We now want to sketch out some solutions to this which will allow us to make some physical conclusions, beginning with a calculation relating it to the classical theory of gravity.

27.1   The Newtonian approximation

We will derive Newton's theory as an approximation to Einstein's theory. Specifically, we will deduce the equation

$$\frac{d^2 x^i}{dt^2} = -\frac{\partial V}{\partial x^i},$$

where V is a potential satisfying the equation

$$\nabla^2 V = 0.$$

Since this will be an approximation to Einstein's theory, we need to be clear on the assumptions that will make the approximation valid.

- <u>Static coordinates</u>: We assume that the metric is a function of $x^0 = t, x^1, x^2, x^3$, such that the components g_{ij} don't depend on the first coordinate t and such that $g_{i0} = 0$ for $i = 1, 2, 3$. As such, we will write the metric as

$$\begin{pmatrix} \Psi^2 & 0 & 0 & 0 \\ 0 & & \ddots & \\ 0 & & (3 \times 3) & \\ 0 & & & \ddots \end{pmatrix}$$

  The physical meaning of this mathematical assumption is that space and time can be "separated" from one another — the system of coordinates will correspond to the time and space measurements of some observer — and that the field is not changing with time.

- <u>Slow particle</u>: The 4-velocity v^i of a particle in this field has v^1, v^2, v^3 small.

  This will mean that Newtonian mechanics applies.

- <u>Weak field</u>: All the first derivatives $g_{ij,k}$ are small.

  This corresponds to a weak gravitational field, since the field strength is given by the derivative of the gravitational potential.

To be precise about what we mean by "small" in the above assumptions, these "small" quantities are such that the product of any two of them is negligible. In other words, we will be doing expansions to first order in these quantities.

The first thing we look for is the equation of motion. This comes from the geodesic equation:

$$\frac{dv^m}{ds} = -\Gamma_{ij}^m v^i v^j.$$

The only term on the right-hand side of this equation which is not of second order is the term with $i, j = 0$, wherefore we get

$$\frac{dv^m}{ds} = -\Gamma_{00}^m (v^0)^2. \tag{27.1}$$

But

$$
\begin{aligned}
\Gamma_{00}^m &= g^{mn}\Gamma_{00n} \\
&= g^{mn}\frac{1}{2}(-(\Psi)_{,n}^2) \\
&= -g^{mn}\Psi\Psi_{,n}, \tag{27.2}
\end{aligned}
$$

by the static coordinates assumption.

The 4-velocity is a tangent to a unit-speed geodesic, that is,

$$g_{ij}v^i v^j = 1,$$

so to first order, we have that

$$\Psi^2 (v^0)^2 = 1,$$

Furthermore,

$$\frac{dv^m}{ds} = \frac{\partial v^m}{\partial x^\mu}v^\mu = \frac{dv^m}{dt}v^0. \tag{27.3}$$

Combining Equations 27.1 and 27.2, we also have

$$\frac{dv^m}{ds} = g^{mn}\Psi\Psi_{,n}(v^0)^2. \tag{27.4}$$

Comparing the right hand sides of Equations 27.3 and 27.4, some cancellation magically occurs, and we get

$$\frac{dv^m}{dt} = g^{mn}\frac{\partial \Psi}{\partial x^n},$$

or, after lowering the index on the left,

$$\frac{dv_n}{dt} = \frac{\partial \Psi}{\partial x^n}.$$

We are seeing the ignorance of the distinction between vectors and covectors in the Newtonian theory, as observed in the Remark of page 117.

We therefore see that the particle is moving as if under a potential Ψ. We have not used Einstein's field equation yet, only that the field is weak and that

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the particle moves along a geodesic. When we introduce the field equation we will obtain a condition on the potential Ψ which will eventually yield Newton's inverse square law.

Einstein's Law tells us that

$$R_{ik} = 0.$$

This equation has ten components, but for our purposes only one of these equations is interesting, namely

$$R_{00} = 0,$$

where,

$$R_{00} = \frac{\partial \Gamma_{00}^i}{\partial x^i} - \frac{\partial \Gamma_{0i}^i}{\partial x^0} + \Gamma_{00}^p \Gamma_{pi}^i - \Gamma_{0i}^p \Gamma_{p0}^i.$$

The last two terms in this expression are always zero — those summands with $i = 0$ sum to zero by symmetry and the rest are second order terms. Recall that these are the terms which are responsible for the non-linearity of Einstein's equations, so we have drastically simplified the mathematics with the above assumptions.

The second term is also zero, since the metric components do not depend on time. We are left with

$$\frac{\partial \Gamma_{00}^i}{\partial x^i} = 0.$$

This becomes

$$g^{mn} g_{00,mn} = 0,$$

In other words, the quantity $\Psi^2 = g_{00}$ satisfies Laplace's Equation.

This is not quite Newton's Law of Gravity, which states that Ψ satisfies the Laplace Equation. This issue is resolved by the weak field assumption, which allows us to approximate Ψ by its first order Taylor expansion, whereby[14] $\Psi = 1 + \psi$ (for some small ψ) and $\Psi^2 = 1 + 2\psi$, and hence one satisfies the Laplace Equation (to first order) if and only if the other does.

Thus, Newton's Law appears as a special case of Einstein's Law. Einstein did this in his 1916 paper, [Ein3].

27.2 Spherically symmetric static solution

The other important solution to Einstein's Equation is the Schwarzschild solution, which we have already studied from a kinematic point of view. This is one of the few examples of explicit solutions that can be obtained.

In order to motivate this solution, compare the standard solution of the Laplace Equation which yields the spherically symmetric Newtonian field around a gravitating body such as the earth or the sun. Once we assume spherical symmetry, Laplace's equation

$$\nabla^2 \Psi = 0.$$

reduces to an ordinary differential equation in the radial potential $V(r)$, which is then easy to solve.

---

[14] We can normalize the metric so that Ψ is approximately 1.

Analogously, we will look for a spherically symmetric metric satisfying Einstein's equations. As an *ansatz* (*ie*, a template for the solution) we take the form

$$ds^2 = e^{2f(r)}dt^2 - e^{2g(r)}dr^2 - r^2(d\theta^2 + \sin^2 \phi d\phi 2).$$

The last two terms are the ordinary spherically symmetric metric, as one has on the sphere. The functions $e^{2f(r)}$ and $e^{2g(r)}$ are representing arbitrary positive functions of r, but in a form that will make the computations most elegant.

Lecture 28

(Wednesday, 19 November 2003)

So, today we undertake an enormous calculation in order to derive the Schwarzschild solution for a spherically symmetric field in general relativity. But before we do so, let us recall a calculation in Newtonian theory: What is the Newtonian gravitational field which is generated by a spherically symmetric body?

Common sense suggests that the gravitational potential resulting from a spherically symmetric distribution of mass will be a spherically symmetric function. So we look for a solution to the Laplace Equation,

$$\nabla^2 \Phi = 0,$$

which is spherically symmetric, *ie*,

$$\Phi = \Phi(r).$$

First, we need to write down the Laplacian in spherical coordinates. We have

$$\frac{\partial \Phi}{\partial x} = \Phi'(r)\frac{\partial r}{\partial x} = \Phi\frac{x}{r},$$

so

$$\frac{\partial^2 \Phi}{\partial x^2} = \frac{1}{r}\Phi'(r) + \Phi''(r)\left(\frac{x}{r}\right)^2 + \Phi'(r)\left(-\frac{1}{r^2}\right)\frac{x^2}{r}.$$

Summing the similar equations for $\frac{\partial^2 \Phi}{\partial y^2}$ and $\frac{\partial^2 \Phi}{\partial z^2}$, we get

$$\nabla^2 \Phi = \frac{2}{r}\Phi'(r) + \Phi''(r) = 0.$$

This gives us an ordinary differential equation for $\Phi(r)$, and what is more it is an ordinary differential equation that is quite easy to solve. Rearranging the above,

$$0 = 2\Phi'(r) + r\Phi''(r) = \frac{1}{r}\frac{d}{dr}(r^2\Phi'(r)).$$

Therefore,

$$r^2\Phi'(r) = m,$$

for some constant m, and hence

$$\Phi(r) = -\frac{m}{r} + C.$$

We fix the constant of integration C by declaring that Φ tends to zero at infinity, giving

$$\Phi(r) = -\frac{m}{r}$$

We were able to find this solution by assuming the form that the solution would take, and hence forcing its spherical symmetry. We do the same thing with Einstein's equations, by assuming the solution to be of the form

$$
\begin{aligned}
g_{00} &= e^{2f(r)} \\
g_{11} &= -e^{2g(r)} \\
g_{22} &= -r^2 \\
g_{33} &= -r^2 \sin\theta,
\end{aligned}
$$

with all other entries are zero. The coordinates are

$$
(x^0, x^1, x^2, x^3) = (t, r, \theta, \phi).
$$

We have to work out the field equations, for which we need to compute the Ricci curvature tensor, for which we need to compute the Riemann curvature tensor, for which we need to compute the Christoffel symbols. It should be clear why the physicists working on general relativity were early proponents of computer algebra.

A straight-forward but lengthy calculation eventually produces

$$
\begin{aligned}
R_{00} &= \left(-f''(r) + f'(r)g''(r) - f'(r)^2 - \frac{2f'(r)}{r}\right)e^{2f-2g} = 0 \quad (28.1) \\
R_{11} &= f''(r) - f'(r)g'(r) + f'(r)^2 - \frac{2g'(r)}{r} = 0 \quad (28.2) \\
R_{22} &= (1 + rf'(r) - rg'(r))e^{-2g(r)} - 1 = 0 \quad (28.3) \\
R_{33} &= \left((1 + rf'(r) + rg'(r))e^{-2g(r)} - 1\right)\sin^2\theta. \quad (28.4)
\end{aligned}
$$

Adding Equations 28.1 and 28.2, we get

$$
f'(r) + g'(r) = 0,
$$

so that

$$
f(r) + g(r) = \text{constant}.
$$

We now make a physical assumption: that space-time is asymptotically flat. In other words, we desire that the curvature of space-time tends to zero at infinity, meaning that the effect of the gravitating body is not felt a great distances. Mathematically, this amounts to

$$
f, g \to 0 \quad \text{as } r \to \infty.
$$

Thus $f(r) + g(r) = 0$. Substituting this into Equation 28.3, we get

$$
(1 + r2f'(r))e^{2f(r)} = 1.
$$

This is an ordinary differential equation for f which we can integrate. We write it as

$$
\frac{d}{dr}(re^{2f(r)}) = 1,
$$

so

$$
re^{2f(r)} = r - 2m,
$$

where $2m$ is just an appropriately named constant of integration. Therefore

$$
e^{2f(r)} = (1 - \frac{2m}{r}),
$$

and hence
$$e^{2g(r)} = (1 - \frac{2m}{r})^{-1}.$$

We still need to check that this solution satisfies the Equations 28.1 and 28.2 (which are now equivalent). We leave this exercise to the reader. We have indeed produced a solution to Einstein's equations.

Lecture 29

(Monday, 12 November 2003)

29.1   The Bianchi Identity

The Bianchi Identity, as is so often the case, was not discovered by Bianchi, but by Ricci. It was published by one of Ricci's students in 1897, then forgotten and published again by another of his students some years later. It is of great importance to the mathematics of Einstein's gravity, although Einstein failed to understand it, as did everyone else, until about 1922, eight years after the general theory of relativity was produced.

The Bianchi identity is a differential symmetry of the curvature tensor. We have already proven several symmetries of the Riemann curvature tensor (Proposition 25.1), but there are more. The symmetries previously noted were all point-wise identities — they depended only on the values of the curvature tensor at a point. The Bianchi identity, which we are about to prove, is a differential symmetry.

In coordinates, the Bianchi identity reads as follows:

$$R^i_{jkl;m} + R^i_{jlm;k} + R^i_{jmk;l} = 0. \tag{29.1}$$

This identity eventually leads to the correct formulation of the Einstein equations in the case where there is matter involved. In fact, this will also yield conservation of energy and momentum as consequences.

[..revisiting dimension counting arguments of a previous lecture, more precisely..]

Einstein's equations of gravity say that

$$R_{ik} = 0.$$

As we have said, these are ten equations for the ten unknowns g_{ij}. By gormless dimension counting, we would expect that this straight-away gives us our well-posed system to be solved. But closer thought reveals that this argument is flawed. The equations are generally covariant — *ie*, geometric in nature — but the unknowns depend on choice of coordinates. Thus, given any solution, we should be able to produce a whole slew of other solutions just by changing coordinates. These symmetries in the solution space correspond to what are called *gauge transformations*.

What does this mean for our system of equations? It means that some of our ten equations must be fakes — there must be some relations between them. Specifically, there must be four relations between the components of the Ricci tensor. These are the Bianchi identities.

Although it is possible to prove these in index notation, we will prove them in coordinate-free notation, which makes for a very elegant presentation. But first, let us revisit commutators.

Given two operators A and B, we can form their *commutator*, which is defined by

$$[A, B] = AB - BA.$$

Whenever we define any such commutator of operators, it will satisfy the *Jacobi identity.*

Proposition 29.1 (Jacobi Identity). *For any A, B and C,*

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0.$$

Proof. One simply expands the identity out to twelve terms, and realizes that they cancel in pairs. □

We are now in a position to prove the Bianchi Identity. Notice that

$$R(X, Y) = [\nabla_X Y, \nabla_Y X] - \nabla_{[X,Y]}.$$

We want to show that

$$(\nabla_X R)(Y, Z) + (\nabla_Y R)(Z, X) + (\nabla_Z R)(X, Y) = 0. \qquad (29.2)$$

This is precisely Equation 29.1, written in terms of general vector fields instead of using indices, since

$$((\nabla_X R)(Y, Z)W)^i = R^i_{jkl;m} X^m Y^j Z^k W^l.$$

Now note that

$$\nabla_X(R(Y, Z)) = (\nabla_X R)(Y, Z) + R(\nabla_X Y, Z) + R(Y, \nabla_X Z) + R(Y, Z)\nabla_X.$$

This is just a complicated version of the product rule, which the reader can take as an exercise. We therefore get

$$
\begin{aligned}
(\nabla_X R)(Y, Z) &= [\nabla_X, R(Y, Z)] - R(\nabla_X Y, Z) - R(Y, \nabla_X Z) \\
&= [\nabla_X, [\nabla_Y, \nabla_Z]] - [\nabla_X, \nabla_{[Y,Z]}] + R(Z, \nabla_X Y) - R(Y, \nabla_X Z) \\
&= [\nabla_X, [\nabla_Y, \nabla_Z]] + \nabla_{[X,[Y,Z]]} - R(X, [Y, Z]) \\
&\quad + R(Z, \nabla_X Y) - R(Y, \nabla_X Z).
\end{aligned}
$$

Now we await a miracle. We sum the three cyclic permutations of this equation, and we get a bunch of cancellations from the Jacobi Identity, leaving us with

$$
\begin{aligned}
\nabla_X R(Y, Z) &+ \nabla_Y R(Z, X) + \nabla_Z R(X, Y) \\
&= -R(X, [Y, Z]) + R(Z, \nabla_X Y) - R(Y, \nabla_X Z) \\
&\quad -R(Y, [Z, X]) + R(X, \nabla_Y Z) - R(Z, \nabla_Y X) \\
&\quad -R(Z, [X, Y]) + R(Y, \nabla_Z X) - R(X, \nabla_Z Y).
\end{aligned}
$$

But, recall that

$$\nabla_Y Z - \nabla_Z Y = [Y, Z],$$

This causes three terms in the right-hand side of the above equation to cancel. The remaining six terms will cancel by cyclic permutations of this observation. Thus,

$$\nabla_X R(Y, Z) + \nabla_Y R(Z, X) + \nabla_Z R(X, Y) = 0,$$

which proves the Bianchi identity.

While we are at it, we can prove the "contracted Bianchi identity". Firstly, we can contract Equation 29.1 in the indices i and m to get

$$R^i_{jkl;i} + R^i_{jli;k} + R^i_{jik;l} = 0.$$

Now, using the fact that the metric tensor is covariant constant (*ie*, $g_{jk;l} = 0$), we can write

$$(g^{jk} R^i_{jkl})_{;i} + (g^{jk} R^i_{jik})_{;l} + (g^{jk} R^i_{jli})_{;k} = 0. \tag{29.3}$$

The contraction of the Ricci curvature,

$$g^{jk} R_{jk},$$

is called the *scalar curvature* and denoted by R (without indices). This appears (differentiated) as the second term in Equation 29.3. The first term can be rewritten by deploying some index raising and lowering tricks:

$$
\begin{aligned}
g^{jk} R^i_{jkl} &= g^{jk} g^{in} R_{njkl} \\
&= -g^{in} g^{kj} R_{jnkl} \\
&= -g^{in} R^k_{nkl} \\
&= -g^{in} R_{nl} \\
&= -R^i_l.
\end{aligned}
$$

The last term can be rewritten similarly (exercise for the reader). The result of these manipulations is that the Bianchi identity, when contracted, results in the following identity:

$$2R^i_{l;i} - R_{;l} = 0.$$

After raising the index l and dividing by two, we get

$$(R^{il} - \frac{1}{2} g^{il} R)_{;i} = 0. \tag{29.4}$$

Definition 29.2. We denote the tensor

$$(R^{il} - \frac{1}{2} g^{il} R)$$

by G^{il}, and call it the *Einstein tensor*.

The Einstein equations as we phrased them previously state that

$$R_{il} = 0.$$

But it turns out that it is equivalent to write this as

$$G^{il} = 0, \tag{29.5}$$

since it is possible to prove that one implies the other. However, it is more elegant to write Einstein's law in the form of Equation 29.5, since then Equation 29.4 exactly describes the four additional conditions which the solution must satisfy.

Lecture 30

(Monday, 1 December 2003)

Last time we produced Einstein's field equations for free space,

$$G^{ij} = 0.$$

Today we discuss what ought to be the right-hand side of Einstein's field equations in the presence of matter.

By analogy, we are asking what corresponds to the density of matter ρ in the Poisson equation,

$$\nabla^2 \Phi = -4\pi\rho.$$

Whatever it is, it will be more complicated than just ρ. This is because the density of matter is not a relativistically invariant quantity. Note that there are *two* reasons why observers in relative motion will disagree about the value of ρ at a point. Firstly, the Fitzgerald contraction will cause relatively moving observers to disagree on lengths, hence volumes, and thus the amount of matter in a given volume. Secondly, there will also be a relativistic disagreement on the mass of the matter in that volume.

To quantify this, suppose, in the realm of special relativity, that an observer is moving with speed v relative to some matter. He observes lengths decreased by a factor of $\sqrt{1-v^2}$ and mass increased by $\frac{1}{\sqrt{1-v^2}}$. The net effect on his observation of the density is

$$\rho_v = \rho_0 \frac{1}{1-v^2},$$

where ρ_0 is the density of the matter in a frame at rest with it.

What, then, is the correct thing to substitute for ρ in Einstein's equation?

To answer this, we will model matter as a swarm of identical particles, each of rest-mass μ. There are two measured quantities that will be relevant to the swarm. Firstly, there is the velocity of a particle at a point in space, which we denote by $\mathbf{v}$. This is not a relativistic invariant, so we will also need to appeal to the 4-velocity

$$\mathbf{V} = \frac{1}{1-v^2}(1, \mathbf{v})$$

of a particle (where $v = |\mathbf{v}|$). Secondly, there is the number density σ, defined as the *number* of particles per unit volume, which is a scalar quantity, and again not relativitistically invariant.

Lecture 31

(Wednesday, 3 December 2003)

Let μ denote the rest mass of the particles from last lecture.

Lemma 31.1. *The quantity*

$$\Sigma = \mu\sigma(1, \mathbf{v}),$$

is a 4-vector, ie, it is a quantity that transforms under a Lorentz transformation according to the transformation law for 4-vectors, and hence is a relativistic invariant.

Proof. Since

$$\mathbf{V} = \frac{1}{\sqrt{1 - v^2}}(1, \mathbf{v})$$

is a 4-vector, we only need to prove that, under a Lorentz transformation, σ changes under by the factor $\frac{1}{\sqrt{1-v^2}}$. But we have made this observation already. $\qquad\square$

Indeed, this shows that

$$
\begin{aligned}
\Sigma &= \mu(\sigma\sqrt{1 - v^2})\mathbf{V} \\
&= \mu\varrho\mathbf{V},
\end{aligned}
$$

where $\varrho(x)$ is the density of particles (in number per volume) as measured in the local rest frame of the particle at a point x.

In particular,

$$\Sigma^0 = \frac{\mu}{\sqrt{1 - v^2}}\varrho,$$

and we call this the "local density of mass/energy". Note that this quantity is a component of a tensor, not a tensor in its own right. It transforms like the density σ. Similarly,

$$\Sigma^1 = \frac{\mu}{\sqrt{1 - v^2}}\varrho v_1,$$

and it makes sense to call this the "local density of the 1-component of momentum."

In general, Σ represents the 4-flux of mass/energy. It describes the mass/energy-flux, not through three-dimensional space, but through four-dimensional space-time. One could compute the flux of the matter across a three-dimensional boundary in space-time by taking the inner product of this quantity with the normal vector to the boundary. Thus, the flux across a completely space-like boundary will represent the density of matter, while the flux across a boundary having two space-dimensions and one time-dimension represents a mass-flux across the two-dimensional surface in space.

<u>Picture</u>: density of world-lines of swarm of particles in low-dimensional space-time diagram: flux across boundaries in space alone = mass-density of particles, flux across boundaries in 2-space and 1-time dimension = mass-flux across 2-dimensional surface..

Definition 31.2. The *energy-momentum* tensor is the symmetric tensor defined by
$$T^{ij} = \mu \mathbf{V}^i \Sigma^j = \mu \varrho \mathbf{V}^i \mathbf{V}^j.$$

Remark. If there are different types of (non-interacting) particles, indexed by τ, then we need to augment this definition by putting

$$T^{ij} = \sum_\tau \mu_\tau \varrho_\tau \mathbf{V}^i_\tau \mathbf{V}^j_\tau.$$

Furthermore, if the particles interact (for instance, by electro-magnetics) we get additional stress-energy tensors from the appropriate physical theory.

The energy-momentum tensor has the following physical interpretation. For an observer with velocity W^i, the measured density of energy of the matter will be given by[15]
$$T_{ij} W^i W^j.$$

Why is this so? By construction it is true for an observer whose 4-velocity is $\mathbf{W} = (1, 0, 0, 0)$, since in that case,

$$T_{ij} W^i W^j = T^{00} = \frac{1}{1 - v^2} \mu \varrho,$$

which we saw at the start of the previous lecture is the measured mass-density. But the quantity $T_{ij} W^i W^j$ is tensorial, and hence the statement must hold true in any reference frame.

31.0.1 Conservation laws

In special relativity, each component of 4-momentum is conserved. The divergence theorem shows that, to demonstrate the conservation of a scalar quantity T, it is equivalent to demonstrate the vanishing of the (4-)divergence of the flux of that quantity. Now, the energy-momentum tensor T^{ij} has the property that, for fixed i, T^{ij} is the *flux* of the ith component of 4-momentum. Thus, if we apply the divergence theorem argument to each component of the 4-momentum, we see that conservation of momentum is equivalent to the statement that

$$T^{ij}_{,j} = 0.$$

---

[15]Note that, since we are working with *special* relativity here, the metric tensor is just
$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, and the difference between raised and lowered indices is easily blurred.

In general relativity, this is replaced by the generally covariant expression

$$T^{ij}_{;j} = 0. \tag{31.1}$$

In general, we assume that this is true for the "total physical energy-momentum tensor", which is the energy-momentum tensor which includes all relevant physical theories, including electromagnetism and so on. The various physical processes may give up energy-momentum to one another.

What does Equation 31.1 mean in the case of our simple model above? In this case,

$$T^{ij}_{;j} = \mu\varrho V^i_{;j}V^j + \mu V^i \Sigma^j_{;j}, \tag{31.2}$$

and we are equating this to zero. Note that saying the second term on the right-hand side vanishes is equivalent to saying that the number of particles is constant — particles are neither created nor destroyed. The first term is the coordinate-dependent expression for $\nabla_V V$, so to say that this vanishes is to say that the particles' paths satisfy the geodesic equation. Therefore, these two physical assumptions combined imply the conservation of energy-momentum, at least in our particular[16] model of matter.

In fact, the converse is also true. If we assume the conservation of energy-momentum as phrased by Equation 31.1, then

$$0 = g_{ik}V^k T^{ij}_{;j} = 0 + \Sigma^j_{;j},$$

since

$$g_{ik}V^k V^i_{;j} = \frac{1}{2}(g_{ik}V^i V^k)_{;j} = 0.$$

We see that the second term of Equation 31.2 vanishes, and hence the first term as well.

The energy-momentum tensor will be the relativistic counterpart of the density ρ from Newtonian theory. Since the left-hand side of Einstein's equation, G^{ij}, satisfies the Bianchi Identity (Equation 29.4), is will follow automatically that T^{ij} must satisfy Equation 31.1. It is therefore a direct consequence of Einstein's field equations that particles travel along geodesics. This gives the theory of general relativity a surprising self-contained nature — that both the dynamics and the kinematics are contained in the one equation. Compare this with Newton's theory in which two independent equations are required: one describing the gravitational field, and one describing the behaviour of particles under its influence.

31.0.2 Einstein's field equation

What then is Einstein's field equation? The equation will be,

$$(R^{ij} - \frac{1}{2}g^{ij}R) = G^{ij} = aT^{ij}, \tag{31.3}$$

where a is a constant which can be determined by comparison with the Newtonian approximation of Section 27.1.

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[16]This is a convenient pun.

What is a? In the Newtonian approximation, we have $\rho = \mu\varrho$, the actual mass-density of matter. Equation 31.3 becomes

$$(R^{ij} - \frac{1}{2}g^{ij}R) = a\rho V^i V^j.$$

Contracting this with g_{ij} gives,

$$-R = a\rho.$$

Considering a static distribution of matter (with $\mathbf{V} = (1,0,0,0)$) in a weak field, we get

$$R^{00} + \frac{1}{2}a\rho = a\rho.$$

So,

$$R^{00} = \frac{1}{2}a\rho.$$

Note that we haven't used the weak field assumption yet, only that the matter is static and that g^{00} is close to one.

In our previous analysis of the Newtonian approximation, we saw that particles move as though under a Newtonian potential Φ, where

$$R^{00} = -\nabla^2\Phi.$$

So we get

$$\nabla^2\Phi = -\frac{1}{2}a\rho.$$

For consistency with the Poisson equation, $a = -8\pi$. Einstein's field equation is thus,

$$(R^{ij} - \frac{1}{2}g^{ij}R) = G^{ij} = -8\pi T^{ij}.$$

As with any physical derivation, we haven't proven Einstein's field equation, just shown that it is plausible. It was Einstein who made the imaginative leap that this is actually the governing equation. History seems to have shown that he was right.

Lecture 32

(Friday, 5 December 2003)

We will now discuss the creation of the universe. We plan to consider the absolutely simplest possible cosmological model. We will then apply Einstein's field equation to the entire universe.

Let us specify our assumptions.

(i)   All the matter in the universe is in the form of a "dust" of galaxies. It has density ρ and velocity field $\mathbf{V}$.

This assumption introduces some privileged observers in the universe, namely those moving with the galaxy at some point of the universe. These observers will each have a notion of time, which we will relate by means of the next assumption.

(ii)   There is a scalar function t, called *cosmic time*, such that

$$\mathbf{V} = \operatorname{grad} t.$$

Thus the universe can be sliced into space-like sections, given by $t = $ constant.

(iii)   The space-like slices $t = $ constant are homogeneous and isotropic.

This assumption is fairly well supported by experiment. It states the (historically counterintuitive) principle that there is nothing special about us — that the universe is more or less the same all over.

Mathematicians have studied manifolds extensively, and as a result of their hard work we can write down a list of all possible homogeneous isotropic three-dimensional manifolds. To start with, we will assume that the space-like slices look like the simplest possible example of these, the Euclidean space $\mathbb{R}^3$.

Under these assumptions, the metric has to be of the form

$$dt^2 - \phi(t)^2 (dx^2 + dy^2 + dz^2).$$

The function $\phi(t)$ is a measure of the expansion of the universe. For consider the two galaxies in our universe at times t_1 and t_2. The distance between them will differ only by the scaling factor $\phi(t)$.

We now appeal to Einstein's equation

$$G^{ij} = -8\pi \rho V^i V^j.$$

Conservation of energy-momentum (or the Bianchi Identity) gives $G^{ij}_{;j} = 0$. Computation shows that this implies

$$\rho \phi(t)^3 = M,$$

for some constant M.

Einstein's equation gives ten equations, of which only two are essentially distinct: one for $i = j = 0$, and one for $i = j =$ anything else. After computing the Christoffel symbols and so forth, these become

$$3\phi(t)^{-1}\phi''(t) = -4\pi\rho,$$

and
$$\phi(t)^{-1}\phi''(t) + 2\phi(t)^{-2}\phi'(t)^2 = 4\pi\rho.$$

We can eliminate the second derivative of ϕ to get
$$6\phi(t)^{-2}\phi'(t)^2 = 16\pi\rho.$$

or
$$\phi'(t)^2 = \frac{8}{3}\pi\rho\phi(t)^2 = \frac{8}{3}M\pi/\phi(t).$$

This is an ordinary differential equation, known as the Friedman Equation.

Rewriting this,
$$\phi(t)^{\frac{1}{2}}\phi'(t) = \sqrt{\frac{8}{3}\pi M}.$$

The left hand side is
$$\frac{d}{dt}\left(\frac{2}{3}\phi(t)^{\frac{3}{2}}\right),$$

so
$$\phi(t) = \left(\sqrt{6\pi M}(t - t_0)\right)^{\frac{2}{3}}.$$

Picture: graph of $\phi(t)$ against t showing singular behaviour at time $t = t_0$ and expansion from then on.

What this simple-minded model suggests, then, is that there was some specific time t_0 in the past at which the universe was all one point, and that the universe suddenly sparked into life and has been expanding ever since.

We want to see some of the qualitative features of this model. Consider the path of light rays in this universe. Without loss of generality, consider a light-ray which passes through the spatial origin, and travels only in the (x, t)-plane. Our experience tells us to look for conserved quantities. The first conserved quantity is the constant (zero) speed of the geodesic path:

$$\left(\frac{dt}{d\lambda}\right)^2 - \phi(t)^2\left(\frac{dx}{d\lambda}\right)^2 = 0.$$

Since we are in two dimensions, we only need one more conserved quantity, which comes from the fact that x is a cyclic coordinate:

$$\phi(t)^2\frac{dx}{d\lambda} = k,$$

for some constant k.

Eliminate $\frac{dx}{d\lambda}$ to get
$$\frac{dt}{d\lambda} = \frac{k}{\phi(t)}.$$

This equation describes a red-shift. Light from the past (say, time t'), or equivalently light from distant galaxies, will be red-shifted when it arrives to us at a time t by a factor of $\frac{\phi(t')}{\phi(t)}$.

This phenomenon was confirmed by Hubble. One can produce estimates of the rate of the expansion of the universe by estimating the linearized quantity

$$H = \frac{\phi'(t)}{\phi(t)}.$$

This is called the *Hubble constant*. Furthermore, the quantity $1/H$ will give an upper bound on the age of the universe.

Einstein didn't like the idea that the universe had a birth at some finite time in the past, and so he did what all good physicists would do — he introduced a fudge factor. This he called the *cosmological constant κ*, and he adjusted his field equation to read

$$G^{ij} + \kappa g^{ij} = -8\pi T^{ij}.$$

If he had not introduced this fudge factor, he would most surely have predicted the red-shift later discovered by Hubble, which may have been one of the most amazing predictions of the general theory of relativity. Einstein later claimed that the cosmological constant was the greatest mistake of his life.

Of course, like so many ideas, the cosmological constant has been more recently re-invigorated. It is now known as "dark matter" — an as yet undetected source of gravitational energy which may be responsible driving the geometry of the universe.

In our model above, the universe begins but it doesn't end. But we made an assumption that the universe is Euclidean. There are other options for homogeneous, isotropic three-dimensional manifolds, most notably the three sphere and three dimensional hyperbolic space. One can therefore reenact the above stunt with any one of those three. The qualitative behaviours differ notably.

If we draw the corresponding picture for a spherical universe, we get

<u>Picture</u>: graph of $\phi(t)$ for spherical universe, with finite end time.

In that case, the present expansion of the universe will eventually slow and then reverse, leading eventually to the death of the universe at a singularity at some finite time in the future. For hyperbolic space, the picture is

<u>Picture</u>: graph of $\phi(t)$ for a hyperbolic universe.

giving an even more rapid growth than the Euclidean case.

How do we know which universe we live in? If we perform a Taylor expansion for in $\phi(t)$ for our model, the first order term is the Hubble constant. This is the same in all three models. The second order term will be the first to give information about the geometry. Unfortunately, to measure the second order term, we need to push our observations out to such enormous distances that there is plenty of room for argument, both in terms of experimental measurement and underlying principles. So danged if I know!

<u>THE END</u>

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