Orthogonal Curvilinear Coordinates

Often it is convenient to use coordinate systems other than rectangular ones. You are familiar, for example, with polar coordinates \((r, \theta)\) in the plane. One can transform from polar to regular (Cartesian) coordinates by way of the transformation equations

\[
x = r \cos \theta, \quad y = r \sin \theta
\]

One can summarize these as \(p = r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j}\), where \(p\) is the position vector of the point \(P = (x, y)\).

Notice that the \(r\) coordinate curves and the \(\theta\) coordinate curves are orthogonal (at right angles) to one another. This convenient property may be expressed by saying that the vector partial derivatives \(\partial p/\partial r\) and \(\partial p/\partial \theta\) are orthogonal, i.e., have zero dot product.

The unit vectors \(e_r\) and \(e_\theta\) in the directions of \(\partial p/\partial r\) and \(\partial p/\partial \theta\) form the moving frame associated to the polar coordinate system. Vector-valued functions may be expressed in terms of this moving frame.
By analogy, we define an orthogonal curvilinear coordinate system in three dimensions to be a ‘non-degenerate’ system of transformation equations

\[ x = x(u, v, w), \quad y = y(u, v, w), \quad z = z(u, v, w) \]

or simply \( p = p(u, v, w) \), where the partial derivatives \( \partial p/\partial u \), \( \partial p/\partial v \), and \( \partial p/\partial w \) should be mutually orthogonal at each point.

**Example 1** Cylindrical Coordinates \((r, \theta, z)\) (book page 842)

\[ x = r \cos \theta, \quad y = r \sin \theta, \quad z = z. \]

**Example 2** Spherical Coordinates \((\rho, \phi, \theta)\) (book page 844)

\[ x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi. \]
Moving frame and scale factors

We put

\[ \frac{\partial \mathbf{p}}{\partial u} = h_u \mathbf{e}_u, \quad \frac{\partial \mathbf{p}}{\partial v} = h_v \mathbf{e}_v, \quad \frac{\partial \mathbf{p}}{\partial w} = h_w \mathbf{e}_w \]

where the \( \mathbf{e}'s \) are (orthogonal) unit vectors — the *moving frame* associated to the coordinate system — and the \( h \)'s are *scale factors* given by

\[ h_u = \left| \frac{\partial \mathbf{p}}{\partial u} \right|, \quad h_v = \left| \frac{\partial \mathbf{p}}{\partial v} \right|, \quad h_w = \left| \frac{\partial \mathbf{p}}{\partial w} \right|. \]

For example consider cylindrical coordinates \((r, \theta, z)\). From the transformation equations,

\[ \mathbf{e}_r = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}, \quad h_r = 1 \]

\[ \mathbf{e}_\theta = -\sin \theta \mathbf{i} + \cos \theta \mathbf{j}, \quad h_\theta = r \]

\[ \mathbf{e}_z = \mathbf{k}, \quad h_z = 1. \]
Calculations in spherical coordinates

For spherical coordinates

\[ x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi. \]

Calculation gives

\[ \mathbf{e}_\rho = \sin \phi \cos \theta \mathbf{i} + \sin \phi \sin \theta \mathbf{j} + \cos \phi \mathbf{k}, \quad h_\rho = 1, \]
\[ \mathbf{e}_\phi = \cos \phi \cos \theta \mathbf{i} + \cos \phi \sin \theta \mathbf{j} - \sin \phi \mathbf{k}, \quad h_\phi = \rho, \]
\[ \mathbf{e}_\theta = -\sin \theta \mathbf{i} + \cos \theta \mathbf{j}, \quad h_\theta = \rho \sin \phi. \]
The gradient in curvilinear coordinates

**Theorem** Let $f$ be a function. In orthogonal curvilinear coordinates $(u, v, w)$ the gradient of $f$ can be expressed by

$$\nabla f = \frac{1}{h_u} \frac{\partial f}{\partial u} e_u + \frac{1}{h_v} \frac{\partial f}{\partial v} e_v + \frac{1}{h_w} \frac{\partial f}{\partial w} e_w.$$

**Proof** Look at the chain rule to see

$$\frac{\partial f}{\partial u} = \nabla f \cdot \frac{\partial p}{\partial u} = h_u \nabla f \cdot e_u$$

with similar calculations for $v$ and $w$.

See Exercises 11,12 on book p.999
The Laplacian in curvilinear coordinates

The *Laplacian* of $f$ is the expression

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

that appears in Laplace’s equation. In curvilinear coordinates this becomes

$$\nabla^2 f = \frac{1}{h_u h_v h_w} \left[ \frac{\partial}{\partial u} \left( \frac{h_v h_w}{h_u} \frac{\partial f}{\partial u} \right) + \frac{\partial}{\partial v} \left( \frac{h_w h_u}{h_v} \frac{\partial f}{\partial v} \right) + \frac{\partial}{\partial w} \left( \frac{h_u h_v}{h_w} \frac{\partial f}{\partial w} \right) \right]$$

The proof will be easier once we have studied the Divergence Theorem.