Lecture 4: Trees

Let $A$ be a set. The set of all finite sequences over $A$ is denoted by $A^{<N}$.

**Definition 4.1:** A tree on $A$ is a set $T \subseteq A^{<N}$ that is **closed under prefixes**, that is

$$\forall \sigma, \tau [\tau \in T \land \sigma \subseteq \tau \Rightarrow \sigma \in T]$$

We call the elements of $T$ nodes.

A sequence $\alpha \in A^N$ is an **infinite path through** or **infinite branch of** $T$ if for all $n$, $\alpha\upharpoonright n = (\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) \in T$. We denote the set of infinite paths through $T$ by $[T]$.

An important criterion for a tree to have infinite paths is the following.

**Theorem 4.2** (König's Lemma): If $T$ is finite branching, i.e. each node has at most finitely many immediate extensions, then

$$T \text{ infinite } \Rightarrow \text{ } T \text{ has an infinite path}.$$  

*Proof sketch.* We construct an infinite path inductively. Let $T_\sigma$ denote the tree “above” $\sigma$, i.e. $T_\sigma = \{\tau \in A^{<N} : \sigma \upharpoonright \tau \in T\}$. If $T$ is finite branching, by the pigeonhole principle, at least one of the sets $T_\sigma$ for $|\sigma| = 1$ must be infinite. Pick such a $\sigma$ and let $\alpha\upharpoonright_1 = \sigma$. Repeat the argument for $T = T_\sigma$ and continue inductively. This yields a sequence $\alpha \in [T]$. □

If $[T] = \emptyset$, we call $T$ **well-founded**. The motivation behind this is that $T$ is well-founded if and only if the inverse prefix relation

$$\sigma \preceq \tau :\iff \sigma \succeq \tau$$

is well-founded, i.e. it does not have an infinite descending chain.

If $T \neq \emptyset$ is well-founded, we can assign $T$ an ordinal number, its **rank** $\rho(T)$.

- If $\sigma$ is a terminal node, i.e. $\sigma$ has no extensions in $T$, then let $\rho_T(\sigma) = 0$.
- If $\sigma$ is not terminal, and $\rho_T(\tau)$ has been defined for all $\tau \supseteq \sigma$, we set $\rho_T(\sigma) = \sup\{\rho_T(\tau) + 1 : \tau \in T, \tau \supseteq \sigma\}$.
- Finally, set $\rho(T) = \sup\{\rho_T(\sigma) + 1 : \sigma \in T\} = \rho_T(\emptyset) + 1$, where $\emptyset$ denotes the empty string.
Orderings on trees

If $A$ itself is linearly ordered, we can extend the inverse prefix ordering to a total ordering on $A^\subseteq\mathbb{N}$. So suppose $\leq$ is a linear ordering of $A$. The (partial) lexicographical ordering $\leq_{\text{lex}}$ of $A^\subseteq\mathbb{N}$ is defined as

$$\sigma \leq_{\text{lex}} \tau \iff \sigma = \tau \text{ or } \exists i < \min\{|\sigma|,|\tau|\}, \left[(\forall j < i)\sigma_j = \tau_j \& \sigma_i < \tau_i\right]$$

This ordering extends to $A^\mathbb{N}$ in a natural way.

**Proposition 4.3:** If $\leq$ is a well-ordering of $A$ and $T$ is a tree on $A$ with $[T] \neq \emptyset$, then $[T]$ has a $\leq_{\text{lex}}$-minimal element, the leftmost branch.

**Proof.** We prune the tree $T$ by deleting any node that is not on an infinite branch. This yields a subtree $T' \subseteq T$ with $[T'] = [T]$. Let $T'_n = \{\sigma \in T': |\sigma| = n\}$. Since $\leq$ is a well-ordering on $A$, $T'_1$ must have a $\leq_{\text{lex}}$-least element. Denote it by $\alpha|_1$. Since $T'$ is pruned, $\alpha|_1$ must have an extension in $T$, and we can repeat the argument to obtain $\alpha|_2$. Continuing inductively, we define an infinite path $\alpha$ through $T'$, and it is straightforward to check that $\alpha$ is a $\leq_{\text{lex}}$-minimal element of $[T']$ and hence of $[T]$. \qed

We can combine the $\leq_{\text{lex}}$-ordering with the inverse prefix order to obtain a linear ordering of $A^\subseteq\mathbb{N}$. This ordering has the nice property that if $A$ is well-ordered and $T$ is well-founded, then the ordering restricted to $T$ is a well-ordering.

**Definition 4.4:** The Kleene-Brouwer ordering $\leq_{\text{KB}}$ of $A^\subseteq\mathbb{N}$ is defined as follows.

$$\sigma \leq_{\text{KB}} \tau \iff \sigma \supseteq \tau \text{ or } \sigma \leq_{\text{lex}} \tau$$

This means $\sigma$ is smaller than $\tau$ if it is a proper extension of $\tau$ or “to the left” of $\tau$.

We now have

**Proposition 4.5:** Assume $(A, \leq)$ is a well-ordered set. Then for any tree $T$ on $A$,

$$T \text{ is well-founded } \iff \leq_{\text{KB}} \text{ restricted to } T \text{ is a well-ordering.}$$

**Proof.** Suppose $T$ is not well-founded. Let $\alpha \in [T]$. Then $\alpha|_0, \alpha|_1, \ldots$ is an infinite descending sequence with respect to $\leq_{\text{KB}}$.

Conversely, suppose $\sigma_0 >_{\text{KB}} \sigma_1 >_{\text{KB}} \ldots$ is an infinite descending sequence on $T$. Then $\sigma_1(0) \geq \sigma_2(0) \geq \ldots$ as a sequence in $A$. Since $A$ is well-ordered, this
sequence must eventually be constant, say $\sigma_n(0) = a_0$ for all $n \geq n_0$. Since
the $\sigma_n$ are descending, by the definition of $\leq_{KB}$ it follows that $|\sigma_n| \geq 2$ for
$n > n_0$. Hence we can consider the sequence $\sigma_{n_0+1}(1) \geq \sigma_{n_0+2}(1) \geq \ldots$ in $A$.
Again, this must be constant $= a_1$ eventually. Inductively, we obtain a sequence
$\alpha = (a_0, a_1, a_2, \ldots) \in [T]$, i.e. $T$ is not well-founded.

Note however that the order type of a well-founded tree under $\leq_{KB}$ is not the
same as its rank $\rho(T)$.

Of course we can also define an ordering on $A^{<\mathbb{N}}$ via an injective mapping from
$A^{<\mathbb{N}}$ to some linearly ordered set $A$. We will use this repeatedly for the case
$A = \mathbb{N}$ and $A = \{0, 1\}$.

For $A = \mathbb{N}$, we can use the standard coding mapping
\[ \pi : (a_0, a_1, \ldots, a_n) \mapsto p_0^{a_0} p_1^{a_1} \cdots p_n^{a_n}, \]
where $p_k$ is the $k$th prime number. This embeds $\mathbb{N}^{<\mathbb{N}}$ into $\mathbb{N}$, and we can
well-order $\mathbb{N}^{<\mathbb{N}}$ by letting $\sigma < \tau$ if and only if $\pi(\sigma) < \pi(\tau)$.

For $A = \{0, 1\}$ we set
\[ \pi : (b_0, b_1, \ldots, b_n) \mapsto \sum_{i=0}^{n} 2^{b_i}. \]

These two embedding allows us henceforth to see trees as subsets of the natural
numbers. If we optimize the coding suitably, we can make it onto, and henceforth
also assume that every subset of $\mathbb{N}$ codes a tree (on $\{0, 1\}$ or $\mathbb{N}$, depending on the
circumstances). This will be an important component in exploring the relation
between topological and arithmetical complexity.

**Trees and closed sets**

Let $A$ be a set with the discrete topology. Consider $A^{\mathbb{N}}$ with the product topology
defined in Lecture 2.

**Proposition 4.6:** A set $F \subseteq A^{\mathbb{N}}$ is closed if and only if there exists a tree $T$ on $A$
such that $F = [T]$.

**Proof.** Suppose $F$ is closed. Let
\[ T_F = \{ \sigma \in A^{<\mathbb{N}} : \sigma \subset \alpha \text{ for some } \alpha \in F \}. \]
Then clearly $F \subset [T_F]$. Suppose $\alpha \in [T_F]$. This means for any $n$, $\alpha \upharpoonright _n \in T_F$, which implies that there exists $\beta_n \in F$ such that $\alpha_n \subseteq \beta_n$. The sequence $(\beta_n)$ converges to $\alpha$, and since $F$ is closed, $\alpha \in F$.

For the other direction, suppose $F = [T]$. Let $\alpha \in A^N \setminus F$. Then there exists an $n$ such that $\alpha \upharpoonright _n \notin T$. Since a tree is closed under prefixes, since implies that no extension of $\alpha \upharpoonright _n$ can be in $T$. This in turn implies $N_{\alpha \upharpoonright _n} \subseteq A^N \setminus F$, and hence $A^N \setminus F$ is open.

**Trees and continuous mappings**

Let $f : A^N \rightarrow A^N$ be continuous. We define a mapping $\varphi : A^{<N} \rightarrow A^{<N}$ by setting

$$\varphi(\sigma) = \text{the longest } \tau \text{ such that } N_\sigma \subseteq f^{-1}(N_\tau).$$

This mapping has the following properties:

1. It is monotone, i.e. $\sigma \subseteq \tau$ implies $\varphi(\sigma) \subseteq \varphi(\tau)$.

2. For any $\alpha \in A^N$ we have $\lim \limits _n |\varphi(\alpha \upharpoonright _n)| = \infty$. This follows directly from the continuity of $f$: For any neighborhood $N_\tau$ of $f(\alpha)$ there exists a neighborhood $N_\sigma$ of $\alpha$ such that $f(N_\sigma) \subseteq N_\tau$. But $\tau$ has to be of the form $\tau = f(\alpha) \upharpoonright _m$, and $\sigma$ of the form $\alpha \upharpoonright _n$. Hence for any $m$ there must exist an $n$ such that $\varphi(\alpha \upharpoonright _n) \supseteq f(\alpha) \upharpoonright _m$.

On the other hand, if a function $\varphi : A^{<N} \rightarrow A^{<N}$ satisfies (1) and (2), it induces a function $\varphi^* : A^N \rightarrow A^N$ by letting

$$\varphi^*(\alpha) = \lim \limits _n \varphi(\alpha \upharpoonright _n) = \text{the unique sequence extending all } \varphi(\alpha \upharpoonright _n).$$

This $\varphi^*$ is indeed continuous: The preimage of $N_\tau$ under $\varphi^*$ is given by

$$(\varphi^*)^{-1}(N_\tau) = \bigcup \{N_\sigma : \varphi(\sigma) \supseteq \tau\},$$

which is an open set.

We have shown

**Proposition 4.7**: A mapping $f : A^N \rightarrow A^N$ is continuous if and only if there exists a mapping $\varphi$ satisfying (1) and (2) such that $f = \varphi^*$.

Again, note that we can completely describe a topological concept, continuity, through a relation between finite strings.