Lecture 18: $\Sigma^1_2$ Sets

In this lecture we extend the results of the previous lecture to $\Sigma^1_2$ sets.

Tree representations of $\Sigma^1_2$ sets

Analytic sets are projections of closed sets. Closed sets are in $\mathbb{N}^\mathbb{N}$ are infinite paths through trees on $\omega$.

We call a set $A \subseteq \mathbb{N}^\mathbb{N}$ $Y$-Souslin if $A$ is the projection $\exists Y^T$ of some $[T]$, where $T$ is a tree on $\mathbb{N} \times Y$, i.e. $A = \exists Y^T = \{ \alpha : \exists y \in Y^N (\alpha, y) \in [T] \}$.

**Theorem 18.1** (Shoenfield, 1961): Every $\Sigma^1_2$ set is $\omega_1$-Souslin. In particular, if $A$ is $\Sigma^1_2$ then there is a tree $T \in L$ on $\mathbb{N} \times \omega_1$ such that $A = \exists (\omega_1)^T$.

**Proof.** Assume first $A$ is $\Pi^1_1$. There is a recursive tree $T$ on $\mathbb{N} \times \mathbb{N}$ (and hence, in $L$, since ‘being recursive’ is definable) such that

$$\alpha \in A \iff T(\alpha) \text{ is well-founded}.$$ 

Hence, $\alpha \in A$ if and only if there exists an order preserving map $\pi : T(\alpha) \to \omega_1$. We recast this in terms of getting an infinite branch through a tree. Let $\{\sigma_i : i \in \mathbb{N}\}$ be a recursive enumeration of $\mathbb{N}^{<\mathbb{N}}$. We may assume for this enumeration that $|\sigma_i| \leq i$. We define a tree $\tilde{T}$ on $\mathbb{N} \times \omega_1$ by

$$\tilde{T} = \{(\sigma, \tau) : \forall i, j < |\sigma| [\sigma_i \supset \sigma_j \wedge (|\sigma_i|, \sigma_i) \in T \to \tau(i) < \tau(j)]\}.$$ 

It is easy to see that $\tilde{T}$ is in $L$, since it is definable from $T$ and $\omega_1$. Furthermore, if $\alpha \in A$, then the existence of an order-preserving map $\pi : T(\alpha) \to \omega_1$ implies that there is an infinite path $(\alpha, \eta)$ through $\tilde{T}$. Conversely, if such a path $(\alpha, \eta)$ exists, then it is easy to see that there is an order preserving map $\pi : T(\alpha) \to \omega_1$. Hence we have

$$\alpha \in A \iff \exists \eta \in (\omega_1)^\mathbb{N} (\alpha, \eta) \in [\tilde{T}] \iff \alpha \in \exists (\omega_1)^\mathbb{N} [\tilde{T}],$$

so $A$ is of the desired form.

Now we extend the representation to $\Sigma^1_2$. If $A$ is $\Sigma^1_2$, then there is a $\Pi^1_1$ set $B \subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$ such that $A = \exists \mathbb{N}^B$. Since $B \in \Pi^1_1$, we can employ the tree representation of $\Pi^1_1$ to obtain a tree $T$ over $\mathbb{N} \times \mathbb{N} \times \omega_1$ such that $B = \exists (\omega_1)^T$. Now we recast $T$ as a tree $T'$ over $\mathbb{N} \times \omega_1$ such that $\exists (\omega_1)^T = \exists (\omega_1)^B$. This
is done by using a bijection between $\mathbb{N} \times \omega_1$ and $\omega_1$. This way we can cast the $\mathbb{N} \times \omega_1$ component of $T$ into a single $\omega_1$ component, and thus transform the tree $T$ into a tree $T'$ over $\mathbb{N} \times \omega_1$ such that $\exists^{(\omega_1)^\mathbb{N}}[T'] = \exists^{(\omega_1)^\mathbb{N}}[B]$. □

**$\Sigma^1_2$ sets as unions of Borel sets**

We can use Shoenfield’s tree representation to extend Corollary 17.8 to $\Sigma^1_2$ sets.

**Theorem 18.2** (Sierpiński, 1925): *Every $\Sigma^1_2$ set is a union of $\aleph_1$-many Borel sets.*

Sierpinski’s original proof used AC. The following proof does not make use of choice.

**Proof.** Let $A \subseteq \mathbb{N}^\mathbb{N}$ be $\Sigma^1_2$. By Theorem 18.1 there exists a tree $T$ on $\mathbb{N} \times \omega_1$ such that $A = \exists^{(\omega_1)^\mathbb{N}}[T]$. For any $\xi < \omega_1$ let

$$T^\xi = \{ (\sigma, \eta) \in T : \forall i \leq |\eta| \eta(i) < \xi \}. $$

Since the cofinality of $\omega_1$ is greater than $\omega$ (this can be proved without using AC), every $d : \omega \to \omega_1$ has its range included in some $\xi < \omega_1$. Thus we have

$$A = \bigcup_{\xi < \omega_1} \exists^{(\omega_1)^\mathbb{N}}[T^\xi].$$

For all $\xi < \omega_1$, the set $\exists^{(\omega_1)^\mathbb{N}}[T^\xi]$ is $\Sigma^1_1$, because the tree $T^\xi$ is a tree on a product of countable sets and hence is isomorphic to a tree on $\mathbb{N} \times \mathbb{N}$. By Corollary 17.9, each $\Sigma^1_1$ set is the union of $\aleph_1$ many Borel sets, from which the result follows.

Again, an immediate consequence of this theorem is (using the perfect set property of Borel sets):

**Corollary 18.3:** *Every $\Sigma^1_2$ set has cardinality at most $\aleph_1$ or has a perfect subset and hence cardinality $2^{\aleph_0}$.***

**Absoluteness of $\Sigma^1_2$ relations**

Shoenfield used the tree representation of $\Sigma^1_2$ sets to establish an important absoluteness result for $\Sigma^1_2$ sets of reals.
Suppose \( A \subseteq \mathbb{N}^N \) is \( \Sigma^1_2 \). Then, by the Kleene Normal Form there exists a bounded formula \( \varphi(\alpha, \beta_0, \beta_1, m) \) such that

\[
\alpha \in A \iff \exists \beta_0 \forall \beta_1 \exists m \varphi(\alpha, \beta_0, \beta_1, m).
\]

Let \( M \) be an inner model of \( ZF \), i.e. \( M \) is transitive and contains all ordinals. Since arithmetical formulas can be interpreted in \( ZF \), \( M \) contains all recursive predicates over \( \mathbb{N} \). In particular, since the truth of the bounded formula \( \varphi \) depends only on finite initial segments of \( \alpha, \beta_0, \beta_1 \), it defines a recursive predicate \( R_\varphi(\alpha, \beta_0, \beta_1, m) = R_\varphi(\sigma, \tau_0, \tau_1, m) \), which in turns defines a subset of \( \mathbb{N}^4 \) that is contained in \( M \). Hence we can define the relativization of \( A \) to \( M \) as

\[
A^M(\alpha) \iff \exists \beta_0 \in M \forall \beta_1 \in M \exists m R(\alpha, \beta_0, \beta_1, m).
\]

We say that \( A \) is **absolute for** \( M \) if for any \( \alpha \in M \),

\[
A^M(\alpha) \iff A(\alpha).
\]

Absoluteness itself can be extended and relativized in a straightforward manner to predicates analytical in some \( \gamma \in \mathbb{N}^N \cap M \).

**Theorem 18.4** (Shoenfield Absoluteness): Every \( \Sigma^1_2(\gamma) \) predicate and every \( \Pi^1_2(\gamma) \) predicate is absolute for all inner models \( M \) of \( ZFC \) such that \( \gamma \in M \). In particular, all \( \Sigma^1_2 \) and \( \Pi^1_2 \) relations are absolute for \( L \).

**Proof.** We show the theorem for \( \Sigma^1_2 \) predicates. For the relativized version, one uses the relative constructible universe \( L[\gamma] \), see Jech [2003] or Kanamori [2003].

Let \( A \) be a \( \Sigma^1_2 \) relation. For simplicity, we assume that \( A \) is unary. Fix a tree representation of \( A \) as a projection of a \( \Pi^1_2 \) set. So, let \( T \) be a recursive tree on \( \mathbb{N} \times \mathbb{N} \times \mathbb{N} \) such that

\[
\alpha \in A \iff \exists \beta T(\alpha, \beta) \text{ is well-founded}.
\]

Note that \( T \) is in \( M \).

Now assume \( \alpha \in M \) and \( \alpha \in A^M \). So there is a \( \beta \in M \) such that \( T(\alpha, \beta) \) is well-founded in \( M \). This is equivalent to the fact that in \( M \) there exists an order preserving mapping \( \pi : T(\alpha, \beta) \to \text{Ord}^M \). Since \( M \) is an inner model and \( T \) is the same in \( V \) and \( M \), such a mapping exists also in \( V \). Hence \( T(\alpha, \beta) \) is well-founded in \( V \) and thus \( \alpha \in A \).
For the converse assume that $\alpha \in A \cap M$. Now we use the tree representation of $A$ given by Theorem 18.1. Let $U \in L \subseteq M$ be a tree on $\mathbb{N} \times \omega_1$ such that $A = \exists^{(\omega_1)^*} U$. This means that for any $\alpha \in \mathbb{N}^\mathbb{N}$,

$$\alpha \in A \iff U(\alpha) \text{ is not well-founded.}$$

So $\alpha \in A \cap M$ implies that there exists no order preserving map $U(\alpha) \to \omega_1$. But then such a map cannot exist in $M$ either. So, $U(\alpha)$ is a tree in $M$ which is ill-founded in the sense of $M$. Thus, by Shoenfield’s Representation Theorem relativized to $M$, $\alpha \in A^M$.

Absoluteness for $\Pi^1_1$ follows by employing the same reasoning, using that the complement is $\Sigma^1_2$.

By analyzing the proof one sees that it actually suffices that $M$ is a transitive $\in$-model of a certain finite collection of axioms ZF such that $\omega_1 \subseteq M$.

The result is the best possible with respect to the analytical hierarchy, since the statement

$$\exists \alpha \ [\alpha \notin L]$$

is $\Sigma^1_3$, but cannot be absolute for $M = L$.

Shoenfield’s Absoluteness Theorem also holds for sentences rather than formulæ, with a similar proof. This means a $\Sigma^1_2$ statement is true in $L$ if and only if it holds in $V$. This has an important consequence regarding the significance of principles like CH for analysis. Many results of classical analysis are $\Sigma^1_2$ statements. The Absoluteness Theorem says that if they can be established under $V = L$ (and hence in a world where CH holds), they can be established in ZF alone.

Another consequence concerns the complexity of reals defined by analytical relations.

**Corollary 18.5:** If $X \subseteq \omega$ is $\Sigma^1_2$, then $X \in L$. In particular, every $\Sigma^1_2$ real (and hence every $\Pi^1_2$ real) is in $L$.

**Proof.** Let $X$ be $\Sigma^1_2$ via some formula $\varphi$. Since $\omega \in L$, and since $L$ is an inner model of ZF, it satisfies the axiom of separation (relativized to $L$) for $\varphi$. So the set $X^L = \{a \in \omega : \varphi^L(a)\}$ is in $L$. It is clear that the representation and absoluteness results also hold for subsets of $\omega$. (Change the notation to include subsets of $\omega$.) Absoluteness for $\varphi$ implies that $X^L \cap L = X \cap L$, but since $X \subseteq \omega$, we have $X = X \cap L$ and $X^L \cap L = X^L$, and hence $X \in L$. \qed
We cannot extend this to $\Sigma^1_2$ sets of reals. In the proof of the Corollary, it is crucial that $\omega$, the set over which we apply separation, is in $L$. This is not longer the case for sets of reals. For example, the set of all reals is clearly $\Sigma^1_2$, but unless $V = L$, it is not contained in $L$. 