Lecture 13: Regularity Properties of Analytic Sets

In this lecture we verify that the analytic sets are Lebesgue measurable (LM) and have the Baire property (BP). Since both properties are closed under complements, they also hold for the class of *co-analytic sets* $\mathbf{\Pi}^1_1$.

The analytic sets also have the perfect subset property (PS). As in the case of the Borel sets, the proof uses different ideas and will therefore be presented in a separate lecture. Besides, the perfect subset property for $\mathbf{\Pi}^1_1$ sets is no longer provable in ZF.

For Borel sets, one proves (LM) and (BP) by showing that the class of sets having (LM) (or (BP), respectively) forms a σ-algebra and contains the open sets. For the analytic sets, this method is no longer available. We can, however, prove a similar property with respect to the Souslin operation $\mathcal{A}$, which can be seen as an extension of basic set theoretic operations into the uncountable.

More specifically, we will achieve the following.

- *Show that the Souslin operation $\mathcal{A}$ is* ***idempotent****, i.e.* $\mathcal{A}\mathcal{A}\,\Gamma = \mathcal{A}\Gamma$. This implies that the analytic sets are closed under $\mathcal{A}$.
- *Show that the family of sets with (LM) (or (BP), respectively), is closed under the Souslin operation.* Since the closed sets have both properties, and the Souslin operator is clearly monotone on classes, this yields the desired regularity results.

Idempotence of the Souslin operation

Theorem 13.1: *For every class Γ of subsets of various Polish spaces,*

$$\mathcal{A}\mathcal{A}\Gamma = \mathcal{A}\Gamma.$$

Proof. We clearly have $\Gamma \subseteq \mathcal{A}\Gamma$, so that we only need to prove $\mathcal{A}\mathcal{A}\Gamma \subseteq \mathcal{A}\Gamma$.

Suppose $A = \mathcal{A}P$ with $P_\sigma \in \mathcal{A}\Gamma$, that is, $P_\sigma = \mathcal{A}Q_{\sigma,\tau}$ mit $Q_{\sigma,\tau} \in \Gamma$. Then

$$\begin{aligned} z \in A \quad &\Leftrightarrow \quad \exists \alpha\, \forall m\, (z \in P_{\alpha|m}) \\ &\Leftrightarrow \quad \exists \alpha\, \forall m\, \exists \beta\, \forall n\, (z \in Q_{\alpha|m,\beta|n}) \\ &\Leftrightarrow \quad \exists \alpha\, \exists \beta\, \forall m\, \forall n\, (z \in Q_{\alpha|m,(\beta)_m|n}), \end{aligned}$$

where $(\beta)_m$ denotes the m-th column of β.

Now we contract the two function quantifiers to a single one, using a (computable) homeomorphism $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$, and the two universal number quantifiers into a single one using the paring function $\langle ., . \rangle$. Then A can be characterized as

$$z \in A \quad \Longleftrightarrow \quad \exists \gamma \, \forall k (z \in R_{\gamma|k})$$

where $R_\sigma = Q_{\varphi(\sigma),\psi(\sigma)} \in \Gamma$ for suitable coding functions φ, ψ. $\square$

Corollary 13.2:

$$\mathcal{A}\mathbf{\Sigma}^1_1 = \mathbf{\Sigma}^1_1.$$

Lebesgue measurability of analytic sets

We start with a lemma that essentially says that we can envelop any set with a smallest (up to measure 0) measurable set.

Lemma 13.3: *For every set $A \subseteq \mathbb{R}$ there exists a set $B \subseteq \mathbb{R}$ so that*

(i) $A \subseteq B$ and B is Lebesgue measurable,

(ii) if B' is such that $A \subseteq B' \subseteq B$ and is Lebesgue measurable, then $B \setminus B'$ has measure 0.

Proof. Suppose first that $\lambda^*(A) < \infty$. For every $n \geq 0$, there exists an open set $O_n \supseteq A$ with $\lambda^*(O_n) = \lambda(O_n) < \lambda^*(A) + 1/n$. Then $B = \bigcap_n O_n$ is measurable, and $\lambda(B) = \lambda^*(A)$. Furthermore, if $A \subseteq B' \subseteq B$, then $\lambda^*(A) \leq \lambda^*(B') \leq \lambda^*(B)$. If B' is also measurable, then

$$\lambda^*(B) = \lambda^*(B \cap B') + \lambda^*(B \setminus B') = \lambda^*(B') + \lambda^*(B \setminus B'),$$

hence $\lambda^*(B \setminus B') = 0$.

If $\lambda^*(A) = \infty$, let $A_n = A \cap [m, m+1)$ for $m \in \mathbb{Z}$. Then $\lambda^*(A_m) \leq 1$, and we can choose $B_m \supseteq A_m$ measurable such that $\lambda^*(B_m) = \lambda^*(A_m)$. Then $B = \bigcup_{m \in \mathbb{Z}} B_m$ has the desired property. $\square$

We now apply the lemma to show that Lebesgue measurability is closed under the Souslin operation. The basic idea is to approximate the local 'branches' of the Souslin operation on a Souslin scheme by measurable sets from outside, in the sense of the lemma. It turns out that the total error we make by this approximation is negligible, and hence the overall result of the Souslin operation differs from a measurable set only by a nullset and hence is measurable.

Proposition 13.4: *The class* **LM** *of all Lebesgue measurable sets* $\subseteq \mathbb{R}$ *is closed under the Souslin operation, i.e.*

$$\mathcal{A}\,\mathbf{LM} \subseteq \mathbf{LM}.$$

Proof. Let $A = (A_\sigma)$ be a Souslin scheme with each A_σ measurable. We can assume that (A_σ) is regular. For each $\sigma \in \mathbb{N}^{<\mathbb{N}}$ we let

$$A^\sigma = \bigcup_{\alpha \supset \sigma} \bigcap_{n \in \mathbb{N}} A_{\alpha|n} \subseteq A_\sigma.$$

Note that $A^{\langle\varnothing\rangle} = \mathcal{A}A$. By the previous lemma, there exist measurable sets $B^\sigma \supseteq A^\sigma$ so that for every measurable $B \supseteq A^\sigma$, $B^\sigma \setminus B$ is null.

By replacing B^σ with $B^\sigma \cap A_\sigma$, we can further assume $B^\sigma \subseteq A_\sigma$. This makes (B^σ) a regular Souslin scheme.

Now let $C_\sigma = B^\sigma \setminus \bigcup_{n\in\mathbb{N}} B^{\sigma^\frown\langle n\rangle}$. Each C_σ is a nullset, by the choice of the B^σ and the fact that $A^\sigma = \bigcup_{n\in\mathbb{N}} A^{\sigma^\frown\langle n\rangle} \subseteq \bigcup_{n\in\mathbb{N}} B^{\sigma^\frown\langle n\rangle}$. Hence $C = \bigcup_\sigma C_\sigma$ is a nullset, too.

It remains to show that

$$B^{\langle\varnothing\rangle} \setminus C \;\subseteq\; A^{\langle\varnothing\rangle} = \mathcal{A}A,$$

for this implies $B^{\langle\varnothing\rangle} \setminus A^{\langle\varnothing\rangle} \subseteq C$ is null, which in turn implies that $A^{\langle\varnothing\rangle}$ is Lebesgue measurable (since it differs from a measurable set by a nullset).

So let $x \in B^{\langle\varnothing\rangle} \setminus C$. Since $x \notin C_{\langle\varnothing\rangle}$, there is an $\alpha(0)$ with $x \in B^{\langle\alpha(0)\rangle}$.

Given $\alpha \mid n$ with $x \in B^{\alpha|n}$, we can choose $\alpha(n)$ so that $x \in B^{\alpha|n+1}$. This is possible because $x \notin C_{\alpha|n}$. This way we construct $\alpha \in \mathbb{N}^\mathbb{N}$ with

$$x \in \bigcap_n B^{\alpha|n} \subseteq \bigcap_n A_{\alpha|n} \subseteq A^{\langle\varnothing\rangle}.$$

□

Corollary 13.5: *Every analytic set is Lebesgue measurable.*

Proof. By the idempotence of $\mathcal{A}$, $\mathcal{A}\mathbf{\Sigma}^1_1 = \mathcal{A}\mathcal{A}\mathbf{\Pi}^0_1 = \mathcal{A}\mathbf{\Pi}^0_1 = \mathbf{\Sigma}^1_1$. On the other hand, we have $\mathcal{A}\mathbf{\Pi}^0_1 \subseteq \mathcal{A}\mathbf{LM} = \mathbf{LM}$, since the Souslin operation is monotone on classes. This yields $\mathbf{\Sigma}^1_1 \subseteq \mathbf{LM}$. □

Universally measurable sets

The previous proof is general enough to work for other kinds of measures on arbitrary Polish spaces.

Given a Polish space X, a **Borel measure** on X is a countably additive set function μ defined on a σ-algebra of the Borel sets in X. A set is **μ-measurable** if it can be represented as a union of a Borel set and a μ-nullset. A measure μ is **σ-finite** if $X = \bigcup_n X_n$, where X_n is μ-measurable with $\mu(X_n) < \infty$. Lebesgue measure is σ-finite Borel measure on the Polish space $\mathbb{R}$.

A set $A \subseteq X$ is **universally measurable** if it is μ-measurable for every σ-finite Borel measure on X.

Theorem 13.6 (Lusin)**:** *In a Polish space, every analytic is universally measurable.*

Baire property of analytic sets

Inspecting the proof of Proposition 13.4, we see that it works for the Baire property as well (with *measure* 0 replaced by *meager*, of course), provided we can prove a Baire category version of Lemma 13.7.

Lemma 13.7: *Let X be a Polish space. For every set $A \subseteq X$ there exists a set $B \subseteq X$ so that*

(i) $A \subseteq B$ and B has the Baire property,

(ii) if $Z \subseteq B \setminus A$ and Z has the Baire property, then Z is meager.

Proof. Let $U_1, U_2, \dots$ be an enumeration of countable base of the topology for X. Given $A \subseteq \mathbb{R}$ set

$$A^* := \{x \in \mathbb{R} \colon \forall i\ (x \in U_i \Rightarrow U_i \cap A \text{ not meager})\}.$$

Note that A^* is closed: If $x \notin A^*$, then there exists i with $x \in U_i$ & $U_i \cap A$ null. If $y \in U_i$, then $y \notin A^*$, since $U_i \cap A$ is null. Hence $U_i \subseteq \neg A^*$.

We have
$$A \setminus A^* = \bigcup \{A \cap U_i : A \cap U_i \text{ meager}\},$$

which is a countable union of meager sets and hence meager.

If we let $B = A \cup A^* = A^* \cup (A \setminus A^*)$, then B is a union of a meager set and a closed set and hence has the Baire property.

Now assume $B' \supseteq A$ has the Baire property. Then $C = B \setminus B'$ has the Baire property, too. Suppose C is not meager, then $U_i \setminus C$ is meager for some i, and hence also $U_i \cap A \subseteq (U_i \setminus C)$. Besides, $U_i \cap C \neq \emptyset$, for otherwise $U_i \subseteq U_i \setminus C$ would be meager. Thus there exists $x \in U_i$ with $x \notin A^*$, which by definition of A^* implies that $U_i \cap A$ is not meager, a contradiction. $\square$

By adapting the proof of Proposition 13.4, we obtain the Baire category version of Proposition 13.4 and hence can deduce that analytic sets have the Baire property.

Proposition 13.8: *In any Polish space X, the class* **BP** *of all sets $\subseteq X$ with the Baire property is closed under the Souslin operation, i.e.*

$$\mathcal{A}\,\mathbf{BP} \subseteq \mathbf{BP}.$$