Lecture 21: Co-analytic Ranks

In the previous lecture we learned about how \( \Pi^m_1 \) set can be analyzed in terms of countable ordinals. In this lecture we will deepen this analysis. We will develop the theory of \( \Pi^m_1 \)-ranks, which is a powerful tool in descriptive set theory. We can view the recursive function \( f \) that we constructed in the proof of Theorem 20.2 as the central fact:

\[
\text{If } R_e \text{ is well-founded, then } \rho(R_e) \leq |f(e)|_0
\]  

(\#)

Boundedness Principles

We start by picking up the observation made in Lemma 20.1. It states that r.e. subsets of \( \emptyset \) are uniformly bounded: Given an index \( e \) of an r.e. subset of \( \emptyset \), we can compute uniformly in \( e \) a ordinal bounding all ordinals denoted by \( W_e \). We can strengthen this to \( \Sigma^m_1 \) sets.

**Theorem 21.1** (Spector): If \( X \subseteq \emptyset \) is \( \Sigma^1_1 \), then there exists \( b \in \emptyset \) such that

\[
\forall x \in X \quad |x|_\emptyset < |b|_\emptyset.
\]

**Proof.** Let \( t \) be a reduction from \( \emptyset \) to \( \text{WF}_n \), that is \( t \) is recursive such that

\[
x \in \emptyset \quad \iff \quad R_{t(x)} \text{ is well-founded}.
\]

The idea is that if \( X \) is unbounded in \( \emptyset \), then we can characterize \( \emptyset \) by a \( \Sigma^1_1 \) formula, contradicting Corollary 20.7. If the desired \( b \) does not exist, then, for each \( x \in \emptyset \), we can find a \( y \in X \) such that there exists an embedding of \( R_{t(x)} \) into \( \emptyset \) below \( y \). Using the proof of Theorem 20.2, we can formulate this as a property \( P(x) \),

\[
P(x) \iff \exists y \ [y \in X \land \exists \gamma \forall z_0, z_1 (R_{t(x)}(z_0, z_1) \Rightarrow \langle \gamma(z_0), \gamma(z_1) \rangle \in W_{g(z)})],
\]

where \( g \) is a recursive function so that \( W_{g(z)} = \{(x, y) : x <_\emptyset y <_\emptyset z\} \) (see Proposition 19.7). If \( X \) is \( \Sigma^1_1 \), then \( P \) is \( \Sigma^1_1 \).

If \( x \in \emptyset \), then \( R_{t(x)} \) is well-founded, hence by (\#), \( \rho(R_{t(x)}) \leq |f(t(x))|_\emptyset \), and thus if \( X \) is unbounded in \( \emptyset \), \( P(x) \) holds. If \( P(x) \) holds on the other hand, then \( R_{t(x)} \) must be well-founded (otherwise such a mapping would not exist), and thus \( x \in \emptyset \). Hence \( P \) would be a \( \Sigma^1_1 \) characterization of \( \emptyset \). \( \square \)
Corollary 21.2: If $X \subseteq \mathbb{N}$ is $\Delta^1_1$, and $h$ is recursive such that $x \in X$ if and only if $h(x) \in \emptyset$, then there exists a $b \in \emptyset$ such that
\[
\forall x \in X \quad |h(x)| < |b|_{\emptyset}.
\]
A similar statement holds with $WF_{\mathbb{N}}$ in place of $\emptyset$.

Boundedness for sets of reals

The key to Spector’s theorem is the fact that $WF_{\mathbb{N}}$ and $\emptyset$ are $m$-complete for the class of $\Pi^1_1$ sets of natural numbers.

We have seen (Theorem 17.6) that the set $WOrd, WF \subseteq \mathbb{N}^N$ are $\Pi^1_1$-complete with respect to Wadge-reducibility. This lets us obtain a similar result for $\Sigma^1_1$ sets of reals.

Theorem 21.3 ($\Sigma^1_1$-boundedness for reals): Let $A \subseteq WOrd$ be $\Sigma^1_1$. Then there exists a $\xi < \omega^CK$ such that
\[
\forall \alpha \in A \quad \|\alpha\| < \xi,
\]
where $\|\alpha\|$ denotes the order type of the well-ordering coded by $\alpha$.

An analogous statement holds for $WF$, with respect to the rank function $\rho$ of a well-founded relation.

Proof. If such a $\xi$ did not exist, then
\[
\alpha \in WOrd \iff \exists \beta \left[ \beta \in A \land WOrd_\rho \right].
\]
The right-hand side is $\Sigma^1_1$, and hence WOrd would be $\Sigma^1_1$, contradiction. $\square$

Rank analysis of co-analytic sets

The previous results constitute a powerful technique when analyzing the complexity of sets. In particular, they give us a method to show that a $\Pi^1_1$ set is not Borel, besides proving that they are $\Pi^1_1$-complete.

If $A \subseteq \mathbb{N}^N$ is $\Pi^1_1$, then there exists a recursive tree $T$ such that
\[
\alpha \in A \iff T(\alpha) \text{ is well-founded.}
\]
Every well-founded \( T(\alpha) \) has a rank \( \rho(T(\alpha)) \). \( \Sigma_1^1 \)-boundedness tells us that if \( A \) is moreover \( \Delta_1^1 \), then the spectrum of these ranks is **bounded by a computable ordinal**. This means that we can show that \( A \) is not \( \Delta_1^1 \) by showing that its ordinal spectrum \( \{ \rho(T(\alpha)) : \alpha \in A \} \) is unbounded in \( \omega_1^{CK} \).

These observations generalize (using relativization) to \( \Pi_1^1 \) sets: Ranks of Borel sets are bounded by an ordinal \( \xi < \omega_1 \).

The downside of this method is that the tree \( T \) associated with a \( \Pi_1^1 \) set is a rather generic object, stemming from the canonical representation of \( \Pi_1^1 \) sets, and it may be rather difficult to prove anything about the ordinals \( \rho(T(\alpha)) \).

In many cases one can replace the canonical rank function with a “custom” one that better reflects the structure of a set.

Given a set \( S \), a **rank** on \( S \) is a map \( \varphi : S \to \text{Ord} \). A rank is called **regular** if \( \varphi(S) \) is an ordinal, i.e. \( \varphi(S) \) is an initial segment of \( \text{Ord} \).

Each rank gives rise to a **prewellordering** \( \leq_\varphi \):

\[
x \leq_\varphi y \iff \varphi(x) \leq \varphi(y).
\]

A prewellordering is a binary relation on \( S \) that is reflexive, transitive, and connected (any two elements are comparable), and every non-empty subset of \( S \) has a \( \leq_\varphi \)-minimal element.

Under AC every set can be well-ordered, which means that every set admits a regular rank function that is one-one. However, we would like a rank function to reflect the complexity and structure of the set. In particular, we would like to preserve the boundedness properties of \( \Sigma_1^1 \) sets. For those to hold it was crucial that the initial segments \( \text{WOrd}_\xi \), \( \xi < \omega_1 \) (and similarly for \( 0 \)) are Borel.

We formulate a similar property that ensures the same for general rank functions.

**Definition 21.4**: Let \( X \) be a Polish space, and suppose \( A \subseteq X \). A rank \( \varphi : A \to \text{Ord} \) is a **\( \Pi_1^1 \)-rank** if there exists a \( \Sigma_1^1 \) relation \( \leq_\varphi^{\Sigma_1} \) and a \( \Pi_1^1 \) relation \( \leq_\varphi^{\Pi_1} \) such that for \( y \in A \),

\[
\{ x \in A : \varphi(x) \leq \varphi(y) \} = \{ x \in X : x \leq_\varphi^{\Sigma_1} y \} = \{ x \in X : x \leq_\varphi^{\Pi_1} y \}.
\]

In other words, the initial segments \( \leq_\varphi \) below a given \( y \in A \) are uniformly \( \Delta_1^1 \).

**Theorem 21.5**: Every \( \Pi_1^1 \) set \( A \subseteq \mathbb{N}^\mathbb{N} \) admits a **\( \Pi_1^1 \)-rank**.
Proof. We first show that WOrd admits a \( \Pi^1 \) -rank. The function \( \varphi \) is obviously \( \varphi(\alpha) = \| \alpha \| \). We have to express \( \| \alpha \| \leq \| \beta \| \) in a \( \Sigma^1 \) and a \( \Pi^1 \) way.

For the \( \Sigma^1 \) relation \( \leq^\Sigma^1 \), let
\[
\alpha \leq^\Sigma^1 \beta \iff E_\alpha \text{ is a linear ordering and}
\exists \gamma [ \gamma \text{ is a one-one, relation preserving mapping } \gamma : E_\alpha \to E_\beta ]
\]
\[
\iff E_\alpha \text{ is a linear ordering and } \exists \gamma \forall m, n [ m E_\alpha n \Rightarrow \gamma(m) E_\beta \gamma(n) ].
\]

Recall that “\( E_\alpha \) is a linear ordering” is \( \Pi^0 \), hence \( \leq^\Sigma^1 \) is \( \Sigma^1 \).

For the \( \Sigma^1 \) relation \( \leq^\Sigma^1 \), let
\[
\alpha \leq^\Sigma^1 \beta \iff E_\alpha \text{ is a well-ordering and}
\exists \gamma [ \gamma \text{ is a one-one, relation preserving mapping } E_\beta \text{ onto an initial segment of } E_\alpha ]
\]
\[
\iff \alpha \in \text{WOrd and } \forall \gamma \exists \exists m, n [ m E_\beta n \Rightarrow \gamma(m) E_\alpha \gamma(n) ].
\]

Since WOrd is \( \Pi^1 \), \( \leq^\Pi^1 \) is \( \Pi^1 \), too.

Now we have for \( \beta \in \text{WOrd},
\]
\[
\alpha \leq^\Sigma^1 \beta \iff \alpha \leq^\Pi^1 \beta \iff \| \alpha \| \leq \| \beta \|,
\]
as desired. \qed

Theorem 21.6 (Boundedness for arbitrary rank functions): Suppose \( A \subseteq X \) is \( \Pi^1 \) but not Borel and \( \varphi : A \to \text{Ord} \) is a \( \Pi^1 \)-rank on \( A \). If \( B \subseteq A \) is \( \Sigma^1 \), then there is an \( x_0 \in A \) such that
\[
\varphi(x) \leq \varphi(x_0) \quad \text{for all } x \in B.
\]

Proof. If not, then
\[
x \in A \iff \exists y [ y \in B \land x \leq^\Sigma^1 y ],
\]
and thus \( A \) would be \( \Sigma^1 \), and thus Borel, a contradiction. \qed

Corollary 21.7: Suppose \( A \subseteq X \) is \( \Pi^1 \) and \( \varphi : A \to \text{Ord} \) is a regular \( \Pi^1 \)-rank.

Then
(a) \( \varphi(A) \leq \omega_1 \);
(b) \( A \) is Borel if \( \varphi(A) < \omega_1 \);
(c) if \( B \subseteq A \) is \( \Sigma^1 \), then \( \sup \{ \varphi(x) : x \in B \} < \omega_1 \).
The Cantor-Bendixson Rank

We illustrate the concept of $\Pi^1_\alpha$-ranks with a rank function that is different from the canonical rank function.

Suppose $T$ is a tree on $\{0, 1\}$. Define the Cantor-Bendixson derivative of $T$ as

$$T' = \{ \sigma \in T : \sigma \text{ has at least two incompatible extensions} \}.$$ 

We can iterate this derivative along the ordinals:

$$T^{(\xi+1)} = (T^{(\xi)})' \quad \text{and} \quad T^{(\lambda)} = \bigcup_{\xi < \lambda} T^{(\xi)} \quad \text{for \lambda \text{ limit}}.$$ 

We clearly have $T^{(\zeta)} \subseteq T^{(\xi)}$ for $\zeta < \xi$. There must exist an ordinal $\xi_0$ such that $(T^{(\xi_0)})' = T^{(\xi_0)}$. Since $T$ is countable, $\xi_0 < \omega_1$. We call the least such $\xi_0$ the Cantor-Bendixson rank of $T$, $\| T \|_{CB}$.

The following is not hard to see.

**Proposition 21.8:** For any tree $T$,

(a) if $[T^{(\| T \|_{CB})}] \neq \emptyset$, then $[T^{(\| T \|_{CB})}]$ is a perfect subset of $\mathbb{N}^\mathbb{N}$;

(b) $T^{(\| T \|_{CB})} = \emptyset$ if and only if $[T]$ is countable.

We hence have a new proof of the Cantor-Bendixson Theorem 2.5 for $2^\mathbb{N}$.

One can show that $\| \cdot \|_{CB}$ is indeed a $\Pi^1_\alpha$-rank on the set of all countable compact subsets of $2^\mathbb{N}$. This follows from the theory of Borel derivatives, which generalizes the Cantor-Bendixson derivative to other settings (see Kechris [1995]).

Since for any given ordinal $\xi < \omega_1$, we can find a tree $T \subseteq 2^{<\mathbb{N}}$ with $\| T \|_{CB} = \xi$, it follows that the set

$$K_\omega(2^\mathbb{N}) = \{ K \subseteq 2^\mathbb{N} : K \text{ countable} \}$$ 

is not Borel.

Using a different derivative, Kechris and Woodin [1986] showed that the set

$$\text{Diff} = \{ f \in C[0, 1] : f \text{ differentiable on } [0, 1] \}$$ 

is not Borel.