Lecture 15: The Constructible Universe

A set $X$ is first-order definable in a set $Y$ (from parameters) if there exists a first-order formula $\varphi(x_0, x_1, \ldots, x_n)$ in the language of set theory (i.e. only using the binary relation symbol $\in$) such that for some $a_1, \ldots, a_n \in Y$,

$$X = \{y \in Y : (Y, \in) \models \varphi[y, a_1, \ldots, a_n]\}.$$  

Here $(Y, \in)$ stands for the interpretation of $Y$ as a structure of the language of set theory, i.e. $Y$ is a set and $\in$ is interpreted as a binary relation over $Y$.

The constructible universe is built as a cumulative hierarchy of sets along the ordinals. In each successor step, instead of adding all subsets of the current set, only the definable ones are added. Formally, $L$ is defined as follows. Given a set $Y$, let

$$P_{\text{DEF}}(Y) = \{X \subseteq Y : X \text{ is first order definable in } Y \text{ from parameters}\},$$

where the underlying language is the language of set theory. Now put

$$L_0 = \emptyset$$

$$L_{\xi+1} = P_{\text{DEF}}(L_\xi)$$

$$L_\xi = \bigcup_{\zeta < \xi} L_\zeta \quad (\xi \text{ limit ordinal})$$

Finally, let

$$L = \bigcup_{\xi \in \text{Ord}} L_\xi.$$  

Basic properties of $L$

The first Proposition tells us that the $L_\xi$ are set-theoretically nice structures and linearly ordered by the $\subseteq$-relation.

**Proposition 15.1**: For each ordinal $\xi$:

1. $L_\xi$ is transitive.
2. For $\zeta < \xi$, $L_\zeta \subseteq L_\xi$.
3. For $\zeta < \xi$, $L_\zeta \in L_\xi$.  

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Proof. We show the first two statements simultaneously by induction. They are clear for $\xi = 0$ and $\xi$ limit, so assume $\xi = \zeta + 1$. Suppose $x \in L_\xi$. Consider the formula $\varphi(x_0) \equiv x_0 \in x$ (here $x$ is a parameter). $\varphi$ defines the set

\[ x' = \{ a \in L_\zeta : L_\zeta \models \varphi[a] \} = \{ a \in L_\zeta : a \in x \}. \]

By induction hypothesis, $L_\zeta$ is transitive, and hence $a \in x$ implies $a \in L_\zeta$, and hence $x' = x$, so $x \in L_{\zeta+1}$. This yields $L_\zeta \subseteq L_{\xi}$. Now if $x \in L_\xi$, then $x \subseteq L_\xi$, and hence $x \subseteq L_{\zeta+1}$. Thus $L_\zeta$ is transitive.

For the third statement, note that the formula $\varphi(x_0) \equiv x_0 = x_0$ defines $L_\xi$ in $L_\xi$, and hence $L_\xi \in L_{\zeta+1}$.

Next, we show that $L$ contains all ordinals and that $\xi$ ‘shows up’ exactly after $\xi$ steps.

**Proposition 15.2:** For any $\xi$,

1. $\xi \in L_\xi$,
2. $L_\xi \cap \text{Ord} = \xi$.

Proof. Clearly, (1) follows from (2). To show (2), one again proceeds by induction. Again, the statement is clear for 0 and limit ordinals, so assume $\xi = \zeta + 1$ and $L_\zeta \cap \text{Ord} = \zeta$. We need to show $L_{\zeta+1} \cap \text{Ord} = \zeta + 1 = \zeta \cup \{ \zeta \}$. Since $L_\zeta \subseteq L_{\zeta+1}$, we have $\zeta \subseteq L_{\zeta+1} \cap \text{Ord}$. On the other hand, since $L_{\zeta+1} \subseteq \mathcal{P}(L_\zeta)$, we have $L_{\zeta+1} \cap \text{Ord} \subseteq \zeta + 1$. It thus remains to show that $\zeta \in L_{\zeta+1}$.

We need a formula $\varphi_{\text{Ord}}$ that defines the ordinals (in $L_\zeta$). Such is formula is easily found by formalizing the statement

“$x$ is transitive and linearly ordered by $\in$."

(Note that we assume that every set is well-founded.) It then seems that we have

$$\zeta = \{ a \in L_\zeta : L_\zeta \models \varphi_{\text{Ord}}[a] \},$$

and hence we can conclude $\zeta \in L_{\zeta+1}$. The problem is that being an ordinal (i.e. satisfying $\varphi_{\text{Ord}}$) in $L_\zeta$ may not be the same as being an ordinal in general (in $V$, the universe of all sets).

The fact that (*) nevertheless is true is a consequence of the absoluteness of $\Delta_0$ formulas for transitive sets. We address this important concept in detail next. □
Given a formula $\varphi$ in the language of set theory and some class $M$, we can relativize $\varphi$ to $\varphi^M$ essentially by restricting all quantifiers occurring in $\varphi$ to range over $M$, i.e. $\exists x \psi$ becomes $(\exists x \in M) \psi^M$, for example. We say a formula $\varphi(x_0, \ldots, x_n)$ is absolute for $M$ if for all $a_0, \ldots, a_n \in M$

$$\varphi^M(a_0, \ldots, a_n) \text{ holds } \iff \varphi(a_0, \ldots, a_n) \text{ holds.}$$

Unfortunately, even simple formulas like $x \subseteq y$ can fail to be absolute. For example, let $M = \{0, a\}$, where $a = \{\{0\}\}$. Then $(a \subseteq 0)^M$ (which is defined as $\forall x \in M (x \in a \rightarrow x \in 0)$) but not $a \subseteq 0$.

However, if $M$ is transitive, then many important formulas are absolute for $M$. A formula is $\Delta_0$ if it contains no or only bounded quantifiers of the form $\forall x \in v$ or $\exists x \in v$, where $x, v$ are set variables.

**Proposition 15.3:** If $M$ is transitive and $\varphi$ is $\Delta_0$, then $\varphi$ is absolute for $M$.

**Proof sketch.** Clearly $x = y$ and $x \in y$ are absolute for any $M$. It is also not hard to see that if $\varphi$ and $\psi$ are absolute for $M$, then so are $\neg \varphi$ and $\varphi \land \psi$. Hence all quantifier free formulas are absolute.

Finally, if $\varphi$ is absolute for $M$, so is $\psi \equiv \exists x \in y \varphi$: If $\psi^M(y, \bar{z})$ holds for $y, \bar{z} \in M$, then we have $[\exists x (x \in y \land \varphi(x, y, \bar{z}))]^M$, i.e., $\exists x \in M(x \in y \land \varphi^M(x, y, \bar{z}))$. Since $\varphi^M(x, y, \bar{z})$ if and only if $\varphi(x, y, \bar{z})$, it follows that $\exists x \in y \varphi(x, y, \bar{z})$, i.e. $\psi$.

Conversely, if for $y, \bar{z} \in M$, $\exists x \in y \varphi(x, y, \bar{z})$, then since $M$ is transitive, $x$ belongs to $M$, and since $\varphi(x, y, \bar{z})$ if and only $\varphi^M(x, y, \bar{z})$, we have $\exists x \in M (x \in y \land \varphi^M(x, y, \bar{z}))$ and so $\psi^M(y, \bar{z})$. \[\square\]

On can show that “$x$ is an ordinal.” is indeed definable by a $\Delta_0$ formula. Furthermore, using the absoluteness of $\Delta_0$ formulas, one can also show that $L$ is a model of ZF. More formally, this means that for every axiom $\sigma$ of ZF, $\text{ZF} \vdash \sigma^L$.

**Theorem 15.4:** For every axiom $\sigma$ of ZF, $\text{ZF} \vdash \sigma^L$.

$L$ is an inner model of ZF, that is, $L$ is transitive, contains all ordinals, and satisfies the axioms of ZF.

We can add to ZF the axiom that all sets are constructible, i.e.

$$\forall x \exists y \ (y \text{ is an ordinal } \land x \in L_y).$$

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This axiom is usually denoted by $V = L$. We may be tempted to think that $L$ is then trivially a model of $ZF + V = L$. But this is not at all clear, since this has to hold relative to $L$, i.e. $(V = L)^L$. This means that

$$\forall x \in L \exists y \in L (y \text{ is an ordinal } \land (x \in L_y)^L).$$

But it might be that $(x \in L_y)^L$ is not absolute, that is, viewed from inside $L$, not every set may be definable. To show that $(x \in L_y)^L$ is indeed absolute, one has to carefully study the notion of definability. In particular, we have to show that definability is definable.

**The definability of $L$**

Fix a Gödel numbering of set theoretic formulae. We can use it to formally define syntactical notions such as the satisfaction relation. More precisely, given a set $X$, let $\text{SAT}_X : X^{<\omega} \times \omega \to \{0, 1\}$ be the binary valued (partial) function that is defined for $((a_1, \ldots, a_n), e)$ iff $e$ is the Gödel number of a formula $\varphi$ with $n$ free variables and in this case

$$\text{SAT}_X(\vec{a}, e) = 1 \quad \text{iff} \quad X \models \varphi[\vec{a}]. \quad (15.1)$$

While Tarski’s Theorem excludes the possibility that a structure $X$ satisfying (a sufficiently large fragment of) ZFC can define its own truth predicate, one can formalize the satisfaction relation and show it works in a relativized environment that has sufficient closure properties. To be more precise, based on the recursive definition of the satisfaction relation one can devise a set theoretic formula $\varphi_{\text{SAT}}$ aiming at describing this relation formally. This formula will “work” in any relativized environment, represented by a set $Y$, as long as $Y$ satisfies some basic closure properties – it has to be transitive, closed under formation of finite sequences, and has to be able to address the Gödel numbers of formulas, i.e. it contains the natural numbers. The latter can be ensured by requiring that $V_\omega \subseteq Y$. Let us call such $Y$ adequate.

**Proposition 15.5:** There exists a set theoretic formula $\varphi_{\text{SAT}}(x_0, x_1)$ such that for all adequate $Y$, whenever $a_0, a_1 \in Y$,

$$Y \models \varphi_{\text{SAT}}[a_0, a_1] \quad \text{iff} \quad (1) \ a_0 \text{ is transitive and}$$

$$(2) \ a_1 = \text{SAT}_{a_0}. \quad (15.4)$$
Based on $\varphi_{\text{SAT}}$, one can devise a formula $\varphi_{\text{DEF}}$ with the following properties:

Suppose $Y$ is adequate. Then for all $a_0, a_1 \in Y$, if $\text{SAT}_{a_0} \subseteq Y$ then

$$Y \models \varphi_{\text{DEF}}[a_0, a_1] \text{ iff } \begin{cases} (1) a_0 \text{ is transitive and} \\ (2) a_1 = P_{\text{DEF}}(a_0). \end{cases}$$

In other words, first-order definability is definable. With regard to absoluteness considerations, it is important to track the complexity of the formulas. It turns out $\varphi_{\text{DEF}}$ is provably equivalent in $\text{ZF}$ to both a $\Sigma_1$ and a $\Pi_1$ formula of set theory. In this case we say the predicate $a_1 = P_{\text{DEF}}(a_0)$ is $\Delta_1$.

It is not hard to see that for limit $\xi$, $L_\xi$ is adequate. One can use these closure properties of $L_\xi$ at limit stages to show that $L_\xi$ can definably “recover” the sequence of $L_\xi$’s leading up to it.

**Proposition 15.6:** Suppose $\xi$ is a limit ordinal, $\xi > 0$. Let $G : \text{Ord} \rightarrow V$ be given by $\xi \mapsto L_\xi$. Then for all $\zeta < \xi$, $G|_{\zeta} \in L_\xi$.

If a formula $\varphi(\bar{x}, y)$ defines a function $F(\bar{x}) = y$, then we say $F$ is absolute for $M$ if $\varphi$ is. (It is not hard to show that this is independent of the particular definition of $F$.) One can in fact show that the function $G$ is absolute for all transitive models of $\text{ZF}$ (it is $\Delta_1$).

**Theorem 15.7:** $L$ is a model of $\text{ZF} + V = L$.

**Proof.** If $x \in L$, then there exists a limit $\xi$ such that $x \in L_\xi$. Since $\text{Ord} \subseteq L$, and since $G$ is absolute,

$$\forall x \in L \exists \xi \in L(x \in L_\xi) \iff \forall x \in L \exists \xi, y \in L(G(\xi) = y \land x \in y) \iff \forall x \in L \exists \xi \in L [(x \in L_\xi)^L].$$

\[ \Box \]

A further consequence is that $L$ is the *smallest* transitive class model of $\text{ZF}$.

**Theorem 15.8:** If $M$ is any transitive proper class model of $\text{ZF}$, then $L = L^M \subseteq M$.

The crucial fact used to prove Proposition 15.6 is that for transitive $X \in L_\xi$,

$\text{SAT}_X$ can be “reached” from $X$

within a finite number of iterations of the $P_{\text{DEF}}$-operator. (15.2)
From this it follows that $\text{SAT}_X \in L_\xi$, and hence $L_\xi$ is closed under the SAT-function.

Note that the proposition is not an immediate consequence of the definition of $L$. Although we have that $L_\zeta \subseteq L_\xi$ for all $\zeta < \xi$, it is not clear at all that in $L_\xi$ one can define the whole ensemble of the $L_\zeta$ in first order terms. The proposition says that we can definably recover them in $L_\xi$: There is a formula $\varphi_L(x_0, x_1)$ such that for limit $\xi$, given $a_0, a_1 \in L_\xi$,

$$ L_\xi \models \varphi_L[a_0, a_1] \text{ iff } a_0 \text{ is an ordinal and } a_1 = L_{a_0}. $$

As a consequence, one can devise a sentence $\varphi_{V=L}$ that identifies precisely the limit levels of the constructible hierarchy: For any transitive set $Y$,

$$ Y \models \varphi_{V=L} \text{ iff } Y = L_\xi \text{ for some limit ordinal } \xi. $$

This last fact has a far-reaching implication.

**Theorem 15.9** (Gödel Condensation Lemma): For every limit ordinal $\zeta$, every elementary substructure of $(L_\zeta, \in)$ is isomorphic to an $(L_\eta, \in)$ for some $\eta \leq \zeta$.

**The canonical well-ordering of $L$**

Every well-ordering on a transitive set $X$ can be extended to a well-ordering of $\mathcal{P}_{\text{DEF}}(X)$. Note that every element of $\mathcal{P}_{\text{DEF}}(X)$ is determined by a pair $(\psi, \bar{a})$, where $\psi$ is a set-theoretic formula, and $\bar{a} = (a_1, \ldots, a_n) \in X^{<\omega}$ is a finite sequence of parameters. For each $z \in \mathcal{P}_{\text{DEF}}(X)$ there may exist more than one such pair (i.e. $z$ can have more than one definition), but by well-ordering the pairs $(\psi, \bar{a})$, we can assign each $z \in \mathcal{P}_{\text{DEF}}(X)$ its least definition, and subsequently order $\mathcal{P}_{\text{DEF}}(X)$ by comparing least definitions. Elements already in $X$ will form an initial segment. Such an order on the pairs $(\psi, \bar{a})$ can be obtained in a definable way: First use the order on $X$ to order $X^{<\omega}$ length-lexicographically, order the formulas through their Gödel numbers, and finally say

$$(\psi, \bar{a}) < (\varphi, \bar{b}) \text{ iff } \psi < \varphi \text{ or } (\psi < \varphi \text{ and } \bar{a} < \bar{b}).$$

Based on this, we can order all levels of $L$ so that the following hold:

1. $<_{L \mid V_\omega}$ is the canonical well-order on $V_\omega$.
2. $<_{L \mid L_\zeta+1}$ is the order on $\mathcal{P}_{\text{DEF}}(L_\zeta)$ induced by $<_{L \mid L_\zeta}$.
3. $<_{L \mid L_\zeta} = \bigcup_{\zeta < \xi} <_{L \mid L_\xi}$ for a limit ordinal $\xi > \omega$. 

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It is straightforward to verify that this is indeed a well-ordering on $L$. But more importantly, for any limit ordinal $\xi > \omega$, $<_L |_{L_\xi}$ is definable over $L_\xi$. To facilitate notation, we denote the restriction of $<_L$ to some $L_\xi$ by $<_\xi$.

**Proposition 15.10:** There is a $\Sigma_1$ formula $\varphi_<(x_0,x_1)$ such that for all limit ordinals $\xi > \omega$, if $a,b \in L_\xi$,

$$L_\xi \models \varphi_<(a,b) \iff a <_{\xi} b.$$ 

The proof of this proposition is similar to the proof that the sequence of $(L_\xi)_{\xi<\xi}$ is definable in $L_\xi$. It relies on the strong closure properties of $L_\xi$ under the SAT-function.

**Theorem 15.11:** If $V = L$ then AC holds.

**The Continuum Hypothesis in $L$**

We show that the Generalized Continuum Hypothesis (GCH) holds if $V = L$.

**Theorem 15.12** (Gödel): If $V = L$, then for any ordinal $\xi$, $2^\alpha = \alpha_{\xi + 1}$.

**Proof sketch.** Suppose $A \subseteq L \cap \mathcal{K}_\xi$. Since we assume $V = L$, there exists limit $\delta > \mathcal{K}_\xi$ such that $A \in L_\delta$. Let $X = \mathcal{K}_\xi \cup \{A\}$. By choice of $\delta$, $X \subseteq L_\delta$. The Löwenheim-Skolem Theorem (and a Mostowski collapse – see Lecture 16) yields a set $M$ such that

- $(M, \in)$ is a transitive, elementary substructure of $(L_\delta, \in)$,
- $X \subseteq M \subseteq L_\delta$,
- $|M| = |X|$.

The Condensation Lemma 15.9 yields that $M = L_\xi$ for some $\xi \leq \delta$.

**Lemma 15.13:** For all $\xi \geq \omega$, $|L_\xi| = |\xi|$.

**Proof of Lemma.** We know that $\xi \subseteq L_\xi$. Hence $|\xi| \leq |L_\xi|$. To show $|\xi| \geq |L_\xi|$, we work by induction on $\xi$.

If $\xi = \delta + 1$, then by Proposition 15.1 (4), $|L_\xi| = |L_\delta| = |\delta| \leq |\xi|$.

If $\xi$ is limit, then $L_\xi$ is a union of $|\xi|$ many sets of cardinality $\leq |\xi|$ (by inductive hypothesis), hence of cardinality $\leq |\xi|$.
Applying the lemma to \( M = L_\zeta \), we obtain
\[
|\zeta| = |L_\zeta| = |M| = |X| = \kappa_\xi < \delta.
\]
Therefore, \( A \subseteq L_\zeta, |\zeta| < \kappa_\xi \), which means that every subset of \( L \cap \kappa_\xi \) appears (is constructed) at an ordinal \( < \kappa_{\xi+1} \), and therefore \( L \cap \mathcal{P}(\kappa_\xi) \subseteq L_{\kappa_{\xi+1}} \), and hence, by the Lemma,
\[
|L \cap \mathcal{P}(\kappa_\xi)| \leq |L_{\kappa_{\xi+1}}| = \kappa_{\xi+1}.
\]
\( \square \)

In the previous proof we have used the Axiom of Choice in various places (Löwenheim-Skolem, proof of the lemma), but since \( V = L \) implies AC, this is not a problem.