Sample Midterm 2 for MATH 185

Problem 1

If the following statements are true, answer "TRUE". If not, give a brief explanation why.

(1) If \( f(z) \) is analytic on a domain \( D \subseteq \mathbb{C} \), and \( \alpha \) is a closed path in \( D \), then \( \int_{\alpha} f(z)dz = 0 \).

Solution. FALSE. If \( D \) is not elementary, this is not necessarily true, e.g. \( 1/z \) on \( D = \mathbb{C} \).

(2) If \( f \) is analytic on the unit disk \( E = \{ z : |z| < 1 \} \), then there exists an \( a \in E \) such that \( |f(a)| \geq |f(0)| \).

Solution. TRUE. If \( f \) is constant, this is true. If \( f \) is non-constant, by the maximal modulus principle, \( f \) cannot take a maximal modulus on \( E \), in particular not in 0.

(3) If \( \sum_n a_n z^n \) has radius of convergence \( R \), then \( \sum_n \text{Re}(a_n) z^n \) has radius of convergence \( \geq R \).

Solution. TRUE. Since \( |\text{Re}(a_n)| \leq |a_n| \), so \( \limsup_n \sqrt[n]{|\text{Re}(a_n)|} \leq \limsup_n \sqrt[n]{|a_n|} \).

(4) If \( f \) and \( g \) are analytic on \( D \), and if they agree on a non-empty set \( S \) which is closed in \( D \), then \( f = g \) in \( D \).

Solution. FALSE. \( S \) might not have an accumulation point in \( D \). E.g. \( D = \mathbb{C}, S = \{0\} \). Then \( f(z) = z \) and \( f(z) = z^2 \) agree on \( S \), but are not identical on \( D \).

Problem 2

Compute the integral

\[ \oint_{|z|=3} \frac{\cos(\pi z)}{z^2 - 1}. \]

Solution. A partial fraction decomposition yields

\[ \frac{\cos(\pi z)}{z^2 - 1} = \frac{1}{2} \left( \frac{\cos(\pi z)}{z - 1} - \frac{\cos(\pi z)}{z + 1} \right). \]

The Cauchy integral theorem yields

\[ \oint_{|z|=3} \frac{\cos(\pi z)}{z - 1} = 2\pi i \cos(\pi) \]

and

\[ \oint_{|z|=3} \frac{\cos(\pi z)}{z + 1} = 2\pi i \cos(-\pi) = 2\pi i \cos(\pi), \]

so the value of the integral is 0.
Problem 3
Let \( f : \mathbb{C} \to \mathbb{C} \) be a non-constant, entire function. Show that \( f(\mathbb{C}) \) is dense in \( \mathbb{C} \), i.e. for every \( a \in \mathbb{C} \) and for every \( \varepsilon > 0 \), \( U_\varepsilon(a) \) contains a point from \( f(\mathbb{C}) \).

Solution. Let \( a \in \mathbb{C} \). Assume there exists an \( \varepsilon > 0 \) such that \( U_\varepsilon(a) \cap f(\mathbb{C}) = \emptyset \). Consider the function \( g(z) = 1/(f(z) - a) \). Obviously, \( g \) is entire. Since \( |f(z) - a| \geq \varepsilon \) for all \( z \in \mathbb{C} \), we have \( |g(z)| \leq 1/\varepsilon \), so \( g \) is bounded, hence constant by Liouville’s Thm. Suppose \( g \equiv c, c \in \mathbb{C} \). But then \( f(z) = \frac{1}{c} - a \) is constant, too – contradiction. ■

Problem 4
Expand \( \frac{1}{z^2 - 1} \) in a Taylor series around \( z = 0 \) and determine the radius of convergence.

Solution. Obviously,
\[
\frac{1}{z^2 - 1} = -\frac{1}{1 - z^2} = -\sum_{n=0}^{\infty} z^{2n}.
\]
The radius of convergence is 1. ■