Bonus Homework for MATH 185

Due: Monday May 7, 3:10 pm in class

Problem 1

Prove the integral representation of the Laurent series coefficients: If \( f(z) = \sum_{n=-\infty}^{\infty} a_n(z-a)^n \) in some annulus \( A = \{ z \in \mathbb{C} : r < |z-a| < R \} \), then
\[
a_n = \frac{1}{2\pi i} \oint_{|z-a|=\rho} \frac{f(z)}{(z-a)^{n+1}} \, dz,
\]
for every \( n \in \mathbb{Z} \) and \( r < \rho < R \).

Problem 2

Does there exist a closed, piecewise smooth curve \( \alpha : [0,1] \to \mathbb{C} \) such that the winding number \( \chi \), interpreted as a function \( \chi_{\alpha}(a) = \chi(\alpha; a) \) from \( \mathbb{C} \) into \( \mathbb{Z} \) takes infinitely many values, i.e. such that the set
\[
\{ \chi_{\alpha}(z) : z \in \mathbb{C} \} \subseteq \mathbb{Z}
\]
is infinite? Justify your answer.

Problem 3

Let \( a \in \mathbb{R}, a > 1 \). Set \( f_a(z) = z + a - e^z \).

(a) Show that \( f_a \) has exactly one zero in the left half-plane \( \{ z \in \mathbb{C} : \text{Re}(z) < 0 \} \).

(b) Show that this zero is on the real line.