Problem 1 [**]

For the following functions, determine the kind of singularity (removable, pole (with order), or essential) in $a$.

(a) $f(z) = \frac{z^3 + 3z - 2i}{z^2 + 1}, a = i$;  
(b) $f(z) = \frac{z}{e^z - 1}, a = 0$;  
(c) $f(z) = \exp(\exp(-1/z)), a = 0$.

Solution.  
(a) $i$ is a solution of $z^3 + 3z - 2i$, so the expression reduces to $\frac{z^{i+20/(-i)}}{z^{i+1}}$. Now it is immediately clear that the singularity in $i$ is removable.

(b) Expanding $e^z$ into a Taylor series, the expression reduces to $1/(1 + \frac{i}{z} + \frac{i^2}{z^2} + \cdots)$. Obviously, the singularity in 0 is removable.

(c) The image of $\exp(\exp(-1/z)), z \neq U_r(0)$ where $r > 0$ arbitrary, is $\mathbb{C}^*$ (periodicity of the exponential function!). Hence, by Casorati-Weierstrass, the singularity is essential. ■

Problem 2 [*]

Let $a$ be a non-essential singularity of the analytic functions $f, g : D \to \mathbb{C}$, where $D$ is a non-empty domain. Show that $a$ is also a non-essential singularity of the functions

$$f \pm g, \; f \cdot g, \; f/g, \; \text{if } g(z) \neq 0 \text{ for all } z \in D \setminus \{a\},$$

and that the following hold:

$$\begin{align*}
\ord(f \pm g; a) &\geq \min\{\ord(f; a), \ord(g; a)\}, \\
\ord(f \cdot g; a) &= \ord(f; a) + \ord(g; a) \\
\ord(f/g; a) &= \ord(f; a) - \ord(g; a)
\end{align*}$$

Solution.  
(1) Let $\ord(f; a) = -k$ and $\ord(g; a) = -l$. Wlog $k \geq l$, so $-k = \min\{\ord(f; a), \ord(g; a)\}$. We have to show that $\ord(f + g; a) \geq -k$. But since $k \geq l$, the function

$$f(z)(z-a)^k + g(z)(z-a)^k = (f(z) + g(z))(z-a)^k$$

has a removable singularity at $a$. It follows that $\ord(f + g; a) \geq -k$. The proof for $f - g$ is completely analogous.

(2) Again, assume $\ord(f; a) = -k$ and $\ord(g; a) = -l$. Then the function

$$f(z)(z-a)^l g(z)(z-a)^l = (f(z)g(z))(z-a)^{k+l}$$

has a removable singularity in $a$. This yields $\ord(f \cdot g; a) \geq \ord(f; a) + \ord(g; a)$. Furthermore, we know that for $h_j(z) = f(z)(z-a)^j$, and $g_j(z) = g(z)(z-a)^j$, the analytic extensions satisfy $h_j(a) \neq 0$ and $g_j(a) \neq 0$. If we set $h_j(a) = (f(z)g(z))(z-a)^{k+l}$, this yields $h_j(a) \neq 0$. Therefore, $\ord(f \cdot g; a) = \ord(f; a) + \ord(g; a)$

(3) The last assertion is proved analogously to (2). ■
Problem 3 [**]

Let $F_1, F_2 \subseteq \mathbb{C}$ be finite, and suppose $f : \mathbb{E} \setminus F_1 \to \mathbb{E} \setminus F_2$ is a bijective mapping such that $f$ and $f^{-1}$ are analytic. (Such a function is also called bianalytic or biholomorphic.)

(a) Show that there exists a unique extension of $f$ to a biholomorphic function $\tilde{f} : \mathbb{E} \to \mathbb{E}$.

**Solution.**

(1) Since $F_1$ is finite and $\mathbb{E}$ is open, $\mathbb{E} \setminus F_1$ is open, so all points in $F_1$ are isolated singularities of $f$. If $a \in F_1$ and $r > 0$ is such that $\delta F_1(a) \subseteq \mathbb{E}$, then $f(\delta F_1(a)) \subseteq \mathbb{E} \setminus F_2$, which is obviously a bounded set. By the Riemann removable singularity theorem, $f$ can be analytically extended to a function $g : \mathbb{E} \to \mathbb{C}$.

(2) Since $g$ is continuous and $a \in \mathbb{E}$, we know that $|g(a)| \leq 1$. But $g$ is analytic, so the image $g(\mathbb{E})$ is open by the open mapping theorem. This implies that $|g(a)| < 1$ (the image cannot have boundary points), and so $g(\mathbb{E}) \subseteq \mathbb{E}$.

(3) By assumption, the same reasoning can be applied to $f^{-1}$, yielding an analytic extension $h : \mathbb{E} \to \mathbb{E}$.

(4) Since $g$ and $h$ agree with $f$ and $f^{-1}$ on $\mathbb{E} \setminus F_1$ and $\mathbb{E} \setminus F_2$, respectively, we have that $h(z) = z$ for all $z \in \mathbb{E} \setminus F_1$ and $g(h(z)) = z$ for all $z \in \mathbb{E} \setminus F_2$. Since $h$ and $g$ are analytic functions on $\mathbb{E}$, and the sets $F_1, F_2$ are discrete in $\mathbb{E}$, we conclude by the identity theorem that the identities hold on all of $\mathbb{E}$. It follows that $g : \mathbb{E} \to \mathbb{E}$ is bianalytic, and that $h = g^{-1}$.

(b) Deduce that $F_1$ and $F_2$ have the same cardinality.

**Solution.** From part (a) it follows that $g$, as an extension of $f$, is a bijection between $\mathbb{E} \setminus F_1$ and $\mathbb{E} \setminus F_2$. Since $g$ is also a bijection between $\mathbb{E}$ and $\mathbb{E}$, it follows that $g$ is a bijection between $F_1$ and $F_2$.

Problem 4 [***]

Let $D_1, D_2, D_3 \subseteq \mathbb{C}$ be domains, $f : D_1 \to D_2$, $g : D_2 \to D_3$, and suppose $f$ is analytic and onto, and $h = g \circ f$ is analytic. Show that $g$ then must be analytic, too.

**Solution.**

(1) We show that the preimage $g^{-1}(U)$ of an open set $U \subseteq D_1$ is open in $D_2$. $h^{-1}(U) \subseteq D_1$ is open, since $h$ is analytic. As $f$ is onto, we have that $g^{-1}(U) = f(h^{-1}(U))$. But the latter set is open due to the open mapping theorem.

(2) Let $F := \{z \in D_1 : f'(z) = 0\}$. By the lemma on which the identity theorem is based, $F$ is discrete in $D_1$ ($f'$ is analytic), so $D_1 \setminus F$ is a domain (see homework 8, problem 2). By the open mapping theorem $f(D_1 \setminus F)$ is a domain.

(3) Let $z \in D_1 \setminus F$. Part 1 of the implicit function theorem implies that there exists a dotted disk $U(z) \subseteq D_1 \setminus F$ such that $f(U(z))$ is one-one. Part 2 of the implicit function theorem now yields a local biholomorphism between $U(z)$ and $f(U(z))$, which is an open set. Using the identity theorem these local biholomorphisms for any $z \in D_1 \setminus F$ combine into a bianalytic mapping $D_1 \setminus F \to f(D_1 \setminus F)$. Now, on $f(D_1 \setminus F)$ we can write $g = h \circ f^{-1}$, which as a composition of analytic functions is analytic.

(4) Now let $w \in D_2 \setminus f(D_1 \setminus F)$. Since $f$ is onto, there exists a $z_0 \in D_1$ such that $f(z_0) = w$. Obviously, $z_0 \in F$. Since $F$ is discrete in $D_1$, we can find $\varepsilon > 0$ such that $\delta F_1(z_0) \subseteq D_1 \setminus F$. Then $f(\delta F_1(z_0)) \subseteq f(D_1 \setminus F)$ and open, so we can find a $\delta > 0$ such that $\delta F_1(w) \subseteq f(\delta F_1(z_0))$. Now consider $g(\delta F_1(w))$. We know that $f^{-1}(g(\delta F_1(w)))$ is contained in $\delta F_1(z_0)$. By analyticity of $h$, $h(\delta F_1(z_0))$ is bounded, so $g(\delta F_1(w)) \subseteq h(\delta F_1(z_0))$ is bounded. Now the Riemann removable singularity condition implies that $g$ can be analytically extended to $w$.

\[\blacksquare\]