Problem 1: WLOG we can assume that $0 \in U$.

If $h(z) = q(z^n)$ is analytic, then it has a power series representation

$$h(z) = \sum_{m=0}^{\infty} h_m z^m$$

that converges in a nbhd of 0.

We want to show that $h_m = 0$ if $m$ is not a multiple of $n$, because then we have

$$h(z) = h_n z^n + h_{2n} z^{2n} + h_{3n} z^{3n} + \ldots \quad n \geq 0$$

and

$$q(z) = h_n z^n + h_{2n} z^{2n} + h_{3n} z^{3n} + \ldots$$

so $q$ is analytic, since it is representable as a convergent power series.

Let $f = e^{2\pi i / n}$. Then $h(fz^n) = h(z^n) = q(z^n) = h(z)$.

We now compute $h_m$ via the Cauchy integral formula:

$$h_m = \frac{1}{2\pi i} \oint_{|z|=\varepsilon} \frac{h(z)}{z^{m+1}} \, dz$$

where $\varepsilon$ is sufficiently small.
Now substitute $z \mapsto w^m$. Then

\[ h_m = \frac{1}{2\pi i} \oint_{|w|=\varepsilon} \frac{h(w^m)}{(w^m)^{m+1}} \, dw = \frac{1}{2\pi i} \oint_{|w|=\varepsilon} \frac{h(w)}{w^{m+1}} \cdot \frac{1}{w^m} \, dw \]

\[ = \frac{h_m}{\varepsilon^m} \]

Hence \( h_m (1 - \varepsilon^m) = 0 \).

We have that \( \varepsilon^m = 1 \iff m \) is a multiple of \( n \),

so \( h_m = 0 \) if \( m \) is not a multiple of \( n \).
Problem 2: If \( f \) is entire, then the Taylor series of \( f \) around 0 has radius of convergence \( \infty \).

\[
f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n
\]

Termwise differentiation yields:

\[
f''(z) = \sum_{n=0}^{\infty} \frac{f^{(n+2)}(0)}{n!} z^n
\]

If \( f(z) + f''(z) = 0 \), then by comparing coefficients we get

\[
f^{(n)}(0) = -f^{(n+2)}(0) \quad \forall n \geq 0.
\]

This allows us to write

\[
f(z) = \sum_{n=0}^{\infty} \frac{f(0)}{2n!} (-1)^n z^{2n} + \sum_{n=0}^{\infty} \frac{f'(0)}{(2n+1)!} (-1)^n z^{2n+1}
\]

\[
= f(0) \cos z + f'(0) \sin z.
\]

Hence all functions \( f \) s.t. \( f + f'' = 0 \) are precisely the functions of the form

\[
\alpha \cos z + \beta \sin z \quad \alpha, \beta \in \mathbb{C}
\]