Problem 2

The integration path is

\[ \int_{\alpha_1} f(z) \, dz = \frac{1}{2i} \left( \int_{\alpha_1} \frac{1}{z-i} \, dz - \int_{\alpha_1} \frac{1}{z+i} \, dz \right) \]

Partial fraction decomposition of \( f \):

\[ f(z) = \frac{1}{2i} \left( \frac{1}{z-i} - \frac{1}{z+i} \right) \]

Hence

\[ \int_{\alpha_1 \oplus \alpha_2} f(z) \, dz = \frac{1}{2i} \left( \int_{\alpha_1 \oplus \alpha_2} \frac{1}{z-i} \, dz - \int_{\alpha_1 \oplus \alpha_2} \frac{1}{z+i} \, dz \right) \]

The mapping \( \mathbb{E} \rightarrow \frac{1}{z+i} \) is analytic in a star-shaped domain containing the full upper semi-disk of radius \( R \).

Hence, by the Cauchy integral theorem,

\[ \int_{\alpha_1 \oplus \alpha_2} \frac{1}{z+i} \, dz = 0 \]

We argue that

\[ \int_{\alpha_1 \oplus \alpha_2} \frac{1}{z-i} \, dz = \oint_{|z|=R} \frac{1}{z-i} \, dz \]

This can be justified by an argument similar to the one in class:

\[ \int = 0 \]
Thus, \[ \int_{\alpha_1 \to \alpha_2} f(z) \, dz = \frac{1}{2i} \int_{|z|=R} \frac{1}{z-i} \, dz = \frac{2\pi i}{2i} = \pi. \]

To prove the second assertion, we use the standard estimate.

\[ \left| \int_{\alpha_{1}^{(R)}} f(z) \, dz \right| \leq \max_{|z|=R} |f(z)| \cdot \min_{0 \leq t \leq \pi} \left| 1/(Re^{it})^2 + 1 \right| \]

\[ = \pi R \cdot \frac{1}{\min_{0 \leq t \leq \pi} |1/(Re^{it})^2 + 1|} \]

\[ \leq \pi R \cdot \frac{1}{R^2 - 1} \quad R \to \infty \quad \longrightarrow \quad 0 \]

Combining the two results, we obtain

\[ \lim_{R \to \infty} \int_{-R}^{R} \frac{1}{1+tt^2} \, dt = \lim_{R \to \infty} \left[ \int_{\alpha_{1}^{(R)}} \frac{1}{1+z^2} \, dz + \int_{\alpha_{2}^{(R)}} \frac{1}{1+z^2} \, dz \right] \]

\[ = \frac{\pi}{2} + 0 = \frac{\pi}{2}. \]
Problem 3: (a) partial fraction decomposition:

\[ \frac{z^2 - 1}{z^2 + 1} = 1 + \frac{i}{z - i} - \frac{i}{z + i} \]

Hence

\[ \oint \frac{z^2 - 1}{z^2 + 1} \, dz = \oint 1 + i \oint \frac{1}{z - i} - i \oint \frac{1}{z + i} = 0 \]

(b) \[ \oint \frac{\sin(\exp(z))}{z} \, dz = \sin(\exp(0)) \cdot 2\pi i = 2\pi i \cdot \sin(1) \]

(c) \[ \oint \left( \frac{z}{z - 1} \right)^n \, dz = n! \cdot 1 \cdot \frac{2\pi i}{(n-1)!} \]

with \( f(z) = z^n \)

\[ = n! \cdot 1 \cdot \frac{2\pi i}{(n-1)!} = 2\pi i \cdot n \]
Problem 4: Proof by induction.

\( n = 1 \): The verification of the identity

\[
(*) \quad \frac{F_1(z) - F_1(a)}{z - a} - \frac{1}{2\pi i} \int_C \frac{\psi(s)}{(s-a)^2} \, ds = \frac{z - a}{2\pi i} \int_C \frac{\psi(s)}{(s-a)^2(s-z)} \, ds
\]

is straightforward. (Note that the identity

\[
(\ast \ast) \quad \frac{1}{(s-z)^m} = \frac{1}{(s-z)^{m-1}(s-a)} + \frac{z - a}{(s-z)^m(s-a)}
\]

also holds for \( m = 2 \).)

It remains to show that the RHS of \((*)\) goes to 0 for \( z \to a \).

For this, it is sufficient to show that the integral is bounded by a constant as \( z \to a \).

Note that \( \psi \) is continuous, so \( |\psi(s)| \) attains a maximum on \( \gamma \).

Furthermore, \( \alpha([0,1]) \) is closed, so if \( \alpha \notin \text{Im}(\alpha) \), then

\[
\text{there exists an } \epsilon > 0 \text{ s.t. } \mathcal{U}_{2\epsilon}(a) \cap \text{Im}(\alpha) = \emptyset.
\]

Hence, if all \( z \in \mathcal{U}_{\epsilon}(a) \), the distance of \( z \) to the curve is at least \( \epsilon \).
Hence we can infer
\[
\left| \frac{\psi(\xi)}{(\xi-a)^2} \right| \leq \frac{M}{\varepsilon^3} \quad \text{for all } \xi \in \mathcal{U}_\varepsilon(a)
\]
\[\xi \text{ on } \kappa.\]

This yields that
\[
\frac{z-a}{2\pi i} \int_{\kappa} \frac{\psi(\xi)}{(\xi-a)^2 (\xi-\xi)} \, d\xi \quad \overset{z \to a}{\longrightarrow} \quad 0
\]

\[n \to n+1: \]

\[
\frac{F_{n+1}(z) - F_{n+1}(a)}{z-a} = \frac{1}{2\pi i} \left[ \frac{1}{(z-\xi)^{n+1}} \right] \int_{\kappa} \frac{\psi(\xi)}{(z-\xi)} \, d\xi - \frac{1}{2\pi i} \left[ \frac{1}{(z-\xi)^{n+1}} \right] \int_{\kappa} \frac{\psi(\xi)}{(z-a)} \, d\xi
\]

\[\ (\star \star) \]

\[
= \frac{1}{2\pi i} \left[ \psi(\xi) \right] \frac{1}{(z-\xi)^{n+1}} \, d\xi - \frac{1}{2\pi i} \left[ \psi(\xi) \right] \frac{1}{(z-\xi)^{n+1}} \, d\xi + \frac{1}{2\pi i} \frac{\psi(\xi)}{z-a} \, d\xi
\]

Let \(\tilde{\psi}(\xi) = \frac{\psi(\xi)}{z-a} \). This mapping is continuous on \(\kappa\), since \(a\) is not on \(\kappa\). Write \(\tilde{F}_n(z) = \frac{1}{2\pi i} \int_{\kappa} \frac{\tilde{\psi}(\xi)}{(z-\xi)^n} \, d\xi\). Thus we have

\[
\frac{F_{n+1}(z) - F_{n+1}(a)}{z-a} = \frac{\tilde{F}_n(z) - \tilde{F}_n(a)}{z-a} + \tilde{F}_{n+1}(z) \quad (\dagger)
\]
Using the induction hypothesis on $\tilde{F}_n$, we see that

$$\frac{\tilde{F}_n(z) - \tilde{F}_n(a)}{z-a} \xrightarrow{z \to a} \tilde{F}_n(a) = n \cdot \tilde{F}_{n+1}(a)$$

$$= n \cdot \int_{\alpha} \frac{\varphi(\zeta)}{(\zeta-a)^{n+1}} d\zeta = n \, F_{n+2}(a)$$

It remains to show that $\tilde{F}_{n+1}(z)$ is continuous in $a$, because this implies

$$\tilde{F}_{n+1}(z) \xrightarrow{z \to a} \tilde{F}_{n+1}(a) = \int_{\alpha} \frac{\varphi(\zeta)}{(\zeta-a)^{n+2}} d\zeta = F_{n+2}(a),$$

and so we have for (4)

$$\frac{\tilde{F}_{n+1}(z) - \tilde{F}_{n+1}(a)}{z-a} \xrightarrow{z \to a} n \cdot F_{n+2}(a) + F_{n+2}(a) = (n+1) \, F_{n+2}(a)$$

It is clear that it suffices to show that $\tilde{F}_{n+1}$ is continuous in a, since $\varphi$ is an arbitrary continuous function on $\text{Im}(\alpha)$.

Applying identity (*) once more, we get

$$\tilde{F}_{n+1}(z) - \tilde{F}_{n+1}(a) = \frac{1}{2\pi i} \int_{\alpha} \frac{\varphi(\zeta)}{(\zeta-z)^{n+1}} d\zeta - \frac{1}{2\pi i} \int_{\alpha} \frac{\varphi(\zeta)}{(\zeta-a)^{n+1}} d\zeta$$

$$= \frac{1}{2\pi i} \int_{\alpha} \varphi(\zeta) \left( \frac{1}{(\zeta-z)^{n}} + \frac{z-a}{(\zeta-z)^{n+1} (\zeta-a)} \right) d\zeta - \frac{1}{2\pi i} \int_{\alpha} \frac{\varphi(\zeta)}{(\zeta-a)^{n+1}} d\zeta$$
\[
= \frac{1}{2\pi i} \int_{\infty}^{\gamma} \frac{\varphi(\xi)}{(\xi-z)\xi} \, d\xi - \frac{1}{2\pi i} \int_{\gamma}^{\infty} \frac{\varphi(\xi)}{(\xi-a)\xi} \, d\xi + \frac{z-a}{2\pi i} \int_{\infty}^{\gamma} \frac{\varphi(\xi)}{(\xi-z)^{n+1}(\xi-a)\xi} \, d\xi
\]

\[
\Rightarrow \quad F_n(z) - F_n(a) + \frac{z-a}{2\pi i} \int_{\infty}^{\gamma} \frac{\varphi(\xi)}{(\xi-z)^{n+1}(\xi-a)} \, d\xi
\]

The inductive hypothesis yields that \( \tilde{F}_n(z) \to a \tilde{F}_n(a) \).

Furthermore, a similar argument as in the case \( n=1 \) bounds the integral \( \left| \int_{\infty}^{\gamma} \frac{\varphi(\xi)}{(\xi-z)^{n+1}(\xi-a)} \, d\xi \right| \) by \( \frac{M}{z^{n+2}} \cdot L(a) \), so the second term goes to 0 as \( z \to a \).

This proves the continuity. \( \square \)

Now we use the lemma to infer the Cauchy integral formula:

The case \( n=0 \) has been proved in class:

\[ n \to n+1: \quad \frac{f^{(n)}(z) - f^{(n)}(a)}{z-a} = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\xi)}{(\xi-z)^{n+1}} \, d\xi = \frac{n!}{2\pi i} \int_{\gamma} \frac{f(\xi)}{(\xi-z)^{n+1}} \, d\xi \]

\[ \Rightarrow \quad \frac{1}{z-a} \left[ F_{n+1}(z) - F_{n+1}(a) \right] \quad \text{(where } \varphi := f \text{)} \]

\[ \Rightarrow \quad n! \, F_{n+1}'(a) = n! \cdot (n+1) \, F_{n+2}(a) = \frac{n+1}{2\pi i} \int_{\gamma} \frac{f(\xi)}{(\xi-a)^{n+2}} \, d\xi \]

\[ \square \]
Problem 5: (a) Assume \( |f(z)| > e^{1z|} \) for all \( z \in \mathbb{C} \). Then \( \left| \frac{1}{f(z)} \right| \leq e^{-1z|} \leq 1 \), and since \( f \) does not have any zeros, \( \frac{1}{f} \) is an entire function. Since it is bounded, \( \frac{1}{f} \) must be constant, by Liouville's theorem. Hence \( f \) is constant, too, contradiction.

(b) If \( A \) is empty, we have \( |f(z)| \geq 1 \) for all \( z \in \mathbb{C} \). Apply the same reasoning as in (a) to conclude that \( f \) must be constant.

(c) Assume \( A \subseteq \mathbb{U}_1(0) \). If \( f \) does not have a zero, \( \frac{1}{f} \) is entire. We have that \( z \notin \mathbb{U}_1(0) \Rightarrow |f(z)| \geq 1 \), hence \( z \notin \mathbb{U}_1(0) \Rightarrow \frac{1}{|f(z)|} \leq 1 \).

On the other hand, \( \frac{1}{f} \) is continuous, so the image of the compact set \( \mathbb{U}_1(0) \) under \( \frac{1}{f} \) is bounded, so \( \frac{1}{f} \) is bounded on all of \( \mathbb{C} \). Again, Liouville's theorem implies that \( f \) is constant.

(d) Consider for example \( f(z) = \exp(z) \). \( \square \)