Problem 1

For \( r = 1, 3, 5 \) compute the following integral:

\[
\oint_{|\xi - 2| = r} \frac{\exp(\xi^2)}{(\xi^2 - 6\xi)} \, d\xi.
\]

**Solution.** For \( r = 1 \), the function \( \frac{\exp(\xi^2)}{(\xi^2 - 6\xi)} \) is analytic on the disk \(|z - 2| \leq 1\), so by the Cauchy integral theorem the integral is 0.

For \( r = 3 \), we set \( f(z) = \exp(\xi^2)\frac{(\xi^2 - 6\xi)}{6\xi} \). \( f \) is analytic on \(|z - 2| \leq 3\), so the Cauchy integral formula yields:

\[
\oint_{|\xi - 2| = 3} \frac{\exp(\xi^2)}{(\xi^2 - 6\xi)} \, d\xi = 2\pi i f(0) = \pi i/3.
\]

For \( r = 5 \), we cannot apply this strategy any longer, since both 'critical points' \( z = 0, 6 \) lie inside the disk \(|z - 2| \leq 5\). We resort to a partial fraction decomposition:

\[
\exp(z^2) = \frac{-\exp(z^2)}{6z} + \frac{\exp(z^2)}{6(z - 6)}.
\]

Now we can apply the Cauchy integral formula:

\[
\oint_{|\xi - 2| = 5} \frac{\exp(\xi^2)}{(\xi^2 - 6\xi)} \, d\xi = -\oint_{|\xi - 2| = 5} \frac{\exp(\xi^2)}{6z} \, d\xi + \oint_{|\xi - 2| = 5} \frac{\exp(\xi^2)}{6(z - 6)} \, d\xi = -2\pi i e^0/6 + 2\pi i e^{36}/6 = \pi i (e^{36} - 1)/3.
\]

Problem 2

We will prove that the Cauchy integral formula holds in a much more general form. In particular,

\[
f'(z) = \frac{1}{2\pi i} \oint_{|\xi - z_0| = r} \frac{f(\xi)}{(\xi - z)^2} \, d\xi,
\]

for every \( z \) with \(|z - z_0| < r\).

Use this to show that if \( f : \mathbb{C} \to \mathbb{C} \) is analytic and \( \lim_{z \to \infty} f(z)/z = 0 \), then \( f \) is constant.

**Solution.** First observe that \( |f(z)/z| \to 0 \) for \( |z| \to \infty \) implies \( |f(z)/(z - a)| \to 0 \) for \( |z| \to \infty \). Let \( a \in \mathbb{C} \). We claim that \( f'(a) = 0 \), so it follows that \( f \) is constant. Let \( \varepsilon > 0 \). By assumption, there exists an \( R > 0 \) such that \( |f(z)/(z - a)| < \varepsilon \) for all \( z \) such that \(|z - a| > R\). The Cauchy integral formula says that

\[
f'(a) = \frac{1}{2\pi i} \oint_{|\xi - a| = R} \frac{f(\xi)}{\xi - a} \, d\xi.
\]

We can use the standard estimate to obtain

\[
|f'(a)| \leq \frac{1}{2\pi} \frac{\varepsilon}{R} 1(|\xi - a| = R) = \frac{1}{2\pi} \frac{\varepsilon}{R} 2\pi R = \varepsilon.
\]

Since \( \varepsilon \) was arbitrary, the result follows.