Problem 1
Verify the identities
\[
\cos(z) = \frac{\exp(iz) + \exp(-iz)}{2} \quad \sin(z) = \frac{\exp(iz) - \exp(-iz)}{2i}
\]
and use them to show that \(\sin(C) = C\) and \(\cos(C) = C\).

Solution. We use the fact that \(\exp(C) = C^*\). Given \(z \in C\), we want to find \(w \in CC\) such that
\[
z = \sin(w) = \frac{\exp(iw) - \exp(-iw)}{2i}.
\]
Substitute \(a = \exp(iw)\), hence \(\exp(-iw) = a^{-1}\). The equation
\[
a - a^{-1} = 2iz
\]
transforms into \(a^2 - 2iz a - 1 = 0\) (note that \(a \neq 0\)), which has a solution \(a \neq 0\). The mapping \(z \mapsto iz\) is a bijection of \(C\), so there exists a \(w\) such that \(\exp(iw) = a\). Then it holds that \(\sin(w) = z\).

Determine all \(z \in C\) such that \(\sin(z) = 12i/5\).

Solution. It holds that \(\sin(z) = 12i/5\) iff \(\exp(iz) - \exp(-iz) = -24/5\). Substitution \(\exp(iz) \mapsto a\) yields the equation \(a^2 + 24a/5 - 1 = 0\), which has solutions \(1/5\) and \(-5\). Hence
\[
\sin(z) = 12i/5 \iff |\exp(iz) = 1/5 or \exp(iz) = -5|
\]
\[
\iff |iz = \log(1/5) + i \text{Arg}(1/5) + 2\pi ik or iz = \log(-5) + i \text{Arg}(-5) + 2\pi ik for some } k \in \mathbb{Z}
\]
\[
\iff |z = i \log(5) - 2\pi k or z = i \log(5) - \pi - 2\pi k for some } k \in \mathbb{Z}
\]

Problem 2
Determine all points at which the function
\[
f : C \to C , \quad z = x + iy \mapsto f(z) := x^3y^2 + ix^2y^3 , \quad x , y \in \mathbb{R}
\]
is complex differentiable.

Solution. Obviously, we have \(u(x,y) = x^3y^2\) and \(v(x,y) = x^2y^3\). The partial derivatives are \(\partial_1 u = \partial_2 v = 3x^2y^2\), \(\partial_2 u = 2x^3y\) and \(\partial_1 v = 2xy^3\). \(f\) is obviously differentiable in the sense of real analysis, so \(f\) is complex differentiable in exactly those points where the Cauchy-Riemann differential equations hold. This is true here iff \(\partial_2 u(x,y) = -\partial_1 v(x,y)\), or \(2xy^3 = -2x^3y\), which is easily seen to hold iff \(x = 0\) or \(y = 0\). So \(f\) is complex differentiable exactly on the coordinate axes \(\mathbb{R}\) and \(i\mathbb{R}\).

Does there exist a non-empty open set \(D \subseteq \mathbb{C}\) on which \(f\) is analytic?

Solution. No, since this would require \(f\) to be complex differentiable on a full disc \(U_\delta(z)\) for some \(z \in \mathbb{C}\). But obviously, the coordinate axes do not contain a full open disc for any radius \(\delta\).

Problem 3
Show that the function \(f : C \to C\),
\[
f(z) = \begin{cases} 
\exp(-1/z^2) & \text{for } z \neq 0, \\
0 & \text{for } z = 0,
\end{cases}
\]
satisfies the Cauchy-Riemann equations for all \(z \in \mathbb{C}\) and is complex differentiable for all \(z \in \mathbb{C^*} = \mathbb{C}\{0\}\), but not at the origin.
Solution. If \( z \neq (0,0) \), then \( f \) is analytic in \( z \) as a composition of two analytic functions, so the Cauchy-Riemann equations hold. 

If \( z = (0,0) \), we compute the partial derivatives: In all cases (note \( i^4 = 1 \)), this reduces to computing the limit 

\[
\lim_{h \to 0} \frac{\exp(-1/h^4)}{h}.
\]

But it is known from calculus/analysis that this goes to 0, hence the Cauchy-Riemann hold for \( z = (0,0) \), too.

Nevertheless, \( f \) is not differentiable at the origin, since this would imply that \( f \) is bounded in a neighborhood of 0. But consider the directional limit 

\[
\lim_{t \searrow 0} f(t(1+i)).
\]

Since \((1+i)^4 = -4\), we have \( f(t(1+i)) = \exp(1/(4t^4)) \), which clearly is unbounded if \( t \searrow 0 \). 

Problem 4

(a) Let \( D = \mathbb{C}^* \) and \( u : D \to \mathbb{R} \) with \( u(x,y) = \frac{x}{x^2+y^2} \). Show that \( u \) is harmonic and find an analytic function \( f : D \to \mathbb{C} \) with \( \text{Re}(f) = u \).

Solution. Verification that \( \Delta(u) = 0 \) is a little cumbersome, but routine. A possible \( f \) is \( 1/z \). 

(b) Given two harmonic functions \( u_1, u_2 : \mathbb{R}^2 \to \mathbb{R} \), prove or disprove (counterexample) the following statements.

1.) \( u_1 + u_2 \) is harmonic.
2.) \( u_1 \cdot u_2 \) is harmonic.

Solution. \( u_1 + u_2 \) is harmonic, because the partial differential operator is linear. Consider \( u_1 = u_2 = x \). Then \( \Delta(u_1) = \Delta(u_2) = 0 \), but \( u_1 \cdot u_2 = x^2 \), which is not harmonic. 
