Problem 1

Let $D \subseteq \mathbb{C}$ be a domain, $a \in D$, and suppose $f, g : D \setminus \{a\} \to \mathbb{C}$ are analytic functions with non-essential singularities in $a$. Show that the following assertions hold.

(a) If $a$ is a pole of order $k$ (i.e. $\text{ord}(f; a) = -k$), then

$$\text{Res}(f; a) = \lim_{z \to a} \frac{h^{(k-1)}(z)}{(k-1)!}$$

where $h(z) = (z-a)^k f(z)$.

**Solution.** $f$ has a Laurent series near $a$ of the form

$$f(z) = \sum_{n=-k}^\infty a_n(z-a)^n.$$

The function $h(z)(z-a)^k$ has removable singularity at $a$, and for the Taylor series of $h$ near $a$ it holds that

$$h(z) = \sum_{n=0}^\infty a_{n-k}(z-a)^n.$$

But we also know that the Taylor series of an analytic function is of the form

$$h(z) = \sum_{n=0}^\infty \frac{h^{(n)}(a)}{n!}(z-a)^n.$$

Now the desired equality follows by comparing coefficients.

(b) If $\text{ord}(f; a) = l$ and $\text{ord}(g; a) = l + 1, l \geq 0$, then

$$\text{Res}(f/g; a) = (l + 1) \frac{f^{(l)}(a)}{g^{(l+1)}(a)}.$$

**Solution.** Near $a$, we have $f(z) = \sum_{n=0}^\infty a_n(z-a)^n$ and $g(z) = \sum_{n=0}^\infty b_n(z-a)^n$, where $a_n, b_n \neq 0$. The function $h(z) = z f(z)$ has a removable singularity in $a$ and $h(0) \neq 0$, so $f/g$ has a pole of order $1$ in $a$. Hence $\text{Res}(f/g; a) = h(a)$. It holds that

$$h(a) = \frac{a_l}{b_{l+1}} = \frac{f^{(l)}(a)/l!}{g^{(l+1)}(a)/(l+1)!} = (l + 1) \frac{f^{(l)}(a)}{g^{(l+1)}(a)}.$$

(c) If $f \neq 0$, then $\text{Res}(f'/f; a) = \text{ord}(f; a)$.

**Solution.** Assume first $\text{ord}(f; a) = 0$. Then $\text{ord}(f'; a) = 0$, and $f'/f$ is analytically extendable to $a$, hence $\text{Res}(f'/f; a) = 0$.

Assume now $\text{ord}(f; a) = k > 0$. Then $\text{ord}(f'; a) = k - 1$. We can apply part (b) with $f = f'$ and $g = f$ and obtain $\text{Res}(f'/f; a) = k f'(a)/f(a) = k = \text{ord}(f; a)$.

Finally, assume $\text{ord}(f; a) = -k, k > 0$. Then the Laurent representation of $f'$ in a punctured disk around $a$ is

$$f'(z) = -k a_{-k}(z-a)^{-k-1} - \cdots - a_{-1}(z-a)^{-2} + a_1 + a_2(z-a) + \cdots$$

where $a_{-k}(z-a)^{-k} - \cdots - a_{-1}(z-a)^{-2} + a_0 + a_1(z-a) + \cdots$ is the Laurent series for $f$. We have that

$$h(z) := (z-a) f'(z) = \frac{(z-a)^{k+1}}{f(z)} f(z) = \frac{-ka_{-k} + (-k+1)a_{-k+1}(z-a) + \cdots}{a_{-k} + a_{-k+1}(z-a) + \cdots},$$

which has a removable singularity in $a$. Hence

$$\text{Res}(f'/f; a) = h(a) = -\frac{ka_{-k}}{a_{-k}} = -k = \text{ord}(f; a).$$
Problem 2

Compute the residues of the following functions at the indicated points:

(a) \( \frac{\exp(z^2)}{z - 1}, a = 1 \)

(b) \( \frac{\exp(z^2)}{(z - 1)^2}, a = 1 \)

(c) \( \left( \frac{\cos(z) - 1}{z} \right)^2, a = 0 \)

(d) \( \frac{z^2}{z^4 - 1}, a = \exp(\pi i/2) \)

(e) \( \frac{\exp(z) - 1}{\sin(z)}, a = 0 \)

(f) \( \frac{1}{\exp(z) - 1}, a = 0 \)

(g) \( \frac{z + 2}{z^2 - 2z}, a = 0 \)

(h) \( \frac{1 + \exp(z)}{z^4}, a = 0 \)

(i) \( \frac{\exp(z)}{(z^2 - 1)^2}, a = 1 \)

Solution.

(a) Pole of order 1 in 1, hence residue given by \( h(1) \) where \( h(z) = f(z)(z - 1) \), thus \( \text{Res}(f; 1) = \exp(1) = e \).

(b) Pole of order 2 in 1, hence residue given by \( h'(1) \), so \( \text{Res}(f; 1) = \exp(1^2) 2 = 2e \).

(c) Pole of order 2 in 0, hence residue given by \( h'(1) = 2(\cos(1) - 1) \sin(1) \), so \( \text{Res}(f; 0) = 0 \).

(d) Pole of order 1 in \( i \), hence residue given by \( h(i) \), so \( \text{Res}(f; 1) = 1/|i(1)(i+1)(2i)| = 1/4i \).

(e) \( \text{ord}(\sin, 0) = 1 \), \( \text{ord}(\exp(z) - 1; 0) \geq 0 \), hence residue given by \( g(0)/h'(0) = (\exp(0) - 1)/\cos(0) = 0 \).

(f) \( \text{ord}(\exp(z) - 1; 0) = 1 \), so \( \text{Res}(f; 0) = 1/\exp(0) = 1 \).

(g) Pole of order 1 in 0, so \( \text{Res}(f; 0) = (0 + 2)/(0 - 2) = -1 \).

(h) Pole of order 4 in 0, so \( \text{Res}(f; 0) = h(3)(0)/3! \) where \( h(z) = \exp(z) + 1 \). Thus \( \text{Res}(f; 0) = \exp(0)/3! = 1/6 \).

(i) Pole of order 2 in 0, so \( \text{Res}(f; 1) = h'(1) \) where \( h(z) = \exp(z)/(z + 1)^2 \). Hence \( \text{Res}(f; 1) = e(2^2 - 2(1 + 1))/2^4 = 0 \).

Problem 3

Evaluate the integral

\[ \oint_{|z|=7} \frac{1 + z}{1 - \cos(z)} \, dz. \]

Solution. \( 1 - \cos(z) \) is 0 if and only if \( z \) is an integer multiple of \( 2\pi \). From the Taylor series for \( \cos(z) \) we conclude that \( 2\pi k, k \in \mathbb{Z}, \) is a zero of order 2 for \( 1 - \cos(z) \), hence a pole of order 2 for \( f(z) := (1 + z)/(1 - \cos(z)) \). Only the poles \( a_1 = -2\pi, a_2 = 0, a_3 = 2\pi \) lie inside the circle of radius 7 around 0.

We now compute the residue of \( f \) at these points. Let

\[ f(z) = a_{-2}(z - a_2)^{-2} + a_{-1}(z - a_1)^{-1} + a_0 + \cdots \]

be the Laurent series of \( f \) around \( a_j \). It the holds that

\[ 1 + z = (1 + a_j) + (z - a_j) = (a_{-2}(z - a_2)^{-2} + a_{-1}(z - a_1)^{-1} + a_0 + \cdots)((z - a_j)^2/2! - (z - a_j)^4/4! + \cdots) \]

Expanding the right hand side and comparing coefficients, we obtain

\[ 2 = a_{-1} = \text{Res}(f; a_j). \]

Hence, by the residue theorem (the winding number is clearly 1),

\[ \oint_{|z|=7} \frac{1 + z}{1 - \cos(z)} \, dz = 2\pi i(2 + 2) = 12\pi i. \]
Problem 4

Do exercise III.6.2 on page 172. Use the hint. Justify your steps carefully and precisely.

Solution. The winding number at $a$ is defined as

$$\chi(\alpha; a) = \frac{1}{2 \pi i} \int_{\alpha} \frac{1}{z-a} \, dz.$$ 

Define the function $G : [0,1] \to \mathbb{C}$ by

$$G(t) = \int_{0}^{t} \frac{\alpha'(s)}{\alpha(s)-a} \, ds.$$ 

By definition of the path integral in $\mathbb{C}$, $G(1) = 2\pi i \chi(\alpha; a)$.

Furthermore, define $F(t) := (\alpha(t) - a) \exp(-G(t))$. $F$ is differentiable, since $\alpha$ is smooth and $G$ is differentiable by the fundamental theorem of calculus.

It holds that

$$F'(t) = \alpha'(t) \exp(-G(t)) + (\alpha(t) - a) \exp(-G(t))(-G'(t)).$$

The fundamental theorem of calculus yields that

$$G'(t) = \frac{d}{dt} \int_{0}^{t} \frac{\alpha'(s)}{\alpha(s)-a} \, ds = \frac{\alpha'(s)}{\alpha(s)-a}.$$ 

This yields $F'(t) = 0$ for all $t \in [0,1]$. Since $[0,1]$ is connected, $F$ is constant. In particular, it holds that $F(0) = F(1)$. By definition of $G$, $G(0) = 0$, so $F(0) = (\alpha(0) - a)$. Since $\alpha$ is a closed curve, we have $\alpha(0) = \alpha(1)$, and thus

$$(\alpha(0) - a) = F(0) = F(1) = (\alpha(0) - a) \exp(-G(1)).$$

Since $\alpha(0) - a \neq 0$, we must have $\exp(-G(1)) = 1$, which means $G(1)$ is an integer multiple of $2\pi i$. From this it follows immediately that $\chi(\alpha; a) = G(1)/2\pi i$ is an integer. \hspace{1cm} $$ Since $\alpha \notin \text{Image}(\alpha)$, and $\alpha$ is a smooth curve, the function $t \mapsto$ is well-defined and integrable.