Homework 11 for MATH 185

Due: Friday April 27, 3:10 pm in class

Problem 1

Let $D \subseteq \mathbb{C}$ be a domain, $a \in D$, and suppose $f, g : D \setminus \{a\} \to \mathbb{C}$ are analytic functions with non-essential singularities in $a$. Show that the following assertions hold.

(a) If $a$ is a pole of order $k$ (i.e. $\text{ord}(f; a) = -k$), then

$$\text{Res}(f; a) = \lim_{z \to a} \frac{h^{(k-1)}(z)}{(k-1)!},$$

where $h(z) = (z - a)^k f(z)$.

(b) If $\text{ord}(f; a) = l$ and $\text{ord}(g; a) = l + 1$, $l \geq 0$, then

$$\text{Res}(f/g; a) = (l + 1) \frac{f^{(l)}(a)}{g^{(l+1)}(a)}.$$

(c) If $f \not\equiv 0$, then $\text{Res}(f'/f; a) = \text{ord}(f; a)$.

Problem 2

Compute the residues of the following functions at the indicated points:

1. \(\frac{\exp(z^2)}{z-1}, a = 1\)
2. \(\frac{\exp(z^2)}{(z-1)^2}, a = 1\)
3. \(\left(\frac{\cos(z) - 1}{z}\right)^2, a = 0\)
4. \(\frac{z^2}{z^4 - 1}, a = \exp(\pi i/2)\)
5. \(\frac{\exp(z) - 1}{\sin(z)}, a = 0\)
6. \(\frac{1 + \exp(z)}{z^4}, a = 0\)
7. \(\frac{\exp(z)}{(z^2 - 1)^2}, a = 1\)

Problem 3

Evaluate the integral

$$\oint_{|z|=7} \frac{1 + z}{1 - \cos(z)} \, dz.$$

Problem 4

Do exercise III.6.2 on page 172. Use the hint. Justify your steps carefully and precisely.