EFFECTIVE HAUSDORFF DIMENSION

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§1. Introduction. Generally speaking, Hausdorff dimension is a generalization of Lebesgue measure. In the early 20th century, Hausdorff [8] used Caratheodory's construction of measures to define a whole family of outer measures. So, for examining a set of a peculiar topological or geometrical nature for which Lebesgue measure or topological dimension are too coarse to investigate this features, one may 'pick' a measure from this family that is suited to study this particular set. This is one reason why Hausdorff measure and dimension became a prominent tool in fractal geometry.

Hausdorff dimension is extensively studied in the context of dynamical systems, too. On the Cantor space, the space of all infinite binary sequences, which is itself a "fractal" (it is homeomorphic to the well known middle-third Cantor set in the unit interval), the interplay between dimension and concepts from dynamical systems such as entropy becomes really close. Results of Besicovitch [2] and Eggleston [6] early brought forth a correspondance between the Hausdorff dimension of frequency sets (i.e. sets of sequences in which one occurs with a certain frequency) and the entropy of a process creating such sequences as typical outcomes. Besides, under certain requirements the Hausdorff dimension of a set in Cantor space equals the topological entropy of this set, viewn as a shift space.

An effective version of measure and entropy has been developed since the middle of the 20th century. Martin-Löf [12] effectivized the notion of a Lebesgue nullset in order to characterize objects (sequences) that are algorithmically random (namely those that do not have effective measure 0). The theoy of *Kolmogorov complexity* (see [9] for a thorough introduction), on the other hand, can be regarded as an effective version of entropy, which makes it possible to determine the entropy of individual objects, just as Martin-Löf randomness can declare individual sequences as random. Consequently, ways have been found how to characterize randomness in terms of Kolmogorov complexity.

The border between randomness and non-randomness is quite stiff. If a set has (effective) measure 0, it cannot be random. However, it might be that a set behaves "close to random" although it is not. The theory of Martin-Löf randomness offers no possibility to distinguish between different "degrees"

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of randomness. In contrast to this, the notion of entropy can be interpreted to describe the degree of randomness of a dynamical system. Kolmogorov complexity, an algorithmic version of entropy, does the same for finite binary sequences.

By developing an effective version of Hausdorff measure and dimension, one may hope to obtain an effect similar to classical case: a family of measures to pick a suitable one from when investigating a set. Furthermore, if the known relations between dimension and entropy carry over, a tool for classifying sets and sequences according to their degree of algorithmic randomness should arise. Besides, as Hausdorff dimension is also a geometrical notion (it is invariant under bi-Lipschitz transformations), effectivizing the theory might point new techniques stemming from fractal geometry for use in algorithmic measure and information theory.

These ideas, of course, are not entirely new. Brudno [3] and White [17] studied the relationship between the entropy of a symbolic dynamical system and the Kolmogorov complexity of an individual trajectory of a system. Ryabko [14], Staiger [15] and Cai and Hartmanis [4] observed close links between the Hausdorff dimension of a set and the Kolmogorov complexity of its members. Lutz [11] was the first to explicitely define an effective notion of Hausdorff dimension. He also introduced a resource bounded version ([10], see also [1]).

In this article, we further develop effective Hausdorff measure and dimension along the line of Lutz [10, 11]. The outline of the paper is as follows. In Section 2 we give a short overview over classical Hausdorff dimension on the Cantor space. In Section 3 we present effective Hausdorff measure and dimension, along with the effectivization of some of their important properties. To achieve this, the close connection of effective dimension to Kolmogorov complexity will be used. In Section 4 we employ the notion of effective dimension to sample objects arrising in the context of computability theory, such as degrees or spans.

Our notation is fairly standard. $\{0,1\}^\infty$ denotes the Cantor space, the set of all infinite binary sequences. The greek letters ζ, η, ξ and ω denote elements of the Cantor space. We write $\xi(n)$, $n \in \mathbb{N}$, to denote the n-th bit of the sequence ξ, and $\xi|_n$ denotes the n-bit initial segment of ξ, that is $\xi|_n = \xi(0)\xi(1)\dots\xi(n-1)$. We identify subsets of the natural numbers with their characteristic sequence, so sometimes we will regard them as elements of the Cantor space, too. Therefore, subsets of the Cantor space are also called *classes*. The lower case roman letters i, j, k, m, n denote natural numbers, whereas v, w, x, y, z usually denote finite binary strings, $l(x)$ denotes the length of a string, so $x = x(0)\dots x(l(x)-1)$, and $\{0,1\}^*$ denotes the set of all finite binary strings. We write $x \prec y$ if x is an initial segment of y, i.e., $l(x) < l(y)$ and $\forall i < l(x)\, x(i) = y(i)$. $x \prec \xi$, $\xi \in \{0,1\}^\infty$, is defined analogously.

§2. Classical Hausdorff Dimension. The basic idea behind Hausdorff dimension is to generalize the process of measuring a set by approximating (covering) it with sets whose measure is already known. Especially, the size of the sets used in the measurement process will be manipulated by certain transformations, thus making it harder or easier to approximate a set with a covering of small accumulated measure. This gives rise to the notion of *Hausdorff measure*.

We will introduce this notion on the Cantor space $\{0,1\}^\infty$ directly, where we can make use of some of its special features in order to simplify some definitions. For a general treatment of Hausdorff dimension and measure on metric r measure spaces, see the textbooks by Edgar [5], Falconer [7] or Mattila [13].

We endow the Cantor space with the usual metric d for sequences. For two sequences $\xi, \omega \in \{0,1\}^\infty$, define $c(\xi,\omega)$ to be their *maximal common initial segment*. Now let

$$d(\xi,\omega) = 2^{-l(c(\xi,\omega))}$$

We write $l(\xi,\omega)$ short for $l(c(\xi,\omega))$The *diameter* $d(X$ of a class $X \subseteq \{0,1\}^\infty$ is then defined by

$$d(X) = \sup\{d(\xi,\omega) : \ \xi, \omega \in X\}$$

The standard topology of $\{0,1\}^\infty$ is generated by the *basic open cylinders*:

$$C_w = \{\xi \in \{0,1\}^\infty : \ w \prec \xi\}, \quad w \in \{0,1\}^*$$

Assigning each of these cylinders the measure

$$\lambda(C_w) = 2^{-l(w)} = d(C_w)$$

induces the Lebesgue measure on $\{0,1\}^\infty$, which measure-theoretically isomorphic to the standard Lebesgue measure on $[0,1)$, the unit interval.

Now we can introduce Hausdorff measures. Let $X \subseteq \{0,1\}^\infty$, $\delta > 0$. A (countable) family $\{C_{w_i}\}_{i\in\mathbb{N}}$ is a *δ-cover* of X, if $(\forall i)\ d(C_{w_i} \leq \delta$ and $X \subseteq \bigcup C_{w_i}$.

For $s \geq 0$, define

$$\mathcal{H}^s_\delta(X) = \inf\{\sum_{i\in\mathbb{N}} d(C_{w_i})^s : \ \{C_{w_i}\}\ \delta\text{-cover of } X\}$$

As δ decreases, there are fewer δ-coverings available, therefore $\mathcal{H}^s_\delta$ is nondecreasing. Consequently, the value

$$\mathcal{H}^s(X) = \lim_{\delta\to\infty} \mathcal{H}^s_\delta(X)$$

is well defined, but may be infinite.

$\mathcal{H}^s(X)$ is called the *s-dimensional Hausdorff measure* of X. It can be shown that $\mathcal{H}^s$ is an outer measure and that the Borel classes of $\{0,1\}^\infty$ are $\mathcal{H}^s$-measurable.

Of course, for $s = 1$ we get the usual Lebesgue outer measure. For later use, the following characterization of Hausdorff measure 0 will be particularly useful.

PROPOSITION 2.1. *A class $X \subseteq \{0,1\}^\infty$ has s-dimensional Hausdorff measure* 0 *if and only if for every $n \in \mathbb{N}$ there exists a cover $\{C_{w_i}^{(n)}\}$ such that*

$$\sum_{i\in\mathbb{N}} d(C_{w_i}^{(n)})^s = \sum_{in\in\mathbb{N}} 2^{-l(w_i)} \leq 2^{-n}.$$

The class functions $\mathcal{H}^s$ have an important property.

PROPOSITION 2.2. *Let $X \subseteq \{0,1\}^\infty$. If, for some $s \geq 0$, $\mathcal{H}^s(X) < \infty$, then $\mathcal{H}^t(X) = 0$ for all $t > s$*

PROOF. Let $\mathcal{H}^s(X) < \infty$, $t > s$. If $\{C_{w_i}\}$ is a δ-cover of X, $\delta > 0$ we have

$$\sum_{i\in\mathbb{N}} d(C_{w_i})^t \leq \delta^{t-s} \sum_{i\in\mathbb{N}} d(C_{w_i})^s$$

so, taking infima, $\mathcal{H}_\delta^t(X) \leq \delta^{t-s}\mathcal{H}_\delta^t(X)$. As $\delta \to 0$, the result follows. ⊣

This means that there can exist only one point $s \geq 0$ where a given class might have finite positive s-dimensional Hausdorff measure. This point is the *Hausdorff dimension* of the class.

DEFINITION 2.3. For a class $X \subseteq \{0,1\}^\infty$, define the *Hausdorff dimension* of X as

$$\dim_{\mathrm{H}}(X) = \inf\{s \geq 0 : \mathcal{H}^s(X) = 0\}$$

In the following, we list some characteristic properties of Hausdorff dimension.

Refinement of measure 0. If $\lambda(X) \neq 0$ then $\dim_{\mathrm{H}}(X) = 1$. This follows from the fact that $\mathcal{H}^1$ is the Lebesgue outer measure. In particular, $\mathcal{H}^1(\{0,1\}^\infty) = \lambda(\{0,1\}^\infty) = 1$.

Monotonicity. If $X \subseteq Y$ then $\dim_{\mathrm{H}}(X) \leq \dim_{\mathrm{H}}(Y)$.

Stability. For $X, Y \subseteq \{0,1\}^\infty$ we have

$$\dim_{\mathrm{H}}(X \cup Y) = \max\{\dim_{\mathrm{H}}(X), \dim_{\mathrm{H}}(y)\}.$$

This can be generalized to the case of countable unions.

Countable Stability. Let $\{X_i\}_{i\in\mathbb{N}}$ be a countable family of classes. Then

$$\dim_{\mathrm{H}}\Big(\bigcup_{i\in\mathbb{N}} X_i\Big) = \sup_{i\in\mathbb{N}}\big\{\dim_{\mathrm{H}}(X_i)\big\}.$$

Geometric Invariance. Let $X \subseteq \{0,1\}^\infty$ and $f : X \to \{0,1\}^\infty$ be a bi-Lipschitz transformation, i.e. there exists $c_1, c_2 > 0$ s.t. $c_1 d(\xi,\omega) \leq d(f(\xi), f(\omega)) \leq c_2 d(\xi,\omega)$ for all $\xi, \omega \in X$. Then $\dim_{\mathrm{H}}(f(X)) = \dim_{\mathrm{H}}(X)$. This property follows easily from the behaviour of Hausdorff measure / dimension under *Hölder transformations.*

PROPOSITION 2.4. *Let $X \subseteq \{0,1\}^\infty$. Suppose $f : X \to \{0,1\}^\infty$ satisfies a* Hölder condition*: There exist $c, \alpha > 0$ s.t. $d(f(\xi), f(\omega)) \le cd(\xi,\omega)^\alpha$ for all $\xi, \omega \in X$. Then*

$$\dim_{\mathrm{H}}(f(X)) \le \left(\tfrac{1}{\alpha}\right) \dim_{\mathrm{H}}(X).$$

The primary interest, of course, now lies in determining the Hausdorff dimension of classes and in exposing the structure of sets having non integral dimension. However, the main obstacle for a direct applicability of Hausdorff dimension in the area of computability theory is the property of countable stability, which easily implies that all countable classes have dimension 0. Therefore, as in the case of effective measure theory, the notion of Hausdorff dimension first has to be effectivized, which is done in the next section.

§3. Effective Hausdorff Dimension. The effectivization of Hausdorff dimension resembles the effectivization of Lebesgue measure on $\{0,1\}^\infty$, as it was done by P. MARTIN-LÖF [12] in order to characterize algorithmic randomness. As measure 0 are defined via coverings the crucial step lies in allowing effective, that is r.e. coverings only. Hausdorff measure is an outer measure, defined via coverings as well. Therefore, the same strategy may be applied in effectivizing Hausdorff dimension, using the characterization of Proposition 2.1.

DEFINITION 3.1. Let $s \ge 0$. A class $X \subseteq \{0,1\}^\infty$ has *effective s-dimensional Hausdorff measure* 0, $\mathcal{H}^{1,s}(X) = 0$, if there exists a recursively enumerable set $C \subseteq \mathbb{N} \times \{0,1\}^*$ such that, for $C_n = \{w \in \{0,1\}^* : (n,w) \in C\}$

$$\sum_{w \in C_n} d(C_w)^s = \sum_{w \in C_n} 2^{-l(w)s} \le 2^{-n}.$$

Note that $\mathcal{H}^{1,1}(X) = 0$ means that X is an effective nullclass in the sense of MARTIN-LÖF. We denote this special case by $\lambda^1(X) = 0$.

The proof of Proposition 2.2 shows that an effective version of Proposition 2.2 holds, too, and the definition of effective Hausdorff dimension follows in a straightforward way.

DEFINITION 3.2. The *effective Hausdorff dimension*of a class $X \subseteq \{0,1\}^\infty$ is defined as

$$\dim^1_{\mathrm{H}}(X) = \inf\{s \ge 0 : \mathcal{H}^{1,s}(X) = 0\}.$$

We check some basic properties of effective dimension.

Dimension Conservation. We have $\dim^1_{\mathrm{H}}(\{0,1\}^\infty) = 1$. Obviously, the trivial cover $C = \{(n,w) : l(w) = n\}$ is r.e. and thus $\mathcal{H}^{1,s}_{2^{-n}}(\{0,1\}^\infty) \le 2^{n(1-s)}$ which goes to 0 for every $s > 1$.

Monotonicity. $\dim^1_{\mathrm{H}}(X) \le \dim^1_{\mathrm{H}}(Y)$ for $X \subseteq Y$ follows just as in the non-effective case.

Refinement of effective Lebesgue measure 0. $\lambda^1(X) \neq 0 \Rightarrow \dim_{\mathrm{H}}^1(X) = 1$. This is another straightforward analogy to the classical case.

Classical and effective Hausdorff dimension. $\dim_{\mathrm{H}}^1(X) \geq \dim_{\mathrm{H}}(X)$ follows directly from the definition.

The other important properties of Hausdorff dimension, countable stability and invariance under bi-Lipschitz transformations, require more careful treatment.

One of the great advantages of effective measure is the existence of a maximal effective nullclass, i.e. one that contains all other effective nullclasses. This has as an easy corollary the closure of effective nullclasses under countable unions. In order to get an effective version of countable stability we need a corresponding closure property of $\mathcal{H}^{1,s}$. The problem is, however, that the enumeration of a maximal $\mathcal{H}^{1,s}$-nullclass may not be effective, as s could be a noncomputable real number. Nevertheless, the computable numbers s (for which 2^{-ks} then is computable, too) are dense in $\mathbb{R}$, so for a dense subset of $\mathbb{R}$ we get corresponding maximal $\mathcal{H}^{1,s}$-nullclasses, from which the countable stability of effective dimension follows.

Besides, it now makes sense to consider the effective dimension of an individual sequences (viewed as a singleton class), as these have not automatically effective dimension 0. (In the following, we write $\dim_{\mathrm{H}}^1(\xi)$ for $\dim_{\mathrm{H}}^1(\{\xi\})$, $\xi \in \{0,1\}^\infty$.)

An example are the *Martin-Löf random sequences.* These are precisely the sequences not contained in the maximal λ^1-nullclass. Every single Martin-Löf random sequence has effective dimension 1.

Furthermore, the effective dimension of a class can be characterized in terms of the effective dimension of its members. The following theorem was first proved by Lutz [11].

Theorem 3.3 (Lutz). *For any class $X \subseteq \{0,1\}^\infty$,*

$$\dim_{\mathrm{H}}^1(X) = \sup_{\xi \in X} \dim_{\mathrm{H}}^1(\xi).$$

As regards an effective version of bi-Lipschitz invariance, the problem, of course, lies in the fact that Hölder transformations of the Cantor space need not to be effective, which means that an effective covering of some class does not automatically yield an effective covering of its image. Easy objects could be transformed in to very complicated ones from a computability point of view.

In order to get results of a similar flavour as those of Proposition 2.4 we have to take into account the computational behaviour as well as the geometrical behaviour of a mapping. In this setting, Kolmogorov complexity will prove quite useful.

[INSERT: Kolmogorov Complexity characterization]

Let $f : \{0,1\}^\infty \to \{0,1\}^\infty$ be a Hölder transformation on the Cantor space, i.e.

$$\forall \xi, \omega \in \{0,1\}^\infty \; d(f(\xi), f(\omega)) \le c\, d(\xi, \omega)^\alpha \tag{1}$$

for some $\alpha, c > 0$. This implies (recall the definition of metric d)

$$l(f(\xi), f(\omega)) \ge \alpha\, l(\xi, \omega) - \log c$$

Indeed, one can describe a Hölder transformation by a function operating on strings: Call a mapping $\varphi : \{0,1\}^* \to \{0,1\}^*$ *α-expansive*, $\alpha > 0$, if

(i) $x \preceq y \Rightarrow \varphi(x) \preceq \varphi(y)$,
(ii) $\forall \omega \in \{0,1\}^\infty \; \lim_{n\to\infty} l(\varphi(\omega|_n)) = \infty$,
(iii) $\forall \omega \in \{0,1\}^\infty \; \liminf_{n\to\infty} \frac{l(\varphi(\omega|_n))}{n} \ge \alpha$.

Obviously, properties (i) and (ii) induce a mapping $\hat\varphi : \{0,1\}^\infty \to \{0,1\}^\infty$, (iii) ensures that $\hat\varphi$ is Hölder continuous. On the other hand, one can show that for each Hölder transformation $f : \{0,1\}^\infty \to \{0,1\}^\infty$ there exists an α-expansive φ such that $\hat\varphi = f$.

Theorem [...] allows to analyze the effective dimension of $\hat\varphi(\xi)$ by investigating the asymptotical behaviour of the Kolmogorov complexity of its initial segments, i.e. $K(\varphi(\xi|_n))$. For this purpose, we may employ results from algoritmic information theory concerning the complexity of pairs. Most prominent, we have the symmetry of algorithmic information for prefix complexity: There exists some constant c such that for any $x, y \in \{0,1\}^*$

$$K(x, y) = K(x) + K(y|x, K(x)) + c \tag{2}$$

A proof of this identity can be found in [9]. [FOOTNOTE] Rewriting this in two different ways and replacing y by $\varphi(x)$ we get

$$K(\varphi(x)) = K(x) + K(\varphi(x)|x, K(x)) - K(x|\varphi(x), K(\varphi(x))) + c \tag{3}$$

Now we can examine the behaviour of K under Hölder transformations. We start with the recursive case.

Theorem 3.4. *Let $\varphi : \{0,1\}^* \to \{0,1\}^*$ be a recursive α-expansive mapping. Then it holds that, for any $X \subseteq \{0,1\}^\infty$,*

$$\dim^1_{\mathrm{H}}(\hat\varphi(X)) \le \left(\tfrac{1}{\alpha}\right) \dim^1_{\mathrm{H}}(X) \tag{4}$$

Proof. It suffices to show that, for any $\xi \in X$,

$$\dim^1_{\mathrm{H}}(\hat\varphi(\xi)) \le \left(\tfrac{1}{\alpha}\right) \dim^1_{\mathrm{H}}(\xi).$$

As φ is a recursive function, there is some constant c_1 such that

$$\forall x\, K(\varphi(x)|x, K(x)) \le c_1.$$

Furthermore, there exists some l^* such that $l(\varphi(x)) \geq \alpha l(x) - l^*$. Hence

$$\begin{aligned}\frac{K(\hat{\varphi}(\xi)|_n)}{n} &= \frac{K(\varphi(\xi|_n))}{l(\varphi(\xi|_n)}\\ &= \frac{K(\xi|_n) + K(\varphi(x)|x, K(x)) - K(x|\varphi(x), K(\varphi(x))) + c}{l(\varphi(\xi|_n))}\\ &\leq \frac{K(\xi|_n)}{\alpha l(x) - l^*} + \frac{c_1 + c}{l(\varphi(\xi|_n))}.\end{aligned}$$

It folllows with Theorem (...)

$$\dim^1_{\mathrm{H}}(\hat{\varphi}(\xi)) = \liminf_{n\to\infty} \frac{K(\hat{\varphi}(\xi)|_n)}{n} = \liminf_{n\to\infty} \frac{1}{\alpha} \frac{K(\xi|_n)}{n} = \dim^1_{\mathrm{H}}(\xi)$$

⊣

Now suppose f satisfies a Hölder condition from below:

$$\exists \alpha, c > 0\, \forall \xi, \omega \in \{0,1\}^\infty\, c\, d(\xi,\omega)^\alpha \leq d(f(\xi), f(\omega)). \tag{5}$$

Note that this implies that f is injective, hence an inverse mapping $f^{-1} : f(X) \to X$ with the property

$$d(f^{-1}(\xi), f^{-1}(\omega)) \leq \left(\tfrac{1}{c}\, d(\xi,\omega)\right)^{\frac{1}{\alpha}}$$

exists. Suppose further f has a recursive monotone representaion $\varphi : \{0,1\}^* \to \{0,1\}^*$ that is injective, too.

After having introduced effective dimension, we can now start to determine the dimension of some classes occuring in computability theory such as degrees, spans, etc., especially of those known to have effective measure 0. In particular, we may try to succeed in the following tasks:

- Find some interesting and "natural" *examples of classes of non-integral dimension* (like the *Middle-third Cantor set* in the classical setting).
- Is there a class of effective measure 0 but effective dimension 1?
- Expose some (nontrivial) examples of effective dimension 0.

§4. Some examples for effective dimension. We now present some results on effective Hausdorff dimension. We consider classes arrising in the context of computability theory. First, we further employ identity (2) to get invariance results for other, not necessarily recursive mappings. This will allow us to show that the effective dimension of a degree of a set and its lower span. For this purpose, we have to generalize the notion of a *join* of two sets (sequences).

Let Z be an infinite recursive subset of $\mathbb{N}$ with infinite complement. We shall call Z simply a *recursive partition.* Define the *Z-join* of two sequences $\xi, \omega \in \{0,1\}^\infty$, $\xi \oplus_Z \omega$, to be the unique sequence ζ which satisfies

$$\zeta|_Z = \xi \quad \text{and} \quad \zeta|_{\overline{Z}} = \omega.$$

If $\lim_{n\to\infty} |Z \cap 0, \ldots, n-1|/n = \delta$ exists, then, for given ω, the "insertion mapping" $g : \xi \to \xi \oplus_Z \omega$ satisfies a Hölder condition: Define $\varphi : \{0,1\}^* \to \{0,1\}^*$ by

$$\varphi(\xi|_n) = (\xi \oplus_Z \omega)|_{\pi_Z(n)},$$

where $\pi_Z(n)$ denotes the nth element of Z. Note that for each $\varepsilon > 0$ there exist constants $c_1, c_2 > 0$ such that

$$\frac{n}{\delta + \varepsilon} - c_1 \leq \pi_Z(n) \leq \frac{n}{\delta + \varepsilon} + c_1$$

for all n. Furthermore, $\hat{\varphi} = g$ and both φ and g are injective. Now we can use (3) to determine the complexity of φ. Obviously, since φ is injective and x is contained in $\varphi(x)$, i.e. can be identified in $\varphi(x)$ since Z is recursive, $K(x|\varphi(x), K(\varphi(x)))$ is bounded by a constant. On the other hand, in order to compute $\varphi(x)$ given x, it suffices to specify the bits that are "inserted " into x (at the $\overline{Z}$-positions). These are at most $n - |Z \cap \{0, \ldots, n-1\}|$ bits, and it follows that, for every $\varepsilon > 0$,

$$K(\varphi(x)|x, K(x)) \leq l(x)(1 - (\delta - \epsilon)) + O(1)$$

Hence, if $\delta = 1$, we can conclude that, for every $X \subseteq \{0,1\}^\infty$,

$$\dim^1_{\mathrm{H}}(X) = \dim^1_{\mathrm{H}}(\hat{\varphi}(X)).$$

We now use this ivariance property to code information into classes of sequences without changing the effective dimension of these classes.

Let r be a standard reducibility in coputability theory, i.e. one of 1, m, k-tt, btt, tt, wtt, T. For a set $A \subseteq \mathbb{N}$, let $\deg_r(A)$ and $A^{\leq_r}$ be its r-degree and r-lower span, respectively.

THEOREM 4.1. *For any set $A \subseteq \mathbb{N}$, it holds that*

$$\dim^1_{\mathrm{H}}(\deg_r(A)) = \dim^1_{\mathrm{H}}(A^{\leq_r})$$

PROOF. Let $A \subseteq \mathbb{N}$. Choose some recursive partition Z with $\lim_{n\to\infty} |Z \cap 0, \ldots, n-1|/n = 1$. Define the mapping $f : \{0,1\}^\infty \to \{0,1\}^\infty$ by $f(\xi) = \xi \oplus_Z A$. (As already mentioned, we identify subsets of the natural numbers with their characteristic sequences.) Then, obviously, $f(A^{\leq_r}) = \deg_r(A)$, and by the results of Section 3 the theorem follows. ⊣

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