The Effective Dimension of Cones and Degrees

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Overview

1. Effective Hausdorff Dimension
2. The Dimension of Cone and Degrees
3. A Wtt Lower Cone of Non-Integral Dimension
4. The Turing Case
A premeasure is a function \( \rho : 2^{<\omega} \rightarrow \mathbb{R}_0^+ \cup \{\infty\} \)

One can obtain an outer measure \( \mu_\rho \) from \( \rho \) by letting

\[
\mu_\rho(X) = \inf \left\{ \sum_i \rho(x_i) : \bigcup_i [x_i] \supseteq X \right\}.
\]

\( \mu = \mu_\rho \) is a countably subadditive, monotone set function. Restriction to sets \( \mathcal{A} \) which satisfy

\[
(\forall Y) \mu(Y) = \mu(Y \cap \mathcal{A}) + \mu(Y \setminus \mathcal{A}),
\]

yields the measurable sets.

The measurable sets form a \( \sigma \)-algebra, and \( \mu \) is an additive set function on this \( \sigma \)-algebra.
Effectivizing Measures

Let $\rho$ be a computable premeasure, with $\mu_\rho$ the induced outer measure.

Definition
A set $X \subseteq 2^\omega$ has **effectively $\mu_\rho$-measure zero** if there exists a uniformly computable sequence $(C_n)$ of sets of strings such that for all $n$,

$$X \subseteq \bigcup_{\sigma \in C_n} [\sigma] \quad \text{and} \quad \sum_{\sigma \in C_n} \rho(\sigma) \leq 2^{-n}.$$
— Hausdorff measures $\mathcal{H}^s$ arise from the premeasures $\rho(\sigma) = 2^{-|\sigma|s}$, $s \geq 0$.

— It is obvious that $\mathcal{H}^sX = 0$ implies $\mathcal{H}^tX = 0$ for all $t > s$.

**Definition**

The **Hausdorff dimension** of $X$ is defined as

$$\dim_H X = \inf\{s \geq 0 : \mathcal{H}^sX = 0\}.$$
Effective Hausdorff Dimension

**Definition**

The **effective Hausdorff dimension** of $\mathcal{X}$ is defined as

$$\dim_{H}^{1} \mathcal{X} = \inf\{s \in \mathbb{Q}_{0}^{+} : \mathcal{X} \text{ is effectively } \mathcal{H}^{s}\text{-null}\}.$$  

[Lutz 2000]

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There are single reals of non-zero dimension: every Martin-Löf random real has dimension one.

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Effective dimension has an important **stability property**:

$$\dim_{H}^{1} \mathcal{X} = \sup\{\dim_{H}^{1}\{A\} : A \in \mathcal{X}\}.$$  

[Lutz 2000]
Theorem

For every real $A$,

$$\dim_1^H A = \liminf_{n \to \infty} \frac{K(A \upharpoonright n)}{n} =: K(A).$$

[Ryabko 1984; Mayordomo 2002]
Effective Hausdorff Dimension
The three basic examples

— Given $0 < r < 1$ rational, let $Z_r = \{\lfloor n/r \rfloor : n \in \mathbb{N}\}$. Given a Martin-Löf random set $X$, define $X_r$ by

$$X_r(m) = \begin{cases} X(n) & \text{if } m = \lfloor n/r \rfloor, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\dim^1 H X_r = r$.

— Let $\mu_p$ be a Bernoulli (“coin-toss”) measure with bias $p \in \mathbb{Q} \cap [0, 1]$, and let $B$ be Martin-Löf random with respect to $\mu_p$. Then

$$\dim^1 H B = H(\mu_p) := -[p \log p + p \log(1 - p)].$$
Let $U$ be a universal, prefix-free machine. Given a computable real number $0 < s \leq 1$, the binary expansion of the real number

$$
\Omega^{(s)} = \sum_{\sigma \in \text{dom}(U)} 2^{-\frac{|\sigma|}{s}}
$$

has effective dimension $s$ [Tadaki 2002]. (Note that $\Omega^{(1)}$ is just Chaitin’s $\Omega$.)
Each of the three examples actually computes a Martin-Löf random real.

This is obvious for the “diluted” sequence.

For computable Bernoulli measures, one may use Von-Neumann’s trick to turn a biased random real into a uniformly distributed random real. More generally, Levin (1970) and Kautz (1991) have shown that any real which is random with respect to computable measure computes a Martin-Löf random real.

\( \Omega^{(s)} \) computes a fixed-point free function. It is a left-computable real, and hence it follows from the Arslanov completeness criterion that \( \Omega^{(s)} \) is Turing complete (and thus T-equivalent to a Martin-Löf random real).
The Dimension Problem
Are there “genuine” reals of non-integral dimension?

— The stability property implies that the Turing lower cone of each of the three examples has effective dimension 1.

Question
Are there any Turing lower cones of non-integral dimension?

— This is an open problem. Any such lower cone would come from a real of non-integral dimension for which it is not possible to extract some content of higher degree of randomness effectively.
— For upper cones, the situation is quite clear.

— It is known that the Turing upper cone of a real has Lebesgue measure zero unless the real is computable [Sacks 1963].

**Theorem** For any real $A$, the many-one upper cone of $A$ has (classical) Hausdorff dimension 1.
— The dimension of a lower cone and a degree coincide.

— This follows from the sparse coding technique: Given two reals $A \leq_r B$, choose a computable real $R$ of density $\lim_n |R \cap \{0, \ldots, n-1\}|/n = 1$, and let $C$ equal $A$ on $R$ and $B$ on the complement of $R$.

— $C$ will be $r$-equivalent to $B$ and be of the same dimension as $A$. It follows that the dimension of the degree and the lower cone of a set coincide.
Many-One Reducibility

**Theorem**

Let $\mu_p$ be a computable Bernoulli measure with bias $p$. If $A$ is $\mu_p$-random, then

$$B \leq_m A \Rightarrow \dim_{\mu}^1 B \leq H(\mu_p).$$

[Reimann and Terwijn 2004]

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**Proof.** Given an $m$-reduction $f$, define

$$F = \{n : (\forall m < n) f(m) \neq f(n)\},$$

so $F$ is the set of all positions of $B$, where an instance of $A$ is queried for the first time.

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$F$ induces a **Kolmogorov-Loveland place selection rule**. If $A$ is $\mu_p$-random, this selection rule will yield a new sequence with the same limit frequency as $A$. 
— This technique does not extend to weaker reducibilities, since for Bernoulli measures the Levin-Kautz result holds for a total Turing reduction.

— Stephan (2005) was able to construct wtt-lower cone of non-integral effective dimension in a relativized world: There is a real $A$ and an oracle $B$ such that

$$1/3 \leq \dim_H^B\{D : D \leq_w^B A\} \leq 1/2.$$
Theorem

For each rational $\alpha$, $0 \leq \alpha \leq 1$, there is a real $A \leq_{\text{wtt}} \emptyset'$ such that

$$K(A) = \alpha$$

and

$$(\forall Z \leq_{\text{wtt}} A) \ K(Z) \leq \alpha.$$
The strategy

Requirements:

\[ R_{\langle e, b \rangle} : Z = \Psi_e(A) \Rightarrow \exists (k \geq j) K(Z \upharpoonright k) \leq^+ (\alpha + 2^{-b}) k \]

where \((\Psi_e)\) is a uniform listing of wtt reduction procedures.

We can assume each \(\Psi_e\) also has a certain (non-trivial) lower bound on the use \(g_e\), because otherwise the reduction would decrease complexity anyway.
The strategy

— We construct $A$ inside the $\Pi^0_1$ class

$$\mathcal{P} = \{Z : (\forall n \geq n_0) K(Z \upharpoonright_n) \geq \lfloor \alpha n \rfloor \}$$

(This ensures $A$ has dimension at least $\alpha$.)

— $\mathcal{P}$ is given as an effective approximation through clopen sets $P_s$.

— We approximate longer and longer initial segments $\sigma_j$ of $A$, where $\sigma_j$ is a string of length $m_j$, both $\sigma_j$, $m_j$ controlled by $R_j$. 

A Wtt Lower Cone of Non-Integral Dimension

The strategy

— Define a length $k_j$ where we intend to compress $Z$, and let $m_j = g_e(k_j)$.

— Define $\sigma_j$ of length $m_j$ in a way that, if $x = \Psi_e^{\sigma_j}$ is defined then we compress it down to $(\alpha + 2^{-b_j})k_j$, by enumerating an appropriate request into a Kraft-Chaitin set $L$.

— The opponent’s answer could be to remove $\sigma_j$ from $\mathcal{P}$. ($\sigma_j$ is not of high dimension.)

— In this case, the capital he spent for this removal exceeds what we spent for our request, so we can account our capital against his.

— Of course, usually $\sigma_j$ is much longer than $x$. So we will only compress $x$ when the measure of oracle strings computing it is large.
We assume that $\mathcal{P}$ is effectively approximated by clopen sets $P_s$.

**Lemma**

Let $\mathcal{C}$ be a clopen class such that $\mathcal{C} \subseteq P_s$ and $\mathcal{C} \cap P_t = \emptyset$ for stages $s < t$. Then

$$\Omega_t - \Omega_s \geq (\lambda \mathcal{C})^\alpha.$$
— In the course of the construction, some $R_j$ might have to pick a new $\sigma_j$.

— In this case we have to initialize all $R_n$ of lower priority ($n > j$).

— We have to make sure that this does not make us enumerate too much measure into $L$.

— We therefore have to assign a new length $k_n$ to the strategies $R_n$.

— For this, it is important to know the use of the reduction related to $R_j$. 
The Turing Case

— It remains an open problem whether there exists a Turing lower cone of non-integral effective dimension.

— This case appears to be much harder. It is, for instance, not even known whether there exists a set of non-integral dimension which does not compute a Martin-Löf random set.

Theorem

There exists computable, non-decreasing, unbounded function $f$ and a set $A$ such that

$$K(A|_n) \geq f(n)$$

and $A$ does not compute a Martin-Löf random set.

[Kjos-Hanssen, Merkle, and Stephan 2004; Reimann and Slaman 2004,]