Randomness and Definability Hierarchies

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Question

- A lot of progress has been made in studying properties of random reals (Lebesgue or computable measures)
- Less clear: which reals in 2^ω are random with respect to *some* measure?
- How can we *find* a measure relative to which a given real is random?
- This talk:

randomness $\perp$ presence of an internal definability structure

Randomness

- Suppose μ is a probability measure on 2^ω, and R_μ is a representation of μ. Suppose further that $Z \in 2^\omega$ and $n \geq 1$.
- An **(R_μ, Z, n)-test** is a set $W \subseteq \omega \times 2^{<\omega}$ recursively enumerable in $(R_\mu \oplus Z)^{(n-1)}$ such that

$$\sum_{\sigma \in W_n} \mu[\sigma] \leq 2^{-n},$$

  where $W_n = \{\sigma : (n, \sigma) \in W\}$
- A real X **passes** a test W if $X \notin \bigcap_n \bigcup_{\sigma \in W_n} [\sigma]$.
- A real X is **(R_μ, Z, n)-random** if it passes all (R, Z, n)-tests.
- A real X **(μ, Z, n)-random** if there exists a representation R_μ such that X is (R_μ, Z, n)-random.

Continuous measures

- μ is *continuous* if $\mu\{X\} = 0$ for all X.
- X is random for a continuous measure iff it is random for a *dyadic* continuous measure.
- This way we can avoid some representational issues.
- In the following, all measures are *continuous dyadic*.

Orthogonality (0)

Stair Trainer Lemma: Suppose Z is μ-n-random, $n \geq 2$. If $Y \leq_T \mu^{(n-1)}$ and $Y \leq_T Z \oplus \mu$, then $Y \leq_T \mu$.

(*Generalizes a result by Downey, Nies, Weber, and Yu*)

- Suffiently random reals form a minimal pair with instances of the jump (relative to the measure).

Orthogonality (I)

Stair Trainer Technique: If $n \geq 2$, then for all $k \geq 0$, $\emptyset^{(k)}$ is not n-random with respect to a continuous measure.

- Suppose $\emptyset^{(k)}$ is μ-n-random for some μ. Then
- $\emptyset' \leq_T \emptyset^{(k)}$ and $\emptyset' \leq_T \mu' \leq_T \mu^{(n-1)}$.
- By Lemma, $\emptyset'$ is recursive in μ.
- Apply argument inductively to $\emptyset^{(i)}$, $i \leq k$.

Orthogonality (II)

Stair Trainer Limit Technique: For $n \geq 3$, $\emptyset^{(\omega)}$ is not n-random with respect to a continuous measure.

- Assume for a contradiction that $0^{(\omega)}$ is μ-n-random for $n \geq 3$ and continuous μ.
- By the previous proof, $0^{(k)} \leq_T \mu$ for all k. By Enderton and Putnam, if X is a $\leq_T$-upper bound for $\{0^{(k)} \colon k \in \omega\}$, then $0^{(\omega)} \leq_T X''$.
- Therefore, $0^{(\omega)} \leq_T \mu''$, but since $n \geq 3$ and $0^{(\omega)}$ is μ-n-random, this is impossible.

Randomness vs structure

Two main points:

- Steps in the hierarchy are given by simple, uniformly arithmetic operations.
- One can pass from an upper bound to a uniform limit by an arithmetic operation.

Beyond arithemtic

Jensen's **Master codes** for the constructible universe provide a similarly stratified hierarchy of definability.

Goal: Show that randomness is equally incompatible with such codes.

The J-hiearchy

Cumulative hierarchy defined as

- $J_0 = \emptyset$
- $J_{\alpha+1} = \text{rud}(J_\alpha)$
- $J_\lambda = \bigcup_{\alpha<\lambda} J_\alpha$ for λ limit.

rud(X) is the closure of $X \cup \{X\}$ under *rudimentary* functions (primitive set recursion).

Properties

- Each J_α is transitive and *amenable* (model of a sufficiently large fragment of set theory).
- $\mathrm{rank}(J_{\alpha+1}) = \mathrm{rank}(J_\alpha) + \omega$.
- $L = \bigcup_\alpha J_\alpha$.
- The Σ_n-satisfaction relation over J_α, $\models^{\Sigma_n}_{J_\alpha}$, is Σ_n-definable over J_α, uniformly in α.
- The mapping $\beta \mapsto J_\beta$ $(\beta < \alpha)$ is Σ_1-definable over any J_α.
- There is a formula $\varphi_{\mathsf{V=J}}$ such that for any transitive set M,

$$M \models \varphi_{\mathsf{V=J}} \iff \exists\alpha\, M = J_\alpha.$$

Rudimentary functions

Every rudimentary function is a combination of the following nine functions:

1. $F_0(x, y) = \{x, y\}$,
2. $F_1(x, y) = x \setminus y$,
3. $F_2(x, y) = x \times y$,
4. $F_3(x, y) = \{(u, z, v) \colon z \in x \ \wedge \ (u, v) \in y\}$,
5. $F_4(x, y) = \{(u, v, z) \colon z \in x \ \wedge \ (u, v) \in y\}$,
6. $F_5(x, y) = \bigcup x$,
7. $F_6(x, y) = \mathrm{dom}(x)$,
8. $F_7(x, y) = \in \cap \, (x \times x)$,
9. $F_8(x, y) = \{\{x(z)\} \colon z \in y\}$.

S-operator

$$S(X) = [X \cup \{X\}] \cup \left[\bigcup_{i=0}^{8} F_i[X \cup \{X\}]\right]$$

This gives rise to a finer hierarchy:

- $S_0 = \emptyset$,
- $S_{\alpha+1} = S(S_\alpha)$,
- $S_\lambda = \bigcup_{\alpha<\lambda} S_\alpha$ (λ limit).

Then

$$J_\alpha = \bigcup_{\beta<\omega\alpha} S_\beta = S_{\omega\alpha}.$$

Projecta

Boolos & Putnam: If $\mathcal{P}(\omega) \cap (L_{\alpha+1} \setminus L_\alpha) \neq \emptyset$, then there exists a surjection $f : \omega \to L_\alpha$ in $L_{\alpha+1}$.

Jensen extended and generalized this observation.

- For $n, \alpha > 0$, the *Σ_n-projectum* ρ_α^n is equal to the least $\gamma \leq \alpha$ such that $\mathcal{P}(\omega\gamma) \cap (\Sigma_n(J_\alpha) \setminus J_\alpha) \neq \emptyset$.
- ρ_α^n is equal to the least $\delta \leq \alpha$ such that there exists a function f that is $\Sigma_n(J_\alpha)$-definable over J_α such that $f(D) = J_\alpha$ for some $D \subseteq \omega\delta$

Master codes

A Σ_n *master code* for J_α is a set $A \subseteq J_{\rho_\alpha^n}$ that is $\Sigma_n(J_\alpha)$, such that for any $m \geq 1$,

$$\Sigma_{n+m}(J_\alpha) \cap \mathcal{P}(J_{\rho_\alpha^n}) = \Sigma_m(\langle J_{\rho_\alpha^n}, A\rangle).$$

A Σ_n master code does two things:

1. It "accelerates" definitions of new subsets of $J_{\rho_\alpha^n}$ by n quantifiers.
2. It replaces parameters from J_α in the definition of these new sets by parameters from $J_{\rho_\alpha^n}$ (and the use of A as an "oracle").

Standard codes

Jensen exhibited a uniform, canonical way to define master codes, by iterating Σ_1-definability.

$$A_\alpha^{n+1} := \{(i,x)\colon i \in \omega \ \wedge \ x \in J_{\rho_\alpha^{n+1}} \ \wedge \ \langle J_{\rho_\alpha^n}, A_\alpha^n \rangle \models \varphi_i(x, p_\alpha^{n+1})\},$$

We will call the structure $\langle J_{\rho_\alpha^n}, A_\alpha^n \rangle$ the *standard Σ_n J-structure* for J_α.

From set theory to recursion theory

We want to apply the recursion theoretic "Stair" techniques to countable J-structures. We therefore have to code them as subsets of ω.

If the projectum ρ_α^n is equal to 1, all "information" about the J-structure $\langle J_{\rho_\alpha^n}, A_\alpha^n \rangle$ is contained in the standard code A_α^n, which is simply a real (or rather, a subset of V_ω).

These lend itself directly to recursion theoretic analysis (e.g. Boolos and Putnam [1968], Jockusch Simpson [1976], Hodes [1980]).

From set theory to recursion theory

The problem in our setting is that we want to uniformly work our way through arithmetic copies of J-structures even when the projectum is greater than 1.

For this purpose we have to code **two objects**, the sets J_α (which keep track of the basic set theoretic relations) and the standard codes over each J_α, which keep track of the definable objects quantifier by quantifier.

ω-copies

Let $X \subseteq \omega$. The *relational structure* induced by X is $\langle F_X, E_X \rangle$, where

$$xE_X y \Leftrightarrow \langle x, y \rangle \in X$$

and

$$F_X = \text{Field}(E_X) = \{x \colon \exists y \ (xE_X y \text{ or } yE_X x) \text{ for some } y\}.$$

We will look at structures $\langle X, M \rangle$, where X is a relational structure, and M is a subset of F_X (coding an additional predicate).

ω-copies

An *ω-copy* of a countable set-theoretic structure $\langle S, A\rangle$, $A \subseteq S$, is a pair $\langle X, M\rangle$ of subsets of ω such that the structure coded by X is extensional and there exists a surjection $\pi : S \to \text{Field}(E_X)$ such that

$$\forall x, y \in S\,[\, x \in y \iff \pi(x) E_X \pi(y)\,], \tag{1}$$

and

$$M = \{\pi(x) \colon x \in A\}. \tag{2}$$

If $\rho^n_\alpha = 1$, then standard code can be seen directly as an ω-copy, which we will call the *canonical copy*.

Extracting information from copies

If $\langle X, M\rangle$ is an ω-copy of $\langle J_{\rho_\alpha^{n+1}}, A_\alpha^{n+1}\rangle$, then $(X \oplus M)^{(2)}$ computes ω-copies of

- $\langle J_{\rho_\alpha^n}, A_\alpha^n\rangle, \langle J_{\rho_\alpha^{n-1}}, A_\alpha^{n-1}\rangle, \ldots,$ and $\langle J_{\rho_\alpha^0}, A_\alpha^0\rangle = \langle J_\alpha, \varnothing\rangle = J_\alpha$,
- $S^{(n)}(J_\beta)$, for all $n \in \omega$, $\beta < \alpha$.

Defining copies

We can define ω-copies of new J-structures from ω-copies of given J-structures using suitable versions of the S-operator.

> *There exists a Π^0_5-definable function $\overline{S}(X) = Y$ such that, if X is an ω-copy of a countable set U, $\overline{S}(X)$ is an ω-copy of $S(U)$.*

Putnam-Enderton analysis: If X is an ω-copy of J_α and $Z \geq_T \overline{S}^{(n)}(X)$ for all n, then $Z^{(5)}$ computes an ω-copy of $J_{\alpha+1}$.

Defining copies

We can also arithmetically define an ω-copies of the successor of a standard J-structure.

- Suppose $\langle X, M \rangle$ is an ω-copy of $\langle J_{\rho_\alpha^n}, A_\alpha^n \rangle$. Then there exists an ω-copy of $\langle J_{\rho_\alpha^{n+1}}, A_\alpha^{n+1} \rangle$ $\Sigma^0_{d^{(1)}_{\models}}$-definable in $\langle X, M \rangle$.

Here $d^{(1)}_{\models}$ is the *arithmetic complexity* of the formula defining $\models^{\Sigma_1}$ for transitive, rud-closed structures.

Recognizing copies

Goal: show that the sequence of canonical copies of J-structures with projectum $= 1$ in L_{β_n}, where β_n is the least ordinal such that $L_{\beta_n} \models \mathrm{ZFC}_n^-$, cannot be $G(n)$-random with respect to a continuous measure.

We will assume for a contradiction that such a copy, say $\langle X, M \rangle$, *is* random for a continuous measure μ.

Recognizing copies

Idea: look at the initial segment of ω-copies computable in (some fixed jump of) μ.

Since $\langle X, M \rangle$ is μ-random, it cannot be among those.

But we can "reach" $\langle X, M \rangle$ from the ω-copies of J_α's computable in μ, by iterating arithmetic operations and taking uniform limits.

Then apply the Stair Trainer Technique.

Recognizing copies

Problem: we cannot arithmetically define the set of ω-copies of structures J_α. We can define a set of "pseudocopies", subsets of ω that behave in most respects like actual ω-copies, but that may code structures that are not well-founded.

Pseudocopies

A pseudocopy is defined through the following properties, which are arithmetically definable.

- The relation E_X is non-empty and extensional.
- X is rud-closed.
- The structure coded by X satisfies $\varphi_{V=J}$.
- X contains (a copy of) ω as a element.

Furthermore, we can also prescribe which power sets of ω exist:

$$\exists y(y = \mathcal{P}^{(n)}(\omega)) \wedge \forall z(z \neq \mathcal{P}^{(n+1)}(\omega)).$$

Such a pseudocopy is called an n-pseudocopy.

Comparing pseudocopies

We can also linearly order pseudocopies by comparing their internal J-structures.

- To check whether two pseudocopies appear to code the same structure, we compare their reals, sets of reals, etc., up to n, the largest existing power of ω.
- For two n-copies, we put $X \prec_n Y$ if there exists a J-segment in Y isomorphic to X.
- If comparability fails, we can arithmetically expose an ill-foundedness in one of the pseudocopies, by looking at the ordinal structure in each pseudocopy.

Result: a $\Sigma^0_{c_n}(Z)$-definable total preorder $(\mathcal{PC}^*_n(Z), \prec_n)$ of pseudocopies recursive in Z.

Canonical copies are not random

Theorem: Suppose $N \geq 0$, $\alpha < \beta_N$, and for some $k > 0$, $\rho_\alpha^k = 1$. Then the canonical copy of the standard J-structure $\langle J_{\rho_\alpha^k}, A_\alpha^k \rangle$ is not $G(N)$-random with respect to any continuous measure.

$$[G(N) = 6^{N+2} \cdot (d_{\models}^{(1)} + 2N + 42)]$$

Proof

- Assume for a contradiction the canonical copy $\langle X, M \rangle$ is $G(N)$-random for continuous μ.
- Any pseudocopy in a well-founded initial segment of $\mathcal{PC}^*_N(\mu)$ is a well-founded pseudocopy, and hence a true arithmetic copy of some J-structure.
- μ can arithmetically recognize the longest well-founded initial segment of $\prec_N$.

Lemma: Let $j \geq 0$. Suppose μ is a continuous measure and $\prec$ is a linear order on a subset of ω such that the relation $\prec$ and the field of $\prec$ are both recursive in $\mu^{(j)}$. Suppose further X is $(j+5)$-random relative to μ, and $I \subseteq \omega$ is the longest well-founded initial segment of $\prec$. If I is recursive in $(X \oplus \mu)^{(j)}$, then I is recursive in $\mu^{(j+4)}$.

Proof (continued)

- As $\langle X, M \rangle$ is sufficiently random, we can build up a chain of pseudocopies computable from μ, using the stair trainer technique.
- One complication: μ and $\langle X, M \rangle$ have different copies, so we need to translate between them, which adds complexity at every step.
- This is offset by looking at the projecta, and recycling copies we have built at previous stages.