Hausdorff Measures and Perfect Subsets

How random are sequences of positive dimension?

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Hausdorff Measures

- Caratheodory-Hausdorff construction on metric spaces: let $\mathcal{A} \subseteq 2^\omega$, $h : \mathbb{R} \to \mathbb{R}$ a monotone, increasing, continous on the right function with $h(0) = 0$, and let $\delta > 0$.

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- Define a set function

$$\mathcal{H}^h_\delta(\mathcal{A}) = \inf \left\{ \sum_i h(\text{diam}(U_i)) : \mathcal{A} \subseteq \bigcup_i U_i, \ \text{diam}(U_i) \leq \delta \right\}.$$

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- Letting $\delta \to 0$ yields an (outer) measure.
- The h-dimensional Hausdorff measure $\mathcal{H}^h$ is defined as

$$\mathcal{H}^h(\mathcal{A}) = \lim_{\delta \to 0} \mathcal{H}^h_\delta(\mathcal{A})$$

Properties of Hausdorff Measures

- $\mathcal{H}^{\mathrm{h}}$ is Borel regular:
  all Borel sets are measurable and for $\mathcal{A} \subseteq 2^{\omega}$ there is a Borel set $\mathcal{B} \supseteq \mathcal{A}$ such that $\mathcal{H}^{\mathrm{h}}(\mathcal{B}) = \mathcal{H}^{\mathrm{h}}(\mathcal{A})$.

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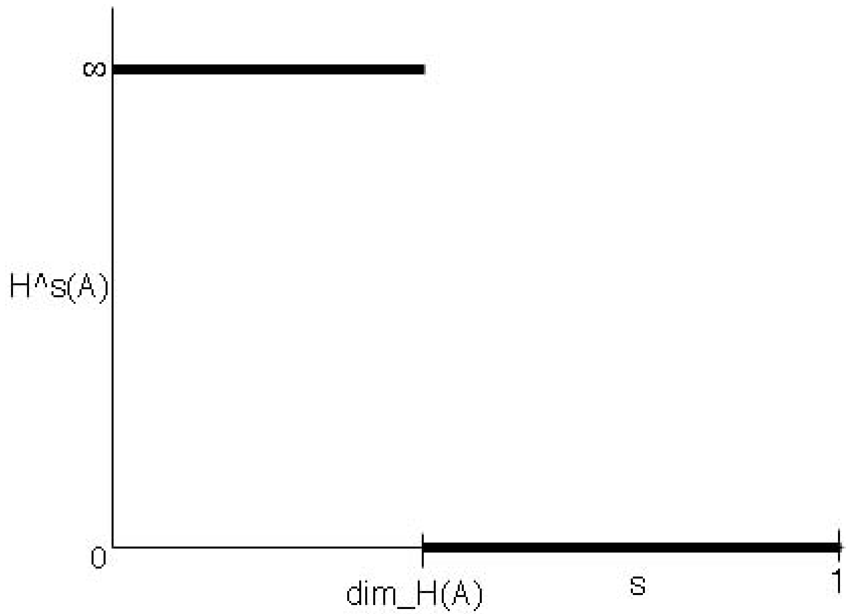
- For $s = 1$, $\mathcal{H}^1$ is the usual Lebesgue measure λ on 2^ω.
- For $0 \leq s < t < \infty$ and $\mathcal{A} \subseteq 2^\omega$,

$$\mathcal{H}^s(\mathcal{A}) < \infty \text{ implies } \mathcal{H}^t(\mathcal{A}) = 0,$$

$$\mathcal{H}^t(\mathcal{A}) > 0 \text{ implies } \mathcal{H}^s(\mathcal{A}) = \infty.$$

Hausdorff dimension

- The situation is depicted in the following graph:



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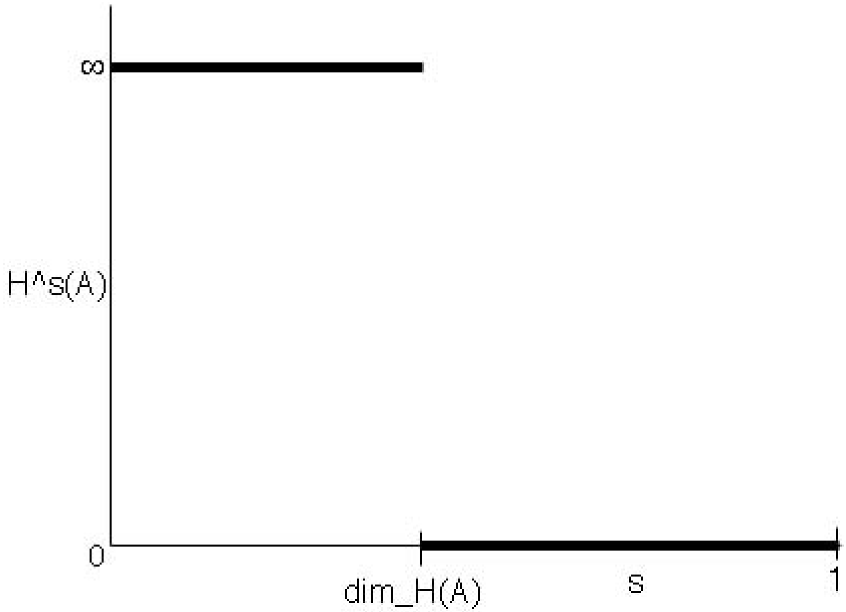

- The Hausdorff dimension of $\mathcal{A}$ is

$$\begin{aligned}\dim_H(\mathcal{A}) &= \inf\{s \geq 0 : \mathcal{H}^s(\mathcal{A}) = 0\} \\ &= \sup\{t \geq 0 : \mathcal{H}^t(\mathcal{A}) = \infty\}\end{aligned}$$

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- Countable Stability: If $\mathcal{A}_1, \mathcal{A}_2, \ldots$ is a sequence of classes then

$$\dim_H(\bigcup \mathcal{A}_i) = \sup\{\dim_H(\mathcal{A}_i)\}.$$

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  (Implies that all countable sets have dimension 0.)
- Geometric transformations: If $f : 2^\omega \to 2^\omega$ is Hölder continuous, i.e. $d(f(\xi), f(\omega)) \leq c\, d(\xi, \omega)^\alpha$ for $c, \alpha > 0$, then

$$\dim_H f(\mathcal{A}) \leq (1/\alpha) \dim_H(\mathcal{A}).$$

Effective Hausdorff measure

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- (Let $s \geq 0$ be rational.) $\mathcal{A} \subseteq 2^\omega$ is Σ_1-$\mathcal{H}^s$ null, Σ^0_1-$\mathcal{H}^s(\mathcal{A}) = 0$, if there is a recursive sequence (C_n) of r.e. sets such that for each n,

$$\mathcal{A} \subseteq \bigcup_{w \in C_n} [w] \quad \text{and} \quad \sum_{w \in C_n} 2^{-|w|s} < 2^{-n}.$$

Effective Hausdorff Dimension

- Definition of effective Hausdorff dimension is straightforward:

$$\mathrm{dim}^1_{\mathrm{H}}(\mathcal{A}) = \inf\{s \geq 0 : \Sigma^0_1\text{-}\mathcal{H}^s(\mathcal{A}) = 0\}.$$

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$$\dim^1_{\mathrm{H}}(\mathcal{A}) = \inf\{s \geq 0 : \Sigma^0_1\text{-}\mathcal{H}^s(\mathcal{A}) = 0\}.$$

- Countable stability takes a particularly nice form:

$$\dim^1_{\mathrm{H}}(\mathcal{A}) = \sup_{\xi \in \mathcal{A}} \dim^1_{\mathrm{H}}(\xi).$$

Some Examples

- Random sequences: ξ Martin-Löf random implies that Σ^0_1-$\mathcal{H}^1(\xi) \neq 0$. Hence $\dim^1_H(\xi) = 1$.

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- Limiting frequency: Let ν be a $(\beta, 1-\beta)$-Bernoulli measure ($0 < \beta < 1$ rational). Then for any ν-random sequence,

$$\dim_H^1(\xi) = H(\beta),$$

with $H(\beta) = -[\beta \log(\beta) + (1-\beta)\log(1-\beta)]$.

’Main Theorem’ of Effective Dimension

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- Theorem: [Ryabko, Staiger, Cai and Hartmanis, Lutz, Tadaki, Mayordomo]
  For any sequence $\xi \in 2^\omega$ it holds that

$$\dim_{\mathrm{H}}^{1}(\xi) = \liminf_{n\to\infty} \frac{H(\xi\restriction n)}{n}.$$

Perfect Subsets

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- Gacs, Kucera: Effective Version – every Π^0_1 class of positive Lebesgue measure can be mapped effectively onto 2^ω (by a process).
- Corollary: Every sequence is Turing reducible to a random one.

Effective Processes

- A function $\phi : 2^{<\omega} \to 2^{<\omega}$ is called monotone, if

$$v \sqsubseteq w \text{ implies } \phi(v) \sqsubseteq \phi(w).$$

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  $\Phi : 2^{\omega} \to 2^{<\omega} \cup 2^{\omega}$.
- If Φ is induced by a computable mapping, it is called a process.
- If $\Phi(A) = B$ via a process Φ, then $B \leq_T A$.

Generalized Reducibility Theorem

- A monotone mapping $\phi : 2^{<\omega} \to 2^{<\omega}$ is weakly Hölder or α-expansive, $\alpha > 0$, if for all $\omega \in \mathrm{dom}(\Phi)$,

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- Theorem: Every Π^0_1 class $\mathcal{A}$ of positive dimension can be mapped onto 2^ω by a computable, weakly Hölder process.
- With some effort, this can be generalized to all Π^0_1 classes $\mathcal{A}$ for which exists a recursive h such that $\mathcal{H}^h(\mathcal{A}) > 0$.

Hausdorff and Probability Measures

- Basic ingredient in the Gacs-Kucera proof:
  If $\lambda(\mathcal{A}) > 2^{-n}$, then there must exists $\xi, \omega \in 2^{\omega}$ such that $d(\xi, \omega) \geq 2^{n-1}$.

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- Theorem: If $\mathcal{A}$ is Π^0_1, and if $\mathcal{A}$ contains at least one random sequence, then one can effectively find $\varepsilon > 0$ such that $\lambda(\mathcal{A}) > \varepsilon$.
- Need: Computable measure 'close' to uniform (Lebesgue measure) which makes $\mathcal{A}$ 'look big'.
- If $\dim_H(\mathcal{A}) > s$, then $\mathcal{A}$ is not $\mathcal{H}^s$-null.
- But for $0 < s < 1$, $\mathcal{H}^s$ is not a probability measure: $\mathcal{H}^s(2^\omega) = \infty$.

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- Frostman's Lemma: Let $\mathcal{B} \subseteq 2^\omega$ be Borel. Then $\mathcal{H}^s(\mathcal{B}) > 0$ if and only if there exists a Radon probability measure μ with compact support contained in $\mathcal{A}$ such that

$$(\forall w \in 2^{<\omega})\ [\mu[w] \leq 2^{-|w|s}].$$

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- Theorem: Let $\mathcal{A} \subseteq 2^\omega$ be Π^0_1. Then $\mathcal{H}^s(\mathcal{A}) > 0$ if and only if there exists a recursive probability measure μ such that $\mu(\mathcal{A}) > 0$ and

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- So, a positive answer to the first question would imply that there are no such spans.

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- Classical results: There exists a universal measure zero class $\mathcal{Z} \subseteq 2^\omega$ (a subset of 2^ω that does not host a non-atomic finite Borel measure) of positive Hausdorff dimension. [Grzegorek, Fremlin, Zindulka]

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  Every 1-generic sequence is improper.
- Theorem: There exists an improper sequence of dimension 1.
- The proof uses a weak Lipschitz join, which means that one sequence is inserted into another at very distant points. (The corresponding monotone mapping is bi-Hölder for every $\alpha > 1$.)

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- It is an area of intensive research in complexity theory how to extract perfect (uniform) randomness from a weakly random source.
- It is not clear to what extend results are helpful in our setting.