Effective Fractal Dimension

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Measures on Cantor Space

Outer measures from premeasures

Approximate sets from outside by open sets and weigh with a general measure function.

- A premeasure is a function $\rho : 2^{<\omega} \to \mathbb{R}_0^+ \cup \{\infty\}$.
- One can obtain an outer measure μ_ρ from ρ by letting

$$\mu_\rho(X) = \inf_{C \subseteq 2^{<\omega}} \left\{ \sum_{\sigma \in C} \rho(\sigma) : \bigcup_{\sigma \in C} N_\sigma \supseteq X \right\},$$

where N_σ is the basic open cylinder induced by σ.
(Set $\mu_\rho(\emptyset) = 0$.)

The resulting $\mu = \mu_\rho$ is a countably subadditive, monotone set function, an outer measure.

Measures on Cantor Space

From outer measures to measures

Measurable sets:

- Restriction to sets A which satisfy

$$(\forall Y)\ \mu(Y) = \mu(Y \cap A) + \mu(Y \setminus A),$$

  yields the measurable sets.
- The measurable sets form a σ-algebra, and μ is an additive set function on this σ-algebra.

The more "well-behaved" ρ is, the better are the regularity properties of μ_ρ.

- In particular, if ρ is already additive on cylinders, then the μ-measurable sets comprise the Borel sets, and μ coincides with ρ on the Borel sets.

Geometric Measures

"Geometric" measures should be translation invariant.

- Geometric outer measures: ρ depends only on $|\sigma|$.
- Most famous example: Lebesgue measure λ given by $\rho(\sigma) = 2^{-|\sigma|}$.
- More general: function h, $h(0) = 0$, right-continuous; $\rho(\sigma) = h(2^{-|\sigma|})$.
- In Cantor space: $h : \mathbb{N} \to \mathbb{R}_0^+$, $h(n) \to \infty$ as $n \to \infty$; $\rho(\sigma) = 2^{-h(|\sigma|)}$.

Measures on Cantor Space

Nullsets

The way we constructed outer measures, $\mu(A) = 0$ is equivalent to the existence of a sequence $(C_n)_{n\in\omega}$, $C_n \subseteq 2^{<\omega}$, such that for all n,

$$A \subseteq \bigcup_{C_n} N_\sigma \quad \text{and} \quad \sum_{C_n} \rho(\sigma) \leqslant 2^{-n}.$$

Thus,

every nullset is contained in a G_δ nullset.

By requiring that the covering nullset is effectively G_δ, we obtain a notion of effective nullsets.

Definition

- Let μ $(= \mu_\rho)$ be an outer measure based on a computable premeasure ρ. A set A is effectively μ-null if there exists a recursive function f such that for all n,

$$A \subseteq \bigcup_{\sigma \in W_{f(n)}} N_\sigma \quad \text{and} \quad \sum_{\sigma \in W_{f(n)}} \rho(\sigma) \leqslant 2^{-n}.$$

- A real $X \in 2^\omega$ is μ-random iff $\{X\}$ is not μ-null.

Hausdorff Measures and Hausdorff Dimension

For $\rho(\sigma) = 2^{-|\sigma|s}$, s a nonnegative real number, we obtain $\mathcal{H}^s$, the s-dimensional Hausdorff measure.

- Note: The actual definition of the Hausdorff measure $\mathcal{H}^h$ is a little more involved. (One wants to ensure that for the resulting measures, all Borel sets are measurable.)
- We are primarily concerned with nullsets. For nullsets the more involved definition coincides with the one given here.

The Hausdorff dimension assigns to every set of reals an "adequate" measure.

Definition

The Hausdorff dimension of $A \subseteq 2^\omega$ is defined as

$$\dim_{\mathrm{H}} A = \inf\{s > 0 : \mathcal{H}^s(A) = 0\}$$

Properties of Hausdorff Dimension

- Lebesgue measure: $\lambda(A) > 0$ implies $\dim_H(A) = 1$.
- Monotony: $A \subseteq B$ implies $\dim_H(A) \leqslant \dim_H(B)$.
- Stability: For $A_1, A_2, \dots \subseteq 2^\omega$ it holds that

$$\dim_H(\bigcup A_i) = \sup\{\dim_H(A_i)\}.$$

- Important geometric properties:
  - If F is Hölder continuous, i.e. if there are constants $c, r > 0$ for which

    $$(\forall x, y)\ d(F(x), F(y)) \leqslant c d(x, y)^r,$$

    then

    $$\dim_H F(A) \leqslant (1/r) \dim_H(A).$$

  - For $r = 1$, F is Lipschitz continuous. If F is bi-Lipschitz, then

    $$\dim_H h(A) = \dim_H(A).$$

Hausdorff Dimension and Martingales

Hausdorff dimension can be expressed in terms of martingales.

- Recall that a martingale is a function $d : 2^{<\omega} \to [0, \infty)$ such that $2d(\sigma) = d(\sigma^\frown 0) + d(\sigma^\frown 1)$.
- Given $s \geqslant 0$, a martingale d is called s-successful on a real $X \in 2^\omega$ if

$$\limsup d(X\restriction_n)/2^{(1-s)n} = \infty.$$

- Note that the usual success-notion for martingales is just being 1-successful.

Theorem (Lutz)

For any set $A \subseteq 2^\omega$,

$$\dim_H A = \inf\{s : \textit{some martingale } d \textit{ is } s\textit{-successful on all } X \in A\}.$$

Packing Dimension

Lutz' martingale characterization allows for an easy characterization of another dimension concept, packing dimension, which can be seen as a dual to Hausdorff dimension.

- Instead of "covering" a set with open balls, "pack" it with disjoint balls.

Given $0 < s \leqslant 1$, a martingale d is strongly s-successful on a real X if

$$\liminf d(X \upharpoonright_n)/2^{(1-s)n} \to \infty.$$

Theorem (Athreya, Hitchcock, Lutz, and Mayordomo)

For any set $A \subseteq 2^\omega$,

$$\dim_P A = \inf\{s : \textit{ some } d \textit{ is strongly } s\textit{-successful on all } X \in A\}.$$

Effective Hausdorff Dimension

Hausdorff dimension can be effectivized using effective Hausdorff measures.

Definition (Lutz)

The Σ^0_1-Hausdorff dimension, or simply 1-dimension (constructive dimension), of $A \subseteq 2^\omega$ is defined as

$$\dim^1_{\mathrm{H}} A = \inf\{s \in \mathbb{Q}^+_0 : A \text{ is effectively } \mathcal{H}^s\text{-null}\}.$$

- There are single reals of non-zero dimension: every λ-random real has dimension one.
- 1-dimension has an important stability property [Lutz]:

$$\dim^1_{\mathrm{H}} A = \sup\{\dim^1_{\mathrm{H}}\{X\} : X \in A\}.$$

Effective Dimension and Algorithmic Entropy

Effective Hausdorff dimension can be interpreted as a degree of incompressibility.

Theorem ((Ryabko); Mayordomo)

For every real X,

$$\dim^1_H X = \liminf_{n\to\infty} \frac{K(X\restriction_n)}{n}.$$

1-packing dimension (constructive strong dimension) can be effectivized using the martingale characterization by Athreya et al.

Theorem (Athreya et al)

For every real X,

$$\dim_{\mathrm{P}}^{1} X = \limsup_{n \to \infty} \frac{\mathrm{K}(X \upharpoonright_n)}{n}.$$

Let $0 < r < 1$ rational. Given a Martin-Löf random set X, define X_r by

$$X_r(m) = \begin{cases} X(n) & \text{if } m = \lfloor n/r \rfloor, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\dim^1_{\mathrm{H}} X_r = r$.

- Geometry: Hölder transformation of Cantor set
- Information theory: Insert redundancy

Effective Hausdorff Dimension

The three basic examples

Let μ_p be a Bernoulli ("coin-toss") measure with bias $p \in \mathbb{Q} \cap [0,1]$, and let X be random with respect to μ_p. Then

$$\dim^1_{\mathrm{H}} X = H(\mu_p) := -[p \log p + p \log(1-p)].$$

[Lutz; Eggleston]

- Kolmogorov complexity can be seen as an effective version of entropy.

Let U be a universal, prefix-free machine. Given a computable real number $0 < s \leqslant 1$, the binary expansion of the real number

$$\Omega^{(s)} = \sum_{\sigma \in \mathrm{dom}(U)} 2^{-\frac{|\sigma|}{s}}$$

has effective dimension s [Tadaki].

- Note that $\Omega^{(1)}$ is just Chaitin's Ω.

Effective Hausdorff Dimension

Each of the three examples actually computes a Martin-Löf random real.

- This is obvious for the "diluted" sequence.
- For recursive Bernoulli measures, one may use Von-Neumann's trick to turn a biased random real into a uniformly distributed random real.
  More generally, any real which is random with respect to a recursive measure computes a Martin-Löf random real [Levin; Kautz].
- $\Omega^{(s)}$ computes a fixed-point free function. It is of r.e. degree, and hence it follows from the Arslanov completeness criterion that $\Omega^{(s)}$ is Turing complete (and thus T-equivalent to a Martin-Löf random real).

The Dimension Problem

Are there reals of “genuine” non-integral dimension?

The stability property implies that the Turing lower cone of each of the three examples has effective dimension 1.

Question

Are there any Turing lower cones of non-integral dimension?

- Any such lower cone would come from a real of non-integral dimension for which it is not possible to extract some content of higher degree of randomness effectively.

Many-One Reducibility

Theorem (Reimann and Terwijn)

Let μ_p be a computable Bernoulli measure with bias p. If X is μ_p-random, then

$$Y \leqslant_{\mathrm{m}} X \;\Rightarrow\; \dim^1_{\mathrm{H}} Y \leqslant H(\mu_p).$$

Proof.

- Given an m-reduction f, define $F = \{n : (\forall m < n) f(m) \neq f(n)\}$, so F is the set of all positions of Y, where an instance of X is queried for the first time.
- F induces a Kolmogorov-Loveland place selection rule. If X is μ_p-random, this selection rule will yield a new sequence with the same limit frequency as X.

WTT-Reducibility

This technique does not extend to weaker reducibilities, since for Bernoulli measures the Levin-Kautz result holds for a total Turing reduction.

Theorem (Reimann and Nies)

For each rational α, $0 \leqslant \alpha \leqslant 1$, *there is a real* $X \leqslant_{\mathrm{wtt}} \emptyset'$ *such that*

$$\dim^1_{\mathrm{H}} X = \alpha \quad \textit{and} \quad (\forall Z \leqslant_{\mathrm{wtt}} X)\ \dim^1_{\mathrm{H}} Z \leqslant \alpha.$$

Requirements:

$$R_{\langle e,j\rangle} : Z = \Psi_e(X) \Rightarrow \exists(k \geqslant j)\, K(Z\restriction_k) \leqslant^+ (\alpha + 2^{-j})k$$

where (Ψ_e) is a uniform listing of wtt reduction procedures.

- We can assume each Ψ_e also has a certain (non-trivial) lower bound on the use g_e, because otherwise the reduction would decrease complexity anyway.

- Define a length k_j where we intend to compress Z, and let $m_j = g_e(k_j)$.
- Define σ_j of length m_j in a way that, if $x = \Psi_e^{\sigma_j}$ is defined then we compress it down to $(\alpha + 2^{-b_j})k_j$, by constructing an appropriate nullset L.
- The opponent's answer could be to remove σ_j from P. (σ_j is not of high dimension.)
- In this case, the capital he spent for this removal exceeds what we spent for our request, so we can account our capital against his.
- Of course, usually σ_j is much longer than x. So we will only compress x when the measure of oracle strings computing it is large.

A Wtt Lower Cone of Non-Integral Dimension

An important Lemma

- We assume that P is effectively approximated by clopen sets P_s.

Lemma

Let C be a clopen class such that $C \subseteq P_s$ and $C \cap P_t = \emptyset$ for stages $s < t$. Then

$$\Omega_t - \Omega_s \geqslant (\lambda C)^\alpha.$$

A Wtt Lower Cone of Non-Integral Dimension

Combining the strategies R_j

- In the course of the construction, some R_j might have to pick a new σ_j.
- In this case we have to initialize all R_n of lower priority $(n > j)$.
- We have to make sure that this does not make us enumerate too much measure into L.
- We therefore have to assign a new length k_n to the strategies R_n.
- For this, it is important to know the use of the reduction related to R_j.

The Turing Case

For the Turing case, the best known result is the following.

Theorem (Kjos-Hanssen, Merkle, and Stephan)

There exists recursive, non-decreasing, unbounded function h and a real X such that for all n,

$$K(X\restriction_n) \geqslant h(n) \qquad (*)$$

and X does not compute a Martin-Löf random real.

- The condition $(*)$ can be interpreted in terms of (generalized) Hausdorff measures. Reals satisfying $(*)$ are called complex.
- A real is complex with recursive bound h iff it is not effectively $\mathcal{H}^{\widetilde{h}}$-null, where $\widetilde{h} = 2^{-h(n)}$.

Diagonally Non-Recursive Functions

A function f is diagonally nonrecursive (dnr) if for all n, $f(n) \neq \varphi_n(n)$.

- Call a function f h-bounded if $f(n) \leqslant h(n)$ for all n.

Theorem (Kumabe)

There exists a minimal degree that contains a dnr function which is bounded by a recursive function.

Theorem (Kjos-Hanssen et al)

If X computes a recursively bounded dnr function f, then X computes a complex real.

Proof:

- Assume $f \leqslant_T X$ is g-bounded dnr. Code f into a real (e.g. via unary representation); since f is rec. bounded, so are the lengths of the codes.
- Let $l \geqslant 0$. For every σ, $|\sigma| \leqslant l$, consider program ψ_σ on input $(e, .)$:
  - Wait till $U(\sigma)$ converges.
  - Check whether $U(\sigma)$ correctly encodes a sequence of natural numbers $\langle y_1, \ldots, y_k \rangle$ with $k \geqslant e$.
  - If so, return y_e.

- By the Recursion Theorem, there exists a number e_σ such that $\psi_\sigma(e_\sigma, x) = \phi_{e_\sigma}(x)$ for all x. The fixed point can be found effectively.
- Let e be the maximum of all e_σ, $|\sigma| \leqslant l$, and let $h(l)$ be larger than the longest possible string needed to code a g-bounded function.
- It follows that $C(A\restriction_{h(l)}) > l$.
  - Suppose that $U(\sigma) = A\restriction_{h(n)}$ for some $|\sigma| \leqslant n$.
  - Then $\psi_\sigma(e_\sigma, e_\sigma)$ returns $f(e_\sigma)$.
  - But e_σ is a fixed point for ψ_σ, so $\psi_\sigma(e_\sigma, e_\sigma) = \varphi_{e_\sigma}(e_\sigma)$, contradicting the assumption that f is dnr.

Possible Strategies

To show that there exists a lower cone of non-integral dimension:

- Construct a minimal degree of positive dimension.
- Combine the wtt-technique with a hyperimmune-free construction.
- $\Omega^{(s)}$-operators?

To show that no such cone exists:

- Show that every real of positive dimension computes a measure for which it is random and apply the Levin-Kautz technique.
- Use sophisticated extractors?