Randomness – Beyond Lebesgue Measure

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Measures on Cantor Space

Outer measures from premeasures

Approximate sets from outside by open sets and weigh with a general measure function.

- A premeasure is a function $\rho : 2^{<\omega} \to \mathbb{R}_0^+ \cup \{\infty\}$.
- One can obtain an outer measure μ_ρ from ρ by letting

$$\mu_\rho(X) = \inf_{C \subseteq 2^{<\omega}} \left\{ \sum_{\sigma \in C} \rho(\sigma) : \bigcup_{\sigma \in C} U_\sigma \supseteq X \right\},$$

where U_σ is the basic open set induced by σ.
(Set $\mu_\rho(\emptyset) = 0$.)

The resulting $\mu = \mu_\rho$ is a countably subadditive, monotone set function, an outer measure.

Measures on Cantor Space

Types of measures

Probability measures: based on a premeasure ρ which satisfies

- $\rho(\emptyset) = 1$ and
- $\rho(\sigma) = \rho(\sigma^\frown 0) + \rho(\sigma^\frown 1)$.

Hausdorff measures: based on a premeasure ρ which satisfies

- If $|\sigma| = |\tau|$, then $\rho(\sigma) = \rho(\tau)$.
- $\rho(n)$ is nonincreasing.
- $\rho(n) \to 0$ as $n \to \infty$.
- For example: $\rho(\sigma) = 2^{-|\sigma| s}$, $s \geqslant 0$.

The way we constructed outer measures, $\mu(A) = 0$ is equivalent to the existence of a sequence $(W_n)_{n\in\omega}$, $W_n \subseteq 2^{<\omega}$, such that for all n,

$$A \subseteq \bigcup_{\sigma \in W_n} U_\sigma \quad \text{and} \quad \sum_{\sigma \in W_n} \rho(\sigma) \leqslant 2^{-n}.$$

Thus,

every nullset is contained in a G_δ nullset.

By requiring that the covering nullset is effectively G_δ, we obtain a notion of effective nullsets.

Definition

- A test relative to $z \in 2^\omega$ is a set $W \subseteq \mathbb{N} \times 2^{<\omega}$ which is c.e. in z.
- Given a natural number $n \geqslant 1$, an n-test is a test which is c.e. in $\emptyset^{(n-1)}$.
- A real x passes a test W if $x \notin \bigcap_n U(W_n)$, where $W_n = \{\sigma : (n, \sigma) \in W\}$.

Hence a real passes a test W if it is not in the G_δ-set represented by W.

To test for randomness, we want to ensure that W actually describes a nullset.

Definition

Suppose μ is a measure on 2^ω. A test W is correct for μ if for all n,

$$\sum_{\sigma \in W_n} \mu(U_\sigma) \leqslant 2^{-n}.$$

Any test which is correct for μ will be called a test for μ.

Accordant definitions for n-tests, arithmetical tests are straightforward.

An effective test for randomness should have access to the measure it is testing for.

- Therefore, represent it by an infinite binary sequence.
- As a measure on 2^ω is completely determined by its values on the cylinder sets (i.e. by the underlying premeasure ρ), it seems reasonable to represent these values via approximation by rational intervals.

Definition

Given a premeasure ρ, define its rational representation r_ρ by letting, for all $\sigma \in 2^{<\omega}$, $q_1, q_2 \in \mathbb{Q}$,

$$\langle \sigma, q_1, q_2 \rangle \in r_\rho \;\Leftrightarrow\; q_1 < \rho(\sigma) < q_2.$$

The condition $q_1 < \rho(\sigma) < q_2$ induces a subbasis for the weak topology on the space of probability measures.

- More general, if a space X is Polish, so is the space $M(X)$ of all probability measures on X (under the weak topology). Also, if X is compact metrizable, so is $M(X)$ (Prokhorov metric).

This yields various ways to represent a measure: Cauchy sequences, list of basic open balls it is contained in, etc.

Definition

Suppose ρ is a premeasure on 2^ω and $z \in 2^\omega$. A real is μ_ρ-z-random if it passes all $r_\rho \oplus z$-tests which are correct for μ_ρ.

Hence, a real x is random with respect to an arbitrary measure μ_ρ if and only if it passes all tests which are enumerable in the representation r_ρ of the underlying premeasure ρ.

The Initial Question

Question

What is the logical/computability theoretic structure of random reals?

Let μ be a probabilty measure and $f : 2^\omega \to 2^\omega$ be a continuous (Borel) function.

Define a new measure μ_f by setting

$$\mu_f(\sigma) = \mu(f^{-1} U_\sigma)$$

Observation: If $\mu\{x\} = 0$ for all $x \in 2^\omega$, then, for

$$F(x) = \mu\{y : \ y <_{\mathbb{R}} x\},$$

it holds that $\mu_F = \lambda$.

Idea

If the transformation f is computable in z, then it should preserve randomness, i.e. it should map a μ-z-random real to a μ_f-z-random one.

Note: If μ is a computable measure, then an atom of μ is μ-random iff it is computable.

Theorem (Levin, Kautz)

If a real is noncomputable and random with respect to a computable probabilty measure, then it is Turing equivalent to a λ*-random real.*

Non-Trivial Randomness

The atomic case

Note that every real is trivially random with respect to some μ if it is an atom of μ.

- We are interested in the case when a real is non-trivially random.

Theorem (Reimann and Slaman)

For any real x, the following are equivalent.

(i) *There exists (a representation of) a measure μ such that $\mu(\{x\}) = 0$ and x is 1-random for μ.*

(ii) *x is not computable.*

Features of the proof:

- Conservation of randomness:
- Randomness of cones:
  - Kucera's coding argument shows that every degree above $\emptyset'$ conatains a λ-random.
  - Relativize this using the Posner-Robinson Theorem.
  - Conclude that every non-recursive real x is Turing equivalent to some λ-z-random real for some real z.
- A basis theorem for relative randomness.

Randomness for Continuous Measures

In the proof there is no control over the measure that makes x random.

- Atoms cannot be avoided.
- Uses a special (though natural) representation of $M(2^\omega)$ as a particular Π^0_1 class.

Question

What if one admits only continuous probability measures?.

Theorem (Reimann and Slaman)

Let x be a real. For any $z \in 2^\omega$, the following are eqivalent.

(i) *x is truth-table equivalent to a λ-z-random real.*

(ii) *x is random for a continuous (dyadic) measure recursive in z.*

(iii) *There exists a functional Φ recursive in z which is an order-preserving homeomorphism of 2^ω such that $\Phi(x)$ is λ-z-random.*

This is an effective version of the classical isomorphism theorem for continuous probability measures.

The Class NCR

Question

Which level of logical complexity guarantees continuous randomness?

Let NCR_n be the set of all reals which are not n-random relative to any continuous measure.

- Kjos-Hanssen and Montalban: Every member of a countable Π^0_1 class is contained in NCR_1. (It follows that elements of NCR_1 can be found at arbitrary high levels of the hyperarithmetical hierarchy.)
- Reimann and Slaman: $\mathrm{NCR}_1 \subseteq \Delta^1_1$ (by arguments tailored for $n = 1$).

Example of higher order: Kleene's $\mathcal{O}$ cannot be 3-random with respect to a continuous measure.

Upper Bounds for Continuous Randomness

In general, can we give a distinct bound on NCR_n like in the case $n = 1$?

- There is some evidence that NCR_n grows very quickly with n.
- Can we give an upper bound?

Theorem (Reimann and Slaman)

For all n, NCR_n is countable.

NCR_n is Countable

Main Features of the Proof

- Show that the complement of NCR_n contains an upper Turing cone.
  - Show that the complement of NCR_n contains a Turing invariant and cofinal Borel set. We can use the set of all y that are Turing equivalent to some $z \oplus R$, where R is $(n+1)$-random relative to z.
  - Use Martin's result on Borel Turing sets to infer that the complement of NCR_n contains a cone.
- Go on to show that the elements of NCR_n are definable at a rather low level of the constructible universe.
  - $\mathrm{NCR}_n \subseteq L_{\beta_n}$, where β_n is the least ordinal such that

    $$L_{\beta_n} \models \mathrm{ZFC}^- + \text{there exist } n \text{ many iterates of the power set of } \omega,$$

    where ZFC^- is Zermelo-Fraenkel set theory without the Power Set Axiom.

NCR_n is Countable

Is the metamathematics necessary?

Question

Do we need to use metamathematical methods to prove the countability of NCR_n?

We make fundamental use of Borel determinacy; this suggests to analyze the metamathematics in this context.

The necessity of iterates of the power set is known from a result by Friedman.

- Martin's proof of Borel determinacy starts with a description of a Borel game and produces a winning strategy for one of the players.
- The more complicated the game is in the Borel hierarchy, the more iterates of the power set of the continuum are used in producing the strategy.

Theorem (Friedman)

$\mathrm{ZFC}^- \nvdash$ *All $\mathbf{\Sigma}^0_5$-games on countable trees are determined.*

Martin later improved this to $\mathbf{\Sigma}^0_4$.

Inductively one can infer from Friedman's result that in order to prove full Borel determinacy, a result about sets of reals, one needs infinitely many iterates of the power set of the continuum.

- The proof works by showing that there is a model of ZFC^- for which $\boldsymbol{\Sigma}^0_4$-determinacy does not hold.
- This model is L_{β_0}.

NCR and Iterates of the Power Set

We can work along similar lines to obtain a result concerning the countability of NCR_n.

Theorem

For every k, the statement

$$\text{For every } n,\ \mathrm{NCR}_n \text{ is countable.}$$

cannot be proven in

$$\mathrm{ZFC}^- + \text{there exists } k \text{ many iterates of the power set of } \omega.$$

- The proof (for $k = 0$) shows that there is an n such that NCR_n is cofinal in the Turing degrees of L_{β_0}. Hence, NCR_n is not countable in L_{β_0}.
- The witnesses for NCR_n are master codes of models L_α for certain $\alpha < \beta_0$.

Hausdorff Measures

Question

What is the computability theoretic relation between randomness for Hausdorff measures and randomness for (continuous) probability measures?

- We denote Hausdorff premeasures by h, and the corresponding measure by $\mathcal{H}^h$.

Hausdorff randomness can be interpreted as a degree of incompressibility.

Theorem

Let h be a computable Hausdorff premeasure. A real x is $\mathcal{H}^h$-random if and only if there exists a constant c such that for all n,

$$\mathrm{K}(x\restriction_n) \geqslant -\log h(n) - c.$$

- K denotes the prefix-free Kolmogorov complexity.

Theorem

For every computable dimension function h there is a real x such that x is $\mathcal{H}^h$-random but not random with respect to any computable measure.

- Proof: Join a 1-generic and a λ-random real with appropriate density.

Theorem (Kjos-Hanssen, Merkle, and Stephan; R. and Slaman)

There exists computable, unbounded, nondecreasing function h and a real x such that for all n,

$$K(x\upharpoonright_n) \geqslant h(n) \tag{*}$$

and x does not compute a Martin-Löf random real.

How close to $h(n) = n$ can h be? (The Dimension Problem)

Theorem (Reimann and Nies)

For each rational r, $0 \leqslant r \leqslant 1$, *there is a real* $x \leqslant_{\mathrm{wtt}} \emptyset'$ *such that*

$$\liminf_n \frac{K(x \upharpoonright_n)}{n} = r \quad \textit{and} \quad (\forall z \leqslant_{\mathrm{wtt}} x) \ \liminf_n \frac{\mathrm{K}(z \upharpoonright_n)}{n} \leqslant r.$$

Probability Measures and Hausdorff Measures

Theorem

For every rational s and every real x such that x is $\mathcal{H}^s$-random, there exists a probability measure μ such that x is μ-random and for some $c > 0$

$$\mu(U_\sigma) \leqslant c 2^{-|\sigma| s} \quad \textit{for all } \sigma \in 2^\omega. \tag{*}$$

- Here $\mathcal{H}^s$ denotes the Hausdorff measure based on the premeasure $2^{-|\sigma| s}$.

Assume that Φ and Ψ are Turing functionals such that

$$\Psi(x) = R \quad \text{and} \quad \Phi(R) = x,$$

where R is λ-random.

Let $\mathrm{Pre}(\sigma) := \{\tau \in 2^\omega : \Phi(\tau) \sqsupseteq \sigma \ \ \& \ \ \Psi(\sigma) \sqsubseteq \tau\}$.

To define a measure μ with respect to which x is random, we satisfy two requirements:

(1) $\mathrm{Pre}(\sigma) := \{\tau \in 2^\omega : \Phi(\tau) \sqsupseteq \sigma \ \ \& \ \ \Psi(\sigma) \sqsubseteq \tau\}$.

(2) $\lambda(U_{\mathrm{Pre}(\sigma)}) \leqslant \mu(U_\sigma) \leqslant \lambda(U_{\Psi(\sigma)})$

The configurations given by (2) induce a Π^0_1 class P of probability measures (with respect to a certain Cauchy representation).

We want to show that for some $\mu \in P$, x is μ-random.

Note that if (V_n) were a μ-test covering x, then $\Phi^{-1}(V_n)$ would be a λ-r_μ-test covering R.

So, what we need to show is that R is λ-r_μ-random for some $\mu \in M$.

Theorem (indep. by Downey, Hirschfeldt, Miller, and Nies)

If $B \subseteq 2^\omega$ is nonempty and Π^0_1, then, for every R which is λ-random there is $z \in B$ such that R is λ-z-random.

The proof is essentially a compactness argument.

Construct an effectively closed set of measures.

- Work along a computable tree $T \subseteq 2^{<\omega}$ whose infinite paths are $\mathcal{H}^s$-random.
- Define sequence of uniformly computable measures $\{\mu^n\}$. Each μ^n can be seen as an approximation to a final μ, knowing only the paths of T up to length n.
- For $n \in \mathbb{N}$, define μ^n_n such that for all $\sigma \in 2^n$

$$\mu^n_n\restriction_{U_\sigma} = \begin{cases} 2^{(1-s)n}\,\lambda\restriction_{U_\sigma}, & \text{if } \sigma \in T, \\ 0, & \text{if } \sigma \notin T. \end{cases}$$

- Modify μ_n^n downward to ensure (*) at all levels $\leqslant n$. Define μ_{n-1}^n by requiring that

$$\mu_{n-k-1}^n \upharpoonright_{U_\sigma} = \gamma(\sigma)\mu_{n-k}^n \upharpoonright_{U_\sigma}$$

  where $\gamma(\sigma) = \min\{1, 2^{-(n-k-1)s}(\mu_{n-k}^n U_\sigma)^{-1}\}$.
- We stop as soon as $[T] \subseteq U_\sigma$ for some $\sigma \in 2^{k_0}$ and define $\mu^n = \mu_{k_0}^n$. k_0 can be determined effectively and μ^n is a computable measure, as the μ_m^n are.
- Finally, note that every real is computable in a λ-random real. (Gacs, Kucera)
- Combine the Gacs-reduction with the basis theorem to obtain the desired continuous probability measure.