Effective Aspects of Diophantine Approximation

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Diophantine Approximation

- **Diophantine Approximation** classifies real numbers by how well they may be approximated by rational numbers.
- Measure in terms of denominator:

$$\left| x - \frac{p}{q} \right| < F(q)$$

- For which F does this have **infinitely many solutions**?

Dirichlet

- For any irrational number α there exist infinitely many rational numbers p/q such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}$$

- Such a sequence is given by the **continued fraction expansion**

$$\alpha = [a_0; a_1, \dots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots}} \qquad a_i \in \mathbb{Z}^+$$

Limits of approximability

- In general, one cannot improve the factor 2 in Dirichlet's theorem.
- A number β is **badly approximable** if there exists a K such that

$$\forall \frac{p}{q} \ \left|\beta - \frac{p}{q}\right| > \frac{K}{q^2}$$

- badly approximable $\Leftrightarrow$ continued fraction *bounded*.

Badly approximable numbers

- Roots of quadratic polynomials are badly approximable (continued fraction is periodic).
  - Golden mean $(1+\sqrt{5})/2 = [1; 1, 1, 1, \ldots]$.
  - $\sqrt{2} = [1;2,2,2,\ldots]$
- Let

$$M(\alpha) = \inf \left\{ M \colon \exists^\infty\, p/q \; |\alpha - p/q| < M/q^2 \right\}.$$

- **Question:** Is $K(\alpha)$ computable for algebraic numbers?
  - Recent work by Chonev, Ouaknine, and Worrell ties this to the (unbounded) *Continuous Skolem Problem*.

Liouville

- Roots of quadratic polynomials are badly approximable (continued fraction is periodic).
- **THM:** If α is algebraic of degree d, then there exists a constant $L(\alpha)$ such that

$$\forall \frac{p}{q} \ \left|\alpha - \frac{p}{q}\right| > \frac{L(\alpha)}{q^d}.$$

- L is computable from the minimal polynomial for α.

Transcendental Numbers

- Liouville used this result to explicitly construct transcendental numbers.
- They are examples of what is now known as a **Liouville number**:

$$\forall n \, \exists \frac{p}{q} \, \left| \alpha - \frac{p}{q} \right| < \frac{1}{q^n}$$

Measuring Irrationality

The **irrationality exponent** of a real number x is defined as

$$\delta(x) = \sup\left\{\delta \colon \exists^\infty \frac{p}{q}\ \left|x - \frac{p}{q}\right| < \frac{1}{q^\delta}\right\}.$$

- Every irrational number has irrationality exponent ≥ 2.
- A Liouville number has $\delta = \infty$.
- Other examples:
  - $\delta(e) = 2$
  - $\delta(\pi) \leq 7.60630853$

The Search For $\delta(\alpha)$

Let α be algebraic of degree d.

- **Thue:** $\delta(\alpha) \leq \frac{1}{2}d + 1$ [1909]
- **Siegel:** $\delta(\alpha) \leq 2\sqrt{d}$ [1921]
- **Roth:** $\delta(\alpha) = 2$ [1955]

Effectivity Issues

While Liouville's proof is completely effective, Thue's method introduced ineffectiveness.

- In particular, for $\delta > \delta(\alpha)$, the exact, finite number of solutions cannot be extracted from the proof.
- **Thm:** [Davenport] There exist a primitive recursive function $\kappa(d)$ ($\frac{1}{2}d + 1 < \kappa(d) < d$) and a computable function $q(x, y)$ such that if $\kappa > \kappa(d)$ and α is algebraic of degree d,

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{\kappa}}$$

has *at most* one solution $q > q(\alpha, \kappa)$.

Effectivity Issues

Question: Is the function

$$(\alpha, \varepsilon) \mapsto \# \text{ of solutions to } \left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{2+\varepsilon}}$$

computable?

The Metric Theory

- Almost every real has irrationality exponent 2.
- Let

$$I_\delta = \{x \colon \delta(x) \geq \delta\}$$

- **Jarník**, independently **Besicovitch**:

$$\dim_H(I_\delta) = \frac{2}{\delta}$$

Jarnik's Fractal

- Let

$$G_q(a) = \left\{ x \in \left(\frac{1}{q^a}, 1 - \frac{1}{q^a} \right) : \exists p \ \left| x - \frac{p}{q} \right| \leq \frac{1}{q^a} \right\}.$$

- For n sufficiently large, $q_1 \neq q_2$ prime and $n < q_1, q_2 \leq 2n$,

$$G_{q_1}(a) \cap G_{q_2}(a) = \varnothing \text{ with gaps } \geq \frac{1}{8n^2}$$

Jarník's Fractal

- If we let

$$H_n(a) = \bigcup_{\substack{q \text{ prime} \\ n<q\leq 2n}} G_q(a),$$

  and let (n_i) be a sufficiently fast growing sequence, then

$$\bigcap H_{n_i}(a)$$

  is a **Cantor set** (after some trimming) containing only reals with irrationality exponent $\geq a$.
- Show that this Cantor set has dimension $2/a$.

Mass Distribution Principle

Let $s > 0, X \subseteq \mathbb{R}$. If μ is a probability measure on $\mathbb{R}$ such that $\mu(X) > 0$, and there exist $\varepsilon, c > 0$ such that for every interval I,

$$|I| < \varepsilon \text{ implies } \mu(I) \leq c\varepsilon^s,$$

then

$$\dim_H(X) \geq s.$$

- We can *uniformly* distribute a (unit) mass along a Cantor set and get a bound for the measure of $|I|$ from
  - the number of subintervals in each step (for Jarník's fractal: prime number theorem),
  - the length of gaps between intervals.

Effective Dimension

- The irrationality exponent reflects how well a real can be approximated by rational numbers.
- The **effective dimension** [Lutz] reflects how well a real can be approximated by computable numbers:

$$\dim(x) = \liminf_{n \to \infty} \frac{C(x \upharpoonright n)}{n}$$

Effective Dimension and Irrationality Exponent

- For a random real x, p/q cannot give significantly more than $2 \log q$ bits of information about x.
  - Hence almost every real has irrationality exponent 2.
- If $x \in (0,1)$ is Liouville, on the other hand, for every n there exist p/q such that $2 \log q$ bits of information give us $n \log q$ bits of x
  - Hence the effective dimension of a Liouville number is 0 [Staiger]
- This line of reasoning can be generalized to obtain

$$\dim(x) \leq \frac{2}{\delta(x)} \qquad \text{[Calude \& Staiger].}$$

  - This gives the upper bound on the Hausdorff dimension of I_δ

Effective Dimension and Irrationality Exponent

Jarník's proof actually shows that

$$\dim_H\{x\colon \delta(x) = \delta\} = \frac{2}{\delta}.$$

Question: Are Hausdorff/effective dimension and irrationality exponent completely independent?

- Can reals have effective dimension $\beta < 2/\delta$ for any choice of β?

Theorem 1

Let $\delta \geq 2$. For every $\beta \in [0, 2/\delta]$ there is a Cantor-like set E such that $\dim_H(E) = \beta$ and for the uniform measure on E, almost all real numbers have irrationality exponent δ.

Features Of The Proof

- Basic approach is to “thin” Jarník’s fractal – use less intervals at each step.
- However, in a straighforward way this only gets us down to dimension $1/\delta$.
  - We can get past this barrier by choosing only a uniformly spaced subset of $G_q(\delta)$ for a single q each step.
- Another problem is that the thinning might concentrate the measure no longer on reals of irrationality exponent δ.

Construction Template

- Define a **family of Cantor sets**

$$\mathcal{E}(\vec{q}, \vec{m}, \vec{\delta})$$

  - $\vec{q}$: controls the choice of subintervals (thinning)
  - $\vec{m}$: controls the branching ratio
  - $\vec{\delta}$: controls the irrationality exponent (width of the intervals)
- Show that for each β, δ, $\beta \leq 2/\delta$, one can find suitable $\vec{q}, \vec{m}, \vec{\delta}$ such that every fractal in $\mathcal{E}(\vec{q}, \vec{m}, \vec{\delta})$ is a subfractal of the corresponding Jarník fractal $\mathcal{E}(\vec{m}, \vec{\delta})$ and has Hausdorff dimension β.

Construction Template

- The family $\mathcal{E}(\vec{q}, \vec{m}, \vec{\delta})$ can be seen as a **tree of Cantor sets**, since identical initial thinning choices up to stage *n* will lead to identical fractals at stage *n*.
- Use a measure-theoretic pigeonhole argument to construct a **path through** $\mathcal{E}(\vec{q}, \vec{m}, \vec{\delta})$ so that the resulting Cantor set has negligible measure on reals with irrationality exponent $> \delta$.

Main Results

Theorem 2

Let $\delta \geq 2$. For every $\beta \in [0, 2/\delta]$ there is a Cantor-like set E such that for the uniform measure on E, almost all real numbers have irrationality exponent δ and effective dimension β.

Construction Template

Modifications needed:

- Since δ and β are arbitrary real number, we have to work with approximations rather than the numbers directly.
- Ensure that the compressibility ratio of every member of E obeys the appropriate upper bound.
  - exhibit an uniformly computable map taking binary sequences of a fixed, computable length onto the k-th step in the construction of E.

Reference

V. Becher, J. Reimann, and T. Slaman, Irrationality exponent, Hausdorff dimension and effectivization, submitted. http://arxiv.org/abs/1601.00153